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# INTRINSICALLY ($n$ – 1)-DIMENSIONAL CELLULAR DECOMPOSITIONS OF $S^n$

BY

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This paper continues investigation into the differences between generalized manifolds and classical manifolds. The generalized $n$-manifolds $X$ studied here arise as cell-like images of an $n$-manifold $M$ under a natural decomposition map $\pi$. In this setting, the difference between $X$ and $M$ is measured by a notion of intrinsic (nonmanifold) dimension—namely, the minimal dimension among the images of the nondegeneracy sets for those cell-like maps from $M$ onto $X$ that approximate $\pi$.

A locally compact, separable metric space $X$ is called a *generalized $n$-manifold* if $X$ is a finite dimensional ANR and if, for each $x \in X$,

$$H_*(X, X - \{x\}; Z) \cong H_*(E^n, E^n\text{-}\{\text{point}\}; Z).$$

It is a consequence of the classical Vietoris-Begle Mapping Theorem [2] that any finite dimensional cell-like image of an $n$-manifold is a generalized $n$-manifold. Of course, those acquainted with the literature will recall that multitudes of non-manifold generalized $n$-manifolds originate as such cell-like images. Within the past few years generalized $n$-manifolds have taken a central position in geometric topology, because of the close tie between manifolds and generalized manifolds provided by cell-like maps, or by cell-like decompositions. In 1977, R. D. Edwards [12, p. 118] proved that each generalized $n$-manifold $X$, $n \geq 5$, that is the cell-like image of an $n$-manifold $M$, is itself a manifold (actually, homeomorphic to $M$) if and only if $X$ satisfies the minimal general position features required to the following Disjoint Discs Property: any two maps of the 2-cell $B^2$ into $X$ can be approximated by maps having disjoint images. Moreover, F. Quinn [18] has announced that, for $n \geq 5$, a finite dimensional space is a generalized $n$-manifold if and only if it is the cell-like image of some $n$-manifold.

Until recently the existent generalized $n$-manifolds were fairly simple, for in several senses they were quite similar to manifolds. Then Cannon-Daverman [5] constructed cell-like, totally noncellular decompositions of

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$n$-manifolds ($n \geq 3$), using them to produce totally wild flows, but thereby setting forth examples of intrinsically $n$-dimensional generalized $n$-manifolds. Daverman-Walsh [11] expanded that construction to produce ghastlier examples. Either of these works can be adapted to provide, for any $k \in \{0, 1, \ldots, n\}$, an intrinsically $k$-dimensional example resulting from a cell-like, noncellular decomposition.

Cellular decompositions, however, cannot be quite so complex. First of all, the associated decomposition spaces are more like manifolds than occurs with arbitrary cell-like decompositions, in that the complement of each point is locally simply connected. Secondly, such decomposition spaces, if finite dimensional, are generalized $n$-manifolds having intrinsic dimension no more than $n - 1$ (see [13] or [17]).

Earlier [9] we described, for $n \geq 3$ and $k \in \{0, 1, \ldots, n - 2\}$, an intrinsically $k$-dimensional cellular decomposition of $E^n$; here we shall describe an intrinsically $(n-1)$-dimensional cellular decomposition of $E^n (n \geq 3)$. The construction delineated in what follows is considerably more intricate than that of [9], due to a necessary intertwining of the chambers comprising a defining sequence. The methodology applied is much closer in spirit to that introduced by Cannon and Daverman [5] than to that we used previously.

In addition to resolving the matter of possible intrinsic dimension in a heretofore missing case, a justification for the efforts expended here stems from a prominent question: is the product of $E^1$ with a (finite dimensional) cell-like image $X$ of an $n$-manifold always an $(n+1)$-manifold. The product must be a manifold if $X$ has intrinsic dimension $\leq n - 3$ [7, Theorem 3.3] [10, Theorem 1]. Had it turned out that cellular decompositions of $n$-manifolds were, at worst, of intrinsic dimension $n - 2$, we might have seemed closer to recognizing the product of the resulting decomposition space with $E^1$ to be a manifold.

Finally, it should be noted that intrinsic dimension serves as just one among several potential measures of the differences between manifolds and generalized manifolds. For many purposes one needs no more than the relatively crude measure provided by the dimension of the set of points at which the space fails to satisfy the definition of a manifold. Another device, more discriminating than that of intrinsic dimension, has been introduced and studied by Garity [14]; it is particularly suited to spaces arising from cellular decompositions, and it involves a generalization of the Disjoint Discs Property from pairs of maps defined on $B^2$ to $k$-tuples of such maps.

