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REMOVABLE SINGULARITIES FOR BLOCH AND BMO FUNCTIONS

BY

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Introduction

This paper is a continuation of a study of the following problem. Let Ω be a domain in $\mathbf{C}^n$ with V a subvariety of Ω. Assume $\mathcal{F}$ is a space of holomorphic functions (on Ω) which satisfy a given property. If one is given a function f holomorphic on $\Omega \setminus V$ what conditions on the growth of f and what geometric properties of V will permit us to extend f to a function $\tilde{f}$ holomorphic on Ω and such that $\tilde{f}$ is in $\mathcal{F}$. In [2] we solved this problem for the ball in $\mathbf{C}^n$ and the space $\mathcal{F}$ represented by the Hardy space. In this paper we pursue this line of investigation and prove the following theorem.

THEOREM 1. *Let Ω be a bounded domain in $\mathbf{C}^n$ and let V be a subvariety of Ω. Assume a function f is holomorphic on $\Omega \setminus V$, with f satisfying the area bounded mean oscillation (BMO) condition on $\Omega \setminus V$. Assume further that V satisfies condition A* (see Section 4). *Then f extends to a function $\tilde{f}$ holomorphic on Ω and $\tilde{f}$ satisfies the area BMO condition.*

Definitions will be given in detail in Section 2. If Ω is strictly pseudo convex then the BMO condition referred to in the theorem coincides with the Bloch condition. A special case of this theorem which motivates the geometric condition on V is the following.

THEOREM 2. *Let $\mathbf{B}_1$ be the unit disc in $\mathbf{C}^1$ and let $V = \{\alpha_j\}_1^\infty$ be a discrete set satisfying the sparsity condition*

$$(1) \qquad \chi(\alpha_j, \alpha_k) = \left| \frac{\alpha_j - \alpha_k}{1 - \overline{\alpha}_j \alpha_k} \right| \geq \delta > 0, \quad j \neq k.$$

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Assume f is holomorphic on $\mathbf{B}_1 \setminus V$ *and satisfies*

(2) $$|f'(z)| \leq c[\min\{\rho(z, V), \rho(z, \partial \mathbf{B}_1)\}]^{-1} \; z \in \mathbf{B}_1 \setminus V$$

for c and δ positive constants and ρ the distance from z to V (in the Euclidean norm). Then f extends holomorphically to a function $\tilde{f}$*, holomorphic on* $\mathbf{B}_1$*, and* $\tilde{f}$ *is a Bloch function, i.e. there exists* $c_1 > 0$ *such that*

$$|\tilde{f}'(z)| \leq c_1(\rho(z, \partial \mathbf{B}_1))^{-1}, \quad z \in \mathbf{B}_1.$$

An example of R. Timoney shows that condition (2) alone is not sufficient to obtain the result of Theorem 2. More detailed comments concerning the one variable case will follow after the proofs of these theorems. Our condition A is in part a covering condition of Whitney type. We note that conditions on the Whitney decomposition of a domain $\Omega \subseteq \mathbf{R}^n$ have been used by Peter Jones [6] in his work on extending *BMO* functions from the domain Ω in $\mathbf{R}^n$ into all of $\mathbf{R}^n$. Some of the research in this paper was carried out while the second author was on sabbatical leave at the University of California, Berkeley.

2. Area BMO functions and Bloch functions

Let Ω be a bounded domain in $\mathbf{R}^n$ and let f be in $L'(\Omega)$. For $Q \in \Omega$ and $r > 0$, $\mathcal{B}(a, r) \equiv \{x \in \mathbf{R}^n: \|x\| < r\}$ and let $\lambda_n = \lambda$ denote Lebesgue measure on $\mathbf{R}^n$. The function f is said to have bounded mean oscillation if there is a number $M > 0$ such that

(3) $$\frac{1}{|\mathcal{B}(Q, r)|} \int_{\mathcal{B}(Q,r)} |f(x) - \left(\frac{1}{|\mathcal{B}(Q, r)|} \int_{\mathcal{B}(Q,r)} f(y) d\lambda(y)\right) | \, d\lambda(x) \leq M$$

for all $Q \in \Omega$ and $r > 0$ such that $\mathcal{B}(Q, r) \subseteq \Omega$ and where $\lambda(\mathcal{B}(Q, r)) = |\mathcal{B}(Q, r)|$.

A function f holomorphic on $\mathbf{B}_1$ is said to be a Bloch function if there is a constant M such that

$$|f'(z)| \leq M(1 - |z|^2)^{-1}, \quad z \in \mathbf{B}_1.$$

In general if Ω is a bounded symmetric domain in $\mathbf{C}^n$ and f is holomorphic on $\mathbf{C}^n$ then f is a Bloch function (on Ω) if

(4) $$\sup\left\{\frac{\langle \nabla f(z), x\rangle}{H_z(x, \bar{x})^{1/2}} : x \neq 0, x \in \mathbf{C}^n, z \in \Omega\right\} < +\infty$$

where $H_z(x, \bar{x})$ denotes the Bergman metric on Ω. If Ω is strictly pseudo convex this definition is equivalent to requiring that

(4′) $$|\nabla f(z)| \leq M(\rho(z, \partial\Omega))^{-1}, \quad z \in \Omega.$$

For the unit disc, as well as strictly pseudo convex domains in $\mathbf{C}^n$ the set of holomorphic BMO functions on Ω coincides with the space of Bloch functions on Ω. This follows from the Basic Lemma proven below. For the $n = 1$ case (and with a slightly different BMO condition) this correspondence is noted in the paper of Coiffman, Rochberg and Weiss [3, p. 47]. For $n > 1$, the proof given here is based on written correspondence with R. Timoney.

BASIC LEMMA. *Let Ω be a bounded domain in $\mathbf{R}^n$ and let f be harmonic on Ω. The function f satisfies the* BMO *condition* (3) *if and only if there is a constant M' such that*

$$|\nabla f(x)| \leq M'(\rho(x, \partial\Omega))^{-1}, \quad x \in \Omega. \tag{5}$$

Proof. Let us assume that g is a function harmonic on the unit ball of $\mathbf{R}^n$ and that

$$|\nabla g(x)| \leq c(1 - |x|)^{-1}.$$

Then

$$g(x) - g(0) = \int_0^1 \left(\frac{d}{dt} g(tx)\right)dt = \int_0^1 \nabla g(tx) \circ x \, dt.$$

It follows then that

$$|g(x) - g(0)| \leq \int_0^1 \frac{|x|}{1 - t|x|} dt = c \, ln(1 - |x|)$$

and

$$\int_{|x|\leq 1} |g(x) - g(0)| d\lambda(x) \leq c \int_{|x|\leq 1} \log(1 - |x|) d\lambda(x) = c'.$$

Now assume f is harmonic on Ω and satisfies (5). Let $a \in \Omega$, $r < \rho(a, \partial\Omega)$. Define $g(y) = f(a + ry)$ for $|y| < 1$. Observe that

$$r(1 - |y|) = \rho(a + ry, \partial\mathcal{B}(a, r)),$$

hence

$$|\nabla g(y)|(1 - |y|) = r(1 - |y|)|\nabla f(a + ry)| \leq M'.$$

With this notation ($x = a + ry$),

$$\frac{1}{|\mathcal{B}(a, r)|} \int_{\mathcal{B}(a,r)} \left|f(x) - \left(\frac{1}{|\mathcal{B}(a, r)|} \int_{\mathcal{B}(a,r)} f(t)dt\right)\right| d\lambda(x)$$

$$= \frac{1}{r^n} \int_{\mathcal{B}(a,r)} |f(x) - f(a)| d\lambda(x) = \int_{|y|<1} |g(y) - g(0)| d\lambda(y).$$

As we have shown above the right side of this inequality is uniformly bounded and this proves that f is in $\mathrm{BMO}(\Omega)$.

Assume for the converse that f is a harmonic function on Ω and satisfies the BMO condition. Let $a \in \Omega$, $r = 1/2\rho(a, \partial\Omega)$ and for $|y| < 1$ set $g(y) = f(a + ry)$. There is a smooth kernel $K(y, \eta)$ such that, for $|y| < 1/2$,

$$g(y) - g(0) = \int_{1/2<|\eta|<3/4} K(y, \eta)(g(\eta) - g(0))d\lambda(\eta).$$

Hence, there is a constant c such that

$$|\nabla g(0)| = \left|\int_{1/2<|\eta|<3/4} \nabla y K(y, \eta)(g(\eta) - g(0))d\lambda(\eta)\right| \leqslant c \int_{|\eta|<1} |g(\eta) - g(0)|d\lambda(\eta).$$