## 1. Preliminaries

We will be considering cell-like (CE) upper semicontinuous (usc) decompositions of $n$-manifolds $M$. If $G$ denotes a decomposition of a space $M$, $H_G$ represents the set whose elements are the nondegenerate elements of $G$, and $N_G$ represents the union of these elements. In general, $\pi$ will be used to denote the natural decomposition map of $M$ onto $M/G$. In case $p$

is a map of $M$ onto a space $X$ and $H$ is the decomposition of $M$ induced by $p$, where $H = \{p^{-1}(x) \mid x \in X\}$, then $N_p$ is defined to equal $N_H$.

We use the symbol $\partial$ to denote the boundary of a manifold with boundary.

A CE map $p: M \to X$ is said to be *one-to-one over a subset* $A$ of $X$ if $p \mid p^{-1}(A)$ is 1-1.

In case $f$ and $h$ denote maps from a space $X$ to a metric space $Y$, the *distance between* $f$ and $h$, written as $\rho(f, h)$, is defined as the supremum, in the extended reals, of $\{\rho(f(x), h(x)) \mid x \in X\}$, where $\rho$ is the metric on $Y$.

The following definitions and theorem, taken from [5], provide the framework to be used in building CE usc decompositions. Throughout the remainder of this section, $M$ will represent a compact $PL$ $n$-manifold.

DEFINITION 1.1. A *defining sequence* (in $M$) is a sequence $\mathcal{S} = \{\mathcal{M}_1, \mathcal{M}_2, \ldots\}$ satisfying the following conditions:

(1) For each positive integer $i$, $\mathcal{M}_i$ is a finite collection $\{A(1), \ldots, A(k_i)\}$ of compact connected $PL$ $n$-manifolds with boundary in $M$ having pairwise disjoint interiors.

(2) For $i > 1$ and each $A \in \mathcal{M}_i$, there exists a unique element Pre $A \in \mathcal{M}_{i-1}$ properly containing $A$.

(3) For each $i \geqslant 1$, each $A \in \mathcal{M}_i$ and each pair of points $x$ and $y$ in $\partial A$, there exists an integer $j > i$ such that no element of $\mathcal{M}_j$ contains both $x$ and $y$.

DEFINITION 1.2. Let $\mathcal{S} = \{\mathcal{M}_1, \mathcal{M}_2, \ldots\}$ be a defining sequence in $M$ and $x \in M$. Then the *star of $x$ in $\mathcal{M}_i$*, written as $st(x, \mathcal{M}_i)$ or $st_1(x, \mathcal{M}_i)$, is defined as

$$st(x, \mathcal{M}_i) = \{x\} \cup \bigcup \{A \in \mathcal{M}_i \mid x \in A\},$$

and, for any integer $e > 1$, the *$e$-th star of $x$ in $\mathcal{M}_i$*, written as $st_e(x, \mathcal{M}_i)$, is defined recursively as

$$st_e(x, \mathcal{M}_i) = \cup \{st(y, \mathcal{M}_i) \mid y \in st_{e-1}(x, \mathcal{M}_i)\}.$$

DEFINITION 1.3. *The decomposition $G$ of $M$ associated with a defining sequence $\mathcal{S} = \{\mathcal{M}_1, \mathcal{M}_2, \ldots\}$ in $M$ is given as follows:* distinct points $x$ and $y$ of $M$ are in the same element of $G$ if and only if there exists an integer $r$, depending only on $x$ and $y$, such that, for each positive integer $j$, $y \in st_r(x, \mathcal{M}_j)$.

THEOREM 1.4 [5, Theorem 1]. *The decomposition $G$ of $M$ associated with any defining sequence $\mathcal{S} = \{\mathcal{M}_1, \mathcal{M}_2, \ldots\}$ is usc, and, for each $x \in M$,*

$$\pi^{-1}\pi(x) = \bigcap_{j=1}^{\infty} st_2(x, \mathcal{M}_j).$$

*Moreover, if $\mathcal{B}$ denotes $\cup\{\partial A \mid A \in \mathcal{M}_j$ for some $j\}$ and if $x \in g \in G$ and either $x \in \mathcal{B}$ or $g \cap \mathcal{B} = \emptyset$, then*

$$\pi^{-1}\pi(x) = \bigcap_{j=1}^{\infty} st(x, \mathcal{M}_j).$$

*In addition, if each $A \in \mathcal{M}_j$ is null homotopic in* Pre $A$ *for every $j \geq 2$, then $G$ is CE.*

## 2. Measuring Intrinsic Dimension

This section presents methods for determining when certain decompositions of an $n$-manifold are intrinsically $(n-1)$-dimensional. These methods will be applied to those decompositions constructed in Section 5.

DEFINITION 2.1. Let $\varepsilon > 0$. A map $f$ from a space $X$ to another space $Y$ endowed with a metric $\rho$ is said to be *$\varepsilon$-approximable* by maps $h$ (usually satisfying additional properties) if there exists such a map $h$: $X \to Y$ with $\rho(h, f) < \varepsilon$, and it is said to be *approximable* by such maps if it is $\varepsilon$-approximable for each $\varepsilon > 0$.

DEFINITION 2.2. Let $G$ be a CE usc decomposition of an $n$-manifold $M$ and let $d$ denote a nonnegative integer. Then $G$ is said to be *secretly $d$-dimensional* if $\pi$: $M \to M/G$ is approximable by CE maps $q$ of $M$ onto $M/G$ such that $q(N_q)$ has dimension less than or equal to $d$, and $G$ is said to be *intrinsically $d$-dimensional* if it is secretly $d$-dimensional but not secretly $(d$-$1)$-dimensional.

The following elementary result sets forth a necessary property of secret dimension.

LEMMA 2.3. *Let $G$ be a CE usc decomposition of an $n$-manifold $M$ that is secretly $d$-dimensional, and let $F_1$, $F_2$ be maps of $B^2$ into $M$ having disjoint images. Then $\pi F_1$, $\pi F_2$: $B^2 \to M/G$ are approximable by maps $f_1$, $f_2$: $B^2 \to M/G$ such that* $\dim(f_1(B^2) \cap f_2(B^2)) \leq d$.

*Proof.* If $q$: $M \to M/G$ is an $\varepsilon$-approximation to $\pi$: $M \to M/G$ such that

$$\dim(q(N_q)) \leq d,$$

then $qF_1$ and $qF_2$ are $\varepsilon$-approximation to $\pi F_1$ and $\pi F_2$, respectively, such that $qF_1(B^2) \cap qF_2(B^2)$ has dimension at most $d$.

PROPOSITION 2.4. *Let $G$ be a cellular usc decomposition of a connected $n$-manifold $M$ such that $M/G$ is finite dimensional, let $w$ and $y$ denote distinct points of $M/G$, and let $F_1$ and $F_2$ be maps of $B^2$ to $M$ having disjoint images. If there exists $\gamma > 0$ such that all $\gamma$-approximations $f_1$, $f_2$ to $\pi F_1$, $\pi F_2$,* *respectively, have the property that*

$$f_1(B^2) \cap f_2(B^2)$$

*separates $w$ from $y$, then $G$ is intrinsically $(n-1)$-dimensional.*

*Proof.* As mentioned in the introduction, cellular decompositions of $n$-manifolds having finite dimensional decomposition spaces are secretly $(n$-$1)$-dimensional (see [13] or [17]). Thus, it suffices to prove that $G$ is not secretly $(n$-$2)$-dimensional. If that were the case, Lemma 2.3 would provide $\gamma$-approximations $f_1$ and $f_2$ to $\pi F_1$ and $\pi F_2$ such that

$$\dim[f_1(B^2) \cap f_2(B^2)] \leq n - 2.$$

But the complement in $M/G$ of the closed set $X = f_1(B^2) \cap f_2(B^2)$ is pathwise connected, since $H_1(M/G, (M/G) - X) = 0$ (see [7, Lemma 2.1]). This contradicts the hypothesis that $X$ must separate $w$ from $y$.

Ultimately the next result will be used to detect the separating sets called for in Proposition 2.4.