By the earlier part of the proof,

$$\rho(a, \partial\Omega)|\nabla f(a)| = 2|\nabla g(0)| \leqslant c' \int_{|\eta|<1} |g(\eta) - g(0)|d\lambda(\eta) \leqslant c''\|f\|_{\mathrm{BMO}}.$$

3. Proof of Theorem 2

Condition (1) on the pseudo-hyperbolic metric for the set $V = \{\alpha_j\}$ is called a sparsity condition (See Vinogradov and Khavin [10], Sarason [7, p. 22]). It is equivalent to the following, there is a $\delta' > 0$ such that

$$|\alpha_j - \alpha_k| \geqslant \delta'(1 - |\alpha_k|), \quad j \neq k. \tag{6}$$

For if $j \neq k$ and condition (6) is satisfied,

$$\left(\frac{|\alpha_j - \alpha_k|}{1 - \bar{\alpha}_j\alpha_k|}\right)^2 \geqslant (\delta')^2 \frac{(1 - |\alpha_j|)(1 - |\alpha_k|)}{|1 - \bar{\alpha}_j\alpha_k|^2}.$$

The equality

$$\left|\frac{a - b}{1 - \bar{a}b}\right|^2 = 1 - \frac{(1 - |a|^2)(1 - |b|^2)}{|1 - \bar{a}b|^2}, \quad \bar{a}b \neq 1,$$

implies

$$\left(\left|\frac{\alpha_j - \alpha_k}{1 - \bar{\alpha}_j\alpha_k}\right|\right)^2 \geqslant (\delta')^2\left(1 - \left|\frac{\alpha_j - \alpha_k}{1 - \bar{\alpha}_j\alpha_k}\right|^2\right)$$

or

$$\left|\frac{\alpha_j - \alpha_k}{1 - \bar{\alpha}_j\alpha_k}\right|^2 (1 + (\delta')^2) \geqslant (\delta')^2.$$

The inequality

$$\left|\frac{\alpha_j - \alpha_k}{1 - \bar{\alpha}_j\alpha_k}\right| \leqslant \frac{|\alpha_j - \alpha_k|}{1 - |\alpha_k|}$$

yields the other half of the equivalence.

We proceed now to the proof of Theorem 2. Assume that f and V satisfy the hypothesis of Theorem 2. For z near α_j our hypothesis implies that f' has an expansion of the form

$$f'(z) = \frac{A_{-1}}{(z - \alpha_j)} + A_0 + A_1(z - \alpha_j) + \ldots$$

for z near α_j. A standard argument shows that f can be continued to a holomorphic function $\tilde{f}$ in a neighborhood of α_j. Hence, f extends to $\tilde{f}$ holomorphic on $\mathbf{B}_1$.

LEMMA 1. *Let D_j denote the Euclidean disc, center α_j and radius*

$$c(1 - |\alpha_j|) = r_j$$

(*where c is a constant smaller than $\delta'/2$*) *and let $\Omega = \mathbf{B}_1/\cup_1^\infty D_j$. Then for $z \in \Omega$, $\rho(z, \partial\mathbf{B}_1) \leq c'(\rho(z, V))$.*

Proof. Assume not and that $z_n \in \mathbf{B}_1 \setminus \cup D_j$ and $t_n \in V$ with

$$\lim_{n\to\infty} \left(\frac{|z_n - t_n|}{1 - |z_n|}\right) = 0.$$

Under this assumption the inequalities

$$\frac{1 - |t_n|}{1 - |z_n|} \leq 1 + \frac{|t_n - z_n|}{1 - |z_n|}$$

and

$$1 - \frac{|t_n - z_n|}{1 - |z_n|} \leq \frac{1 - |t_n|}{1 - |z_n|}$$

imply that

$$\lim_{n\to\infty}\left(\frac{|z_n - t_n|}{1 - |t_n|}\right) = 0.$$

This is inconsistent with our hypothesis and so the lemma is valid.

Lemma 1 shows there is a constant $d_1 > 0$ such that, for $z \in \Omega = \mathbf{B}_1 \setminus \cup_1^\infty D_j$,

$$|f'(z)| \leq d_1(\rho(z, \partial\mathbf{B}_1))^{-1}.$$

Also, for j fixed and $z \in \overline{D}_j$,

$$|\tilde{f}'(z)| \leq \max\{|\tilde{f}'(\zeta)| : \zeta \in \partial D_j\} \equiv M_j.$$

Note that, for $\zeta \in \partial D_j$,

$$|\tilde{f}'(\zeta)| \leq d_1(\rho(\zeta, \partial B_1))^{-1} \leq \frac{d_1}{r_j}$$

and so $M_j \leq d_1/r_j$ for all $j = 1, 2, 3, \ldots$.