LEMMA 2.5. *Let $P$ denote a subpolyhedron of $S^n$ such that* $\mathrm{Cl}(S^n - P)$ *consists of components $C_1, \ldots, C_s$ where $s \geq 2$. There exists $\delta > 0$ such that if $\sigma_1, \ldots, \sigma_k$ are simplexes of some subdivision of $P$ and*

$$\sum_{i=1}^{k} \operatorname{diam} \sigma_i < \delta,$$

*then the set* $Q = \mathrm{Cl}[S^n - (P - \cup_{i=1}^k \sigma_i)]$ *has distinct components $\tilde{C}_1, \ldots, \tilde{C}_s$ with $\tilde{C}_j \supset C_j$ $(j \in \{1, \ldots, s\})$.*

*Proof.* Choose $\delta < \min\{\rho(C_i, C_j) \mid i \neq j\}$. Let $\tilde{C}_j$ denote that component of the set $Q$, as above, containing $C_j$.

It remains to be shown that $\tilde{C}_i \cap \tilde{C}_j = \emptyset$ whenever $i \neq j$. Assume the contrary. Then there exists a PL arc $\alpha$ joining the boundary of some $C_i$ to the boundary of some other $C_j$ and intersecting $P$ only in $\cup_{i=1}^k \sigma_i$. The arc $\alpha$ must contain a subarc $\beta \subset \cup_{i=1}^k \sigma_i$ joining the boundary of some $C_k$ to the boundary of some $C_m$ $(k \neq m)$. This implies that

$$\rho(C_k, C_m) \leq \operatorname{diam} \beta \leq \sum_{i=1}^{k} \operatorname{diam} \sigma_i < \delta,$$

which is impossible.

## 3. Spinning Certain Subsets of $I^3$

In Section 4 we will arrange solid tori in $I^3 = [-1, 1]^3$ according to a carefully devised linking pattern. To obtain comparably linked objects in

$S^n$ that can function as appropriate guides for determining an intrinsically ($n$-1)-dimensional decomposition, we will sweep out $S^n$ by spinning $I^3$ and will work with the resulting spun tori. Before specifying the linking scheme, we review here the spinning operation and investigate how the crucial property performs with spun objects. Our discussion of spinning is a compactified version of the treatment given by Cannon in Appendix III of [4]; see also Section 11 of [8].

One can view $S^n$, $n \geqslant 3$, as the decomposition space $(I^3 \times S^{n-3})/K^n$, where $K^n$ is the decomposition of $I^3 \times S^{n-3}$ consisting of points and the sets $\{b\} \times S^{n-3}$, $b \in \partial I^3$. The space $(I^3 \times S^{n-3})/K^n$ obviously inherits the structure of an abstract simplicial complex and, as Lemma 3.2 will make evident, with this inherited structure the space forms a *PL* $n$-manifold (which can be seen to be *PL* homeomorphic to $S^n$, since the space readily can be expressed as the union of two *PL* $n$-cells intersecting precisely in the boundary of each). Let $p\colon I^3 \times S^{n-3} \to S^n$ denote the quotient mapping induced by $K^n$. There is a *PL* map $\psi\colon S^n \to I^3$ induced by $p$ and the projection $I^3 \times S^{n-3} \to I^3$ that is one-to-one over $\partial I^3$ and that makes the diagram below commutative.

$$\begin{tikzcd}
I^3 \times S^{n-3} \arrow[rr] \arrow[d, dashed, "\text{proj.}"'] & & S^n \arrow[dll, dashed, "\psi"] \\
I^3 & &
\end{tikzcd}$$

DEFINITION 3.1. Let $A$ be a subset of $I^3$. Then the ($n$-3)-*spin of* $A$, written as $Sp^{n-3}(A)$, is the set $\psi^{-1}(A)$.

LEMMA 3.2. *Let $D$ be a PL 3-cell in $I^3$ such that $D \cap \partial I^3 = B$, a 2-cell. Then $Sp^{n-3}(D)$ is a PL $n$-cell in $S^n$.*

*Proof.* Think of $D$ as $B \times [0, 1]$ with $D \cap \partial I^3 = B \times \{0\}$. Then

$$\begin{aligned}
Sp^{n-3}(D) &\cong ((B \times [0, 1]) \times S^{n-3})/K^n \\
&= (B \times [0, 1] \times S^{n-3})/\{\{b\} \times \{0\} \times S^{n-3} | b \in B\} \\
&\cong B \times [([0, 1] \times S^{n-3})/(\{0\} \times S^{n-3})] \\
&\cong B \times (\text{cone on } S^{n-3}) \\
&\cong B \times I^{n-2} \\
&\cong I^n
\end{aligned}$$

The equivalences here can be realized by *PL* homeomorphisms.