LEMMA 2. *For w_1, w_2 in D_j we have constants c_1 and c_2 which satisfy*

$$0 < c_1 \leq \frac{\rho(w_1, \partial \mathbf{B}_1)}{\rho(w_2, \partial \mathbf{B}_1)} \leq c_2$$

where the constants are independent of j.

Proof. For w_1, w_2 in $\overline{D}_j$, we have

$$\frac{1 - |w_1|}{1 - |w_2|} \leq \frac{(1 - |\alpha_j|) + c(1 - |\alpha_j|)}{(1 - |\alpha_j|) - c(1 - |\alpha_j|)} \leq \frac{1 + c}{1 - c}.$$

Similarly,

$$\frac{1 - c}{1 + c} \leq \frac{1 - |w_1|}{1 - |w_2|}$$

The proof of Theorem 2 is completed by observing that for $z \in \overline{D}_j$,

$$|\tilde{f}'(z)| \leq M_j \leq \frac{d_1}{r_j} \leq \frac{d_1}{c}\left(\frac{1 + c}{1 - c}\right)\frac{1}{1 - |z|}.$$

The following example of R. Timoney [9] shows the condition (2) of Theorem 2 is not sufficient and so some geometric condition must be imposed on V to obtain the result. Take $f(z) = (1 - z)^{-1}$ which is not in the Bloch space. For V, choose the sequence

$$V = \bigcup_{n=1}^{\infty}\left(\bigcup_{j=1}^{n^2} \alpha_{j,n}\right) \quad \text{where } \alpha_{j,n} = \left(1 - \frac{1}{n}\right)\exp\left(\frac{-2\pi j}{n^2}\right) i.$$

V is the zero set of some function holomorphic on $\mathbf{B}_1$. The points $\alpha_{j,n}$ are evenly distributed on a system of circles centered at 0. The distance between consecutive circles is approximately $(\sim)1/n^2$. There are n^2 points on the circle of radius $1 - 1/n$ so the distance between adjacent points on the circle is also approximately $1/n^2$. Hence for

$$1 - \frac{1}{n} \leq |z| \leq 1 - \frac{1}{n+1}, \quad \rho(z, V) \sim 1/n^2.$$

But, for

$$1 - \frac{1}{n} \leq |z| \leq 1 - \frac{1}{n+1},$$

we have

$$|f'(z)| = \left|\frac{1}{1 - z}\right|^2 \leq \frac{1}{(1 - |z|)^2} \leq (n + 1)^2 \sim (\rho(z, V))^{-1}.$$

Hence, $|f'(z)|$ satisfies the growth condition (2) but can not be (is not) extended to a Bloch function.

4. Generalization to domains in $\mathbf{C}^n$

The purpose of this section is to generalize the ideas of the previous section. In particular, since the zeros (and poles) of holomorphic functions of several complex variables are not isolated the separation property of Theorem 2 must be replaced by a more refined covering argument in the n (>1) dimensional case. In the following Ω is a bounded domain in $\mathbf{C}^n$ and V is a subvariety of Ω.

DEFINITION. The subvariety V satisfies the A covering condition if the following conditions are satisifed.

(a) There are polydiscs P_α with centers z_α and polyradii $r_\alpha = (r_1(\alpha), \ldots, r_n(\alpha))$ such that $\overline{P}_\alpha \subset \Omega$ and $V \subseteq \cup P_\alpha$.

(b) There are constants d_1 and d_2 such that

$$d_1 r_j(\alpha) \leqslant \rho(P_\alpha, \partial\Omega) \leqslant d_2 r_j(\alpha), \quad j = 1, 2, \ldots, n.$$

(3) There is a constant $d_3 > 0$ such that

$$\rho(V, \partial_0 P_\alpha) \geqslant d_3 \rho(P_\alpha, \partial\Omega) \quad \text{for all } \alpha.$$

The distinguished boundary of a polydisc P_α is written $\partial_0 P_\alpha$.

(d) There is a constant $d_4 > 0$ such that for any $w \in V$ there exists α with $w \in P_\alpha$ and

$$\rho(w, \partial P_\alpha) \geqslant d_4 \rho(P_\alpha, \partial\Omega).$$

Examples of such varieties are given by subvarieties of bounded domains which extend across the boundary of Ω and are smooth near the boundary. More details will be given in the next section. In view of the Basic Lemma, Theorem 1 may be stated in the following form.

THEOREM 1′. *Ω is a bounded domain in $\mathbf{C}^n$ and V is a subvariety of Ω satisfying the A covering condition. Suppose f is holomorphic on Ω/V and*

$$|\nabla f(z)| \leqslant c[\min(\rho(z, \partial\Omega), \rho(z, V))]^{-1}, \quad z \in \Omega \setminus V$$

then f has a holomorphic extension $\tilde{f}$ to Ω and $\tilde{f}$ satisfies

$$|\nabla \tilde{f}(z)| \leqslant c'[\rho(z, \partial\Omega)]^{-1}.$$

Hence, if Ω is strictly pseudo convex $\tilde{f}$ is a Bloch function.

Proof of Theorem 1′. We use the Weierstrass Preparation Theorem to reduce the extension problem to the one variable argument given in the last section (also, see [2, p. 25]). Henceforth, assume $\tilde{f}$ extends f holomorphically to all of Ω.

LEMMA 2. *If w_1, w_2 are in P_α there are constants c_1 and c_2 such that*

$$c_1 \leqslant \frac{\rho(w_1, \partial\Omega)}{\rho(w_2, \partial\Omega)} \leqslant c_2$$

where the c_i are independent of α.

Proof. The assumption on V states that each $r_j(\alpha)$ is equivalent to $\rho(P_\alpha, \partial\Omega)$ (condition (b) of A). Hence

$$\frac{\rho(w_1, \partial\Omega)}{\rho(w_2, \partial\Omega)} \leqslant \frac{\rho(w_1, w_2) + \rho(w_2, \partial\Omega)}{\rho(w_2, \partial\Omega)} \leqslant 1 + \mathrm{c}\,\frac{\rho(P_\alpha, \partial\Omega}{\rho(w_2, \partial\Omega)} \leqslant 1 + c.$$

Similarly, it follows that

$$\frac{\rho(w_1, \partial\Omega)}{\rho(w_2, \partial\Omega)} \geqslant \frac{1}{1 + c}.$$

LEMMA 3. *For $z \in \Omega' = \Omega \setminus \cup P_\alpha$ there is a constant k_1 such that*

$$\rho(z, V) \geqslant k_1\, \rho(z, \partial\Omega).$$

Proof. Suppose the conclusion is false and that there exists sequences $\{z_j\}$ in Ω' and $\{W_j\}$ in V such that

$$\lim_{j\to\infty} \frac{\rho(z_j, w_j)}{\rho(z_j, \partial\Omega)} = 0. \tag{7}$$

Let P_j be a polycylinder associated to w_j according to condition (d) of the A-covering property. We note that

$$\frac{\rho(z_j, \partial\Omega)}{\rho(z_j, w_j) + \rho(z_j, \partial\Omega)} \leqslant \frac{\rho(z_j, \partial\Omega)}{\rho(w_j, \partial\Omega)} \leqslant \frac{\rho(z_j, \partial\Omega)}{\rho(z_j, \partial\Omega) - \rho(z_j, w_j)}. \tag{8}$$

Inequality (8) and equality (7) imply that

$$\lim_{j\to\infty} \frac{\rho(z_j, \partial\Omega)}{\rho(w_j, \partial\Omega)} = 1. \tag{9}$$

Since z_j is not in P_j it follows from Lemma 2 and (9) that

$$\lim_{j\to\infty} \frac{\rho(w_j, \partial P_j)}{\rho(P_j, \partial\Omega)} = 0.$$

This contradicts condition (d) of the A-covering property.