COROLLARY 3.3. *Let $M$ be a PL 3-manifold with boundary in $I^3$ such that $M \cap \partial I^3$ is a 2-manifold. Then $Sp^{n-3}(M)$ is a PL $n$-manifold with boundary in $S^n$.*

Next we state the fundamental properties to be exploited in detecting linking and its complexity. They involve variations to the concept of $I$-essential mapping first introduced in [6].

DEFINITIONS 3.4. Let $M$ be a $PL$ $n$-manifold with boundary, $B$ a 2-cell, and $F: B \to M$ a map with $F(\partial B) \subset \partial M$. Then $F$ is said to be *I-inessential* (an abbreviation of *Interior-inessential*) if there exists a map $\tilde{F}: B \to \partial M$ with $\tilde{F} \mid \partial B = F \mid \partial B$; otherwise, $F$ is said to be *I-essential*. Or, in different terminology, $F: B \to M$ is $I$-essential iff $F \mid \partial B$ is an essential map into $\partial M$.

Now consider a compact connected 2-manifold $H$ in $B^2$ ($H$ is often called a disc with holes). We shall speak of a map $f: H \to M$ with $f(\partial H) \subset \partial M$ as being *virtually I-essential* if $f$ extends to an $I$-essential map $F: B_H \to M$ satisfying $F(B_H - H) \subset \partial M$, where $B_H$ denotes the (unique) 2-cell in $B^2$ for which $H \subset B_H$ and $\partial B_H \subset \partial H$. One should understand that a map $f: H \to M$ with $f(\partial H) \subset \partial M$ is virtually $I$-essential iff $f$ sends the outermost component of $\partial H$ essentially into $\partial M$ but sends all of the remaining boundary components inessentially (null homotopically).

Generally, given an $n$-dimensional $PL$ submanifold $Q$ of Int $M$, we shall say that $f: H \to M$ as above is *virtually I-essential with respect to Q* provided that, for each $PL$ map $\tilde{f}: H \to M$ sufficiently close to $f$ and in general position with respect to $Q$, some component $H_Q$ of $\tilde{f}^{-1}(Q)$ is a disc with holes and $\tilde{f} \mid H_Q: H_Q \to Q$ is virtually $I$-essential.

PROPOSITION 3.5. *Suppose $M$ is a $PL$ $n$-manifold with boundary, $Q$ is a $PL$ $n$-manifold in* Int $M$ *such that the only loops of $\partial(M -$ Int $Q)$ null homotopic in $M -$ Int $Q$ are those loops that are null homotopic in $\partial(M -$ Int $Q)$, and $f: B^2 \to M$ is an $I$-essential map. Then $f$ is virtually $I$-essential with respect to $Q$.*

*Proof.* Consider a $PL$ map $\tilde{f}: B^2 \to M$ in general position with respect to $Q$ and so close to $f$ that $\tilde{f}(\partial B^2) \subset M - Q$ and $\tilde{f} \mid \partial B^2$ is not homotopically trivial in $M -$ Int $Q$. Some component of $\tilde{f}^{-1}(\partial Q)$ is not mapped null homotopically into $\partial Q$ by $\tilde{f}$, for otherwise $\tilde{f}$ could be redefined on each 2-cell $\tilde{B}$ of $B^2 - \tilde{H}$, where $\tilde{H}$ is the component of $B^2 - \tilde{f}^{-1}(\partial Q)$ containing $\partial B^2$, so as to send $\tilde{B}$ into $\partial Q$ and thereby to exhibit a contraction of the loop $\tilde{f}(\partial B^2)$ in $M -$ Int $Q$. From those components of $\tilde{f}^{-1}(\partial Q)$ not mapped null homotopically to $\partial Q$, select an innermost (with respect to $B^2$) one, say $J$, and let $H$ denote the closure of that component of $B^2 - \tilde{f}^{-1}(\partial Q)$ having $J$ as its outermost boundary component. As a result, if $K$ stands for any component of $\partial H - J$, clearly $\tilde{f}$ maps $K$ to $\partial Q$ null homotopically.

We claim that $\tilde{f}(H) \subset Q$. If not, $\tilde{f}(H) \subset M -$ Int $Q$. Then exactly as in the preceding paragraph we would find that $\tilde{f}$ maps $J$ into $M -$ Int $Q$ in homotopically trivial fashion, contrary to the operative hypothesis. Consequently, $\tilde{f} \mid H$ is a virtually $I$-essential map of $H$ to $Q$.

COROLLARY 3.6. *Suppose $M$ and $Q$ are PL manifolds as in Proposition 3.5, and suppose $f: H \to M$ is a virtually I-essential map of a disc with holes $H \subset B^2$. Then $f$ is virtually I-essential with respect to $Q$.*

LEMMA 3.7. *Let $T$ be a PL 3-manifold with boundary in $I^3$ such that $T \cap \partial I^3$ is a 2-manifold, and let $F_0: B^2 \to T$ be an I-essential embedding for which $F_0(\partial B^2) \subset \partial T - \partial I^3$. Then there exists an I-essential embedding $F: B^2 \to Sp^{n-3}(T)$ such that $\psi F = F_0$.*

*Proof.* To do this, simply fix $s_0 \in S^{n-3}$ and prescribe $F(b)$, $b \in B^2$, as the image (under the map $p$) of $F_0(b) \times s_0 \in I^3 \times S^{n-3}$ in $S^n$.

PROPOSITION 3.8. *Let $T$ be a PL 3-manifold with boundary in $I^3$ such that $T \cap \partial I^3$ is a 2-manifold and suppose $f: H \to Sp^{n-3}(T)$, $n \geq 5$, is a map of a disc with holes $H \subset B^2$ such that*

$$f(\partial H) \subset \partial Sp^{n-3}(T) \quad \text{and} \quad f(H) \cap Sp^{n-3}(T \cap \partial I^3) = \emptyset.$$

*Then $f$ is virtually I-essential if and only if $\psi f: H \to T - \partial I^3$ is virtually I-essential.*

*Proof.* It suffices to consider the case where $H$ equals the 2-cell $B^2$.

Obviously, whenever $f: B^2 \to Sp^{n-3}(T)$ fails to be virtually $I$-essential, so does $\psi f: B^2 \to T$, because the image under $\psi$ of a contraction of $f(\partial B^2)$ in $\partial Sp^{n-3}(T)$ provides a contraction of $\psi f(\partial B^2)$ in $\partial T$. On the other hand, when $f: B^2 \to Sp^{n-3}(T)$ is I-essential, then $f$ maps $\partial B^2$ essentially to $\partial Sp^{n-3}(T)$. Since $\partial Sp^{n-3}(T)$ is the image of

$$\mathrm{Cl}(\partial T - \partial I^3) \times S^{n-3}$$

under the map $\rho$ and $n \geq 5$, one can easily verify that $p$ induces an isomorphism between $\pi_1(\mathrm{Cl}(\partial T - \partial I^3) \times S^{n-3})$ and $\pi_1(\partial Sp^{n-3}(T))$. Moreover, because $n \geq 5$, the projection of $\mathrm{Cl}(\partial T - \partial I^3) \times S^{n-3}$ to the first factor induces a $\pi_1$-isomorphism, so the same holds for $\psi$ (restricted). Consequently, $\psi f$ maps $\partial B^2$ essentially into $\mathrm{Cl}(\partial T - \partial I^3)$ and $\psi f$ is (virtually) I-essential.

Remark. With a minor amount of extra work, one can establish 3.8 in case $n = 4$. Since we avoid that case later, there seems to be no reason to bring it up now.

## 4. Linking Tori in $I^3$

At the outset we want to illustrate the linking pattern capitalized upon here. Given an integer $N > 1$, we must place $N$ solid tori in the space $T = S^1 \times I^2$ so that any two of them are linked while the individual ones are each contained in a 3-cell in $T$. Our method for carrying this out is pictured for the case $N = 4$ in Figure 1.