From Lemma 3 it follows that $|\nabla f(z)| \leqslant k'(\rho(z, \partial\Omega))^{-1}$ all z in Ω'. It remains to estimate $|\nabla\tilde{f}|$ for z in P_α (independent of α). For any polydisc P_α with z_0 in the distinguished boundary $(\partial_0 P_\alpha)$ we have

$$\rho(z_0, V) \geqslant d_3\, \rho(P_\alpha, \partial\Omega).$$

Also, from Lemma 2,

$$\rho(z_0, \partial\Omega) \geqslant c_1 \rho(P_\alpha, \partial\Omega).$$

The assumption on ∇f implies

$$|\nabla f(z)| \leqslant c'[\rho(P_\alpha, \partial\Omega)]^{-1}\ z \in \partial_0 P_\alpha.$$

By the maximum principle applied to $\partial f/\partial z_j$ $(j = 1, 2, \ldots, n)$,

$$|\nabla \tilde{f}(z)| \leqslant c''[\rho(P_\alpha, \partial\Omega)]^{-1}, \quad z \in P_\alpha.$$

Using Lemma 3 again, $|\nabla\tilde{f}(z)| \leqslant c'''(\rho(z, \partial\Omega))^{-1}$. This completes the proof.

Recall the example of Timoney presented in Section 3. We can exploit it for an n-dimensional example. Let

$$z = (z_1, z_2, \ldots, z_n) \in \mathbf{B}_n \quad \text{and} \quad f(z) = (1 - z_1)^{-1}$$

with

$$S = \{z \in \mathbf{B}_n : z_1 \in V\}.$$

V is the sequence $\{\alpha_{j,n}\}$ defined in that example. As in $\mathbf{C}^1$ the function f is not in the Bloch space of $\mathbf{B}_n$ yet

$$\|\nabla f(z)\| \leqslant c\{\min[\rho(z, S), \rho(z, \partial\mathbf{B}_n)]\}^{-1}.$$

It suffices to prove the stronger inequality

$$\frac{1}{1 - |z_1|^2} \leqslant c[\min(\rho(z, S)), \rho(z, \partial\mathbf{B}_n))]^{-1}.$$

For

$$1 - \frac{1}{n} \leqslant |z| \leqslant 1 - \frac{1}{n+1},$$

we have

$$\rho(z, S) \leqslant \rho(z, S \cap \{|z_1| = 1 - \frac{1}{n}\}) \leqslant c/n^2.$$

Hence,

$$\min[\rho(z, S), \rho(z, \partial\mathbf{B}_n)] \leqslant \rho(z, V) \leqslant \frac{c}{n^2}$$

and

$$\min[\rho(z, S), \rho(z, \partial\mathbf{B}_n)]^{-1} \geqslant \frac{n^2}{c} \geqslant c'\left(\frac{1}{1 - |z_1|^2}\right).$$

5. The A-covering property

In this section we shall consider the A-covering property in more detail. We shall show that if $V = V' \cap \Omega$ where V' is a submanifold of a neighborhood of $\overline{\Omega}$ then V has the A-covering property. An open question is whether a subvariety of the form $V = V' \cap \Omega$, where V' is a subvariety of a neighborhood of $\overline{\Omega}$, has the A-covering property. We begin with the following pointwise result.

PROPOSITION 1. *Let $V = \{z \in U : h(z) = 0\}$ be the zero set of a holomorphic function h on an open set $U \subset \mathbf{C}^n$. Let z_0 be in V. There exists $\varepsilon_0 > 0$ and there exist constants $\mathscr{C}_1$, $\mathscr{C}_2$, $\mathscr{C}_3$ such that the following is true: for all $\varepsilon \leqslant \varepsilon_0$ there is a polycylinder $P_\varepsilon(z_0)$ centered at z_0 such that*

(1) $C_1\varepsilon \leqslant r_j \leqslant C_2\varepsilon, j = 1, 2, \ldots, n,$

(*where r_j is the j-th polyradius of $P_\varepsilon(z_0)$*) *and*

(2) $\rho(V, \partial_0 P_\varepsilon(z_0)) \geqslant C_3\varepsilon.$

Proof. Take the origin of coordinates at z_0 and choose the z_n coordinate so that $h(0, \ldots, 0, z_n)$ vanishes to minimal order at z_0. Write $h = \phi\, q$ where ϕ is a unit and q is a Weierstrass polynomial in z_n:

$$q(z) = z\frac{\alpha}{n} + \sum_{j=1}^{\alpha} q_{\alpha-j}(z')z_n^{\alpha-}j.$$

Here, $z' = (z_1, \ldots, z_{n-1})$ and this representation is valid for $|z'| < \varepsilon_1$, $|z_n| < \varepsilon_2$. Because of the choice of the z_n coordinate, the coefficient $q_{\alpha-j}(z')$ of $z_n^{\alpha-j}$ vanishes to order at least j at 0 for $j = 1, 2, \ldots, \alpha$. Thus there exists $K > 0$ such that

(11) $$|g_{\alpha-j}(z')| \leqslant K|z'|^j \quad \text{for } j = 1, 2, \ldots, \alpha, \text{ and } |z'| < \varepsilon_1.$$

Now for $z_n \neq 0$ we may write

$$q(z) = z_n^\alpha\left(1 + \sum_{j=1}^{\alpha} \frac{q_{\alpha-j}(z')}{z_n^{\alpha-j}}\right).$$

Choose c such that $K\Sigma_{j=1}^{\alpha}c^{-j} < 1$; we may assume that $c \geqslant 1$. Choose ε_0 small enough that

$$\varepsilon_0 < \min\left(\frac{\varepsilon_1}{2}, \frac{\varepsilon_2}{2c}\right)$$

and also so that $\phi(z) \neq 0$ when z lies in the polycylinder

$$P_\varepsilon(z_0) = \{z \in \mathbf{C}^n : |z_j| < 2\varepsilon_0, \quad j = 1, \ldots, n-1, \text{ and } |z_n| < 2c\varepsilon.\}.$$

Now if $\varepsilon < 2\varepsilon_0$ and z' is fixed so that $|z'| < \varepsilon$ we claim that the α roots $\xi_1(z'), \ldots, \xi_\alpha(z')$ of $q(z', z_n) = 0$ satisfy $|\xi_j(z')| < c \cdot \varepsilon, j = 1, \ldots, \alpha$. This is a consequence of Rouche's Theorem: using the estimates (11) and the choice of c we see that for fixed z' the polynomials $q(z', z_n)$ and z_n^α have the same number of roots inside the circle $|z_n| = c \cdot \varepsilon$. Now if we restrict ε so that $\varepsilon < \varepsilon_0$ we may be sure that for $|z'| < \varepsilon$, the equation $h(z', z_n) = 0$ has no roots ξ which satisfy $c\varepsilon < |\xi| < 2c\varepsilon$.

It suffices therefore to set $c_1 = 1$, $c_2 = 3/2c$, $c_3 = 1/2c$ and

$$P_\varepsilon(z_0) = \{z \mid |z_j| < \varepsilon, j = 1, 2, \ldots, n-1 \text{ and } |z_n| < c_2\varepsilon\}.$$

Properties 1 and 2 then hold for $P_\varepsilon(z_0)$, $0 < \varepsilon < \varepsilon_0$.

In order to use Proposition 1 to show that a given subvariety of Ω satisfies the A-covering property, we need to know that the constants ε_0, c_1, c_2 and c_3 depend continuously on the point z_0. We can establish this only in special cases.

LEMMA 4. *Let M be a complex submanifold of an open set U of $\mathbf{C}^n$. Then in Proposition 1 we may take $c_1 = c_2 = 1$ (i.e., each radius of $\mathcal{P}_\varepsilon(z_0)$ is precisely ε), and the constants ε_0 and c_3 may be taken to the locally independent of z_0.*

Proof. Suppose $\dim_{\mathbf{C}}(M) = n - k$. Let $z_0 \in M$. Choose orthonormal coordinates at z_0 (using the standard inner product on $\mathbf{C}^n$) so that the $z_{n-k+1}, \ldots, z_n$ axes are tangential to M at z_0. By considering local defining functions $h_1, h_2 \ldots, h_k$ for M near z_0 such that $h_j(z) = z_j + \phi_j(z)$ where $\phi_j(z)$ vanishes to order greater than one at z_0, $j = 1, 2, \ldots, k$, we see that there exists $\varepsilon_0 > 0$ such that for $0 < \varepsilon \leqslant \varepsilon_0$ the polycylinder

$$P_\varepsilon(z_0) = \{z : |z_j| < \varepsilon, j = 1, \ldots, n\}$$

satisfies $\rho(M, \partial_0 P_\varepsilon(z_0)) \sim \varepsilon$. It is clear that this family of polycylinders may be translated to nearby points without destroying property (2) of Theorem 3.

Using this lemma we can show:

PROPOSITION 2. *Suppose $\Omega \subset\subset \mathbf{C}^n$ has smooth boundary. Suppose V' is an analytic subvariety of a neighborhood Ω' of Ω such that V' is smooth in a neighborhood U of $\partial\Omega$. Let $V = V' \cap \Omega$. Then V has the A-covering property.*

Remark. V can have at worst point singularities in Ω.

Proof. Let $M = V' \cap U$. Lemma 4 applies to M. Let W be a neighborhood of $\partial\Omega$ such that $\overline{W} \subseteq U$. Referring to the construction in Lemma 4, we set

$\eta = \inf\{\varepsilon_0 > 0, z_0 \in M \cap W\}$. For each $z_0 \in M \cap W$, the polycylinder $P_\eta(z_0)$ is defined by the construction in Lemma 4. All radii of this polycylinder are equal to η, but the direction of the axes depends on z_0. Let

$$V_1 = \{z \in V \cap W : P_\eta(z) \cap \partial\Omega \neq \emptyset\}, \quad V_2 = \{z \in V \cap W : \rho(z, \partial\Omega) \geqslant \eta\}.$$

The set V_2 is compact. Hence it has a finite covering by polycylinders P_1, $P_2, \ldots, P_m$ such that $\overline{P}_j \subset \Omega$ and $\partial_0 P_j \cap V = \phi$. (We make use of the construction in Proposition 1 at each point of V_2, and then use compactness to extract a finite subcovering.) Also it is not hard to see that part (d) of the A-covering property is satisfied for $z \in V_2$. At this point we just have to cover $V \setminus V_2$ in the desired way. It is not hard to see that if $z \in V \setminus V_2$ then $z \in V_1$. The polycylinder $P_\varepsilon(z)$ obtained from Lemma 4 with

$$\varepsilon = \frac{1}{2\sqrt{n}}\rho(z, \partial\Omega)$$

has closure contained in Ω. Clearly $\rho(z, \partial\Omega) \simeq \varepsilon$ and $\rho(\partial_0 P_\varepsilon(z), V) \simeq \varepsilon$. Hence the covering of $V - V_2$ consists of one polycylinder centered at each point $z \in V - V_2$ with radii

$$\frac{1}{2\sqrt{n}}\rho(z, \partial\Omega)$$

and suitably chosen axes. Part (d) of the A-covering property is satisfied since each point $z \in V - V_2$ is actually the center of one of the polycylinders. This completes the proof of Proposition 2.

It would be of interest to see whether Proposition 2 holds under the assumption that $V = V' \cap \Omega$ where V' is an arbitrary subvariety of a neighborhood of $\overline{\Omega}$. This amounts to determining whether the constants in Proposition 1 are locally independent of z_0. We shall indicate how this may be proved if U is a neighborhood of $0 \in \mathbf{C}^2$ and $V = \{z \in U : z_1^2 - z_2^3 = 0\}$. This illustrates some of the considerations which will be relevant to the case of more general singularities. (It follows that Proposition 2 holds when Ω is any smoothly bounded domain in $\mathbf{C}^2$ such that $0 \in \partial\Omega$ and $V = \{z \in \Omega \mid z_1^2 - z_2^3 = 0\}$.)

For each value of $z_2 \neq 0$ there are two values of z_1 such that $(z_1, z_2) \in V$. In modulus they are given by $|z_1| = |z_2|^{3/2}$. For each $\varepsilon < \varepsilon_0$ where ε_0 is some number we want to construct a polycylinder $P_\varepsilon(z_1, z_2)$ centered at (z_1, z_2) with sides equivalent to ($\simeq$) ε and $\rho(V, \partial_0 P_\varepsilon(z_1, z_2)) \simeq \varepsilon$. Now the distance from (z_1, z_2) to the other sheet of the variety is equivalent to $|z_1|$. It is much less than $|z_2|$ if (z_1, z_2) is near 0. For $\varepsilon > k|z_1|$ where k is some constant we choose the polycylinder $P_\varepsilon(z_1, z_2)$ to have fixed proportions, axes parallel to the coordinate axes, and such that both sheets of the variety are contained in the polycylinder. For $\varepsilon < k|z_1|$ we change the proportions of the polycylinders, choosing a smaller multiple of ε for the z_1 radius, so that the polycylinder just includes one sheet of the variety. (Of course when $z = 0$ each polycylinder $P_\varepsilon(0)$ contains both sheets of the variety).

What is involved here is that the local representation of the variety at a given point must be used to construct the family of polycylinders at nearby points.

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