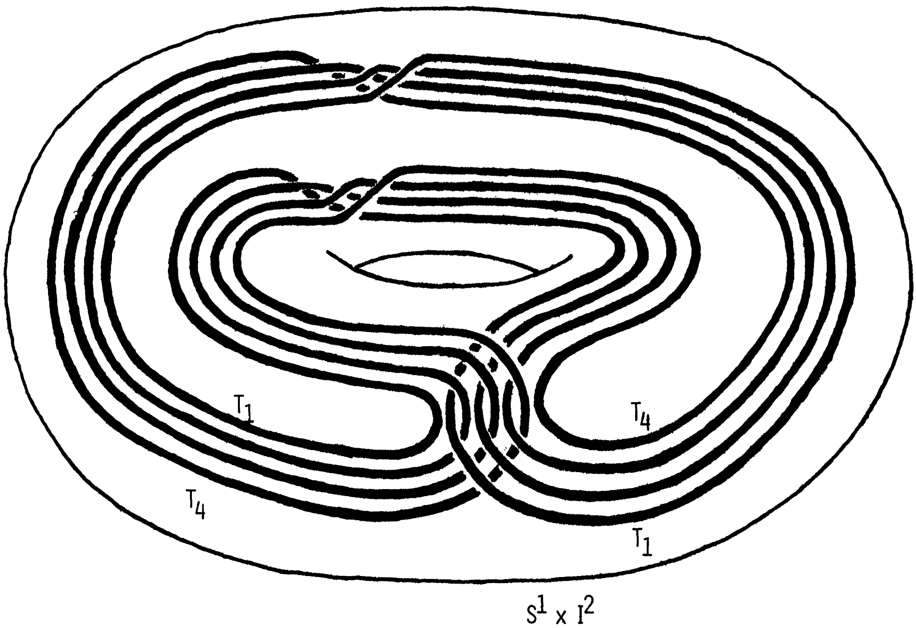


FIGURE 1

It should be clear from Figure 1 that each of the smaller tori there lies interior to a 3-cell in $T$. It should also be clear that any pair $\{T_j, T_k\}$ of the smaller tori is embedded in $T$ exactly like the tori $A_{i1}$ and $A_{i2}$ are embedded in the torus $A_i$ of [2, Figure 3]. It is fairly well known (one can verify this fact directly with the aid of a Wirtinger presentation for the related link) that a loop in

$$\partial A_i \cup \partial A_{i1} \cup \partial A_{i2}$$

is null homotopic in

$$A_i - \mathrm{Int}(A_{i1} \cup A_{i2})$$

iff the loop is null homotopic in the appropriate boundary component. Consequently, the manifolds $M = T$ and $Q = T_j \cup T_k$ satisfy the relevant hypotheses of Proposition 3.5.

LEMMA 4.1. *Let $\varepsilon$ denote a positive number and $K$ a positive integer. There exists a cell decomposition $\Delta$ of $T = S^1 \times I^2$ having mesh less than $\varepsilon$ and there exists a 3-cell $\sigma \in \Delta$ such that, for every integer $N \geqslant K$, $T$ contains a collection of $N$ mutually exclusive PL solid tori, $\{T_1, \ldots, T_N\}$, satisfying:*

(1) *$\sigma \cap T_i = \emptyset$ and $\sigma \cup T_i$ lies interior to a 3-cell in $T$ whenever $i \in \{1, \ldots, K\}$.*

(2) *For each 3-cell* $\tau \in \Delta - \{\sigma\}$, $\tau \cup T_j$ *lies in a 3-cell in* $T$ ($j \in \{1, \ldots, N\}$).

(3) *For each virtually I-essential map* $f$: $H \to T$ *of a disc with holes* $H \subset B^2$, $f$ *is virtually I-essential with respect to at least* $N - 1$ *of the tori* $T_1, \ldots, T_N$.

*Proof.* Let $J$ represent the $PL$ image of $\partial I^2$ mapped into Int $T$ as illustrated in Figure 2, with one point $s$ of singularity (corresponding to exactly two points of $\partial I^2$). The first $K$ tori, $T_1, \ldots, T_K$, lie near $J$ in Int $T - J$; each one is to be contained in a 3-cell in Int $T$ while each pair is to be linked in Int $T$ in the way described earlier. Part of this configuration is pictured in Figure 2.

Thicken $J$ to a $PL$ regular neighborhood $U(J)$ missing each torus $T_i$. Determine a cell decomposition $\Delta$ of $T$ having mesh less than $\varepsilon$ so that, in particular, $\Delta$ has a 3-cell $\sigma$ for which $s \in$ Int $\sigma$ and $\sigma \subset U(J)$ separates $U(J)$ into two components, so that any 3-cell $\sigma' \in \Delta - \{\sigma\}$ with $\sigma' \subset U(J)$ satisfies $\sigma' \cap \partial U(J)$ is an annulus and $\sigma' \cap J$ is an unknotted spanning arc of $\sigma'$, and so that any other 3-cell $\tau \in \Delta$ meets at most one of the tori, say $T_i$, in which case $\tau \cap T_i$ is an unknotted spanning 3-cell of $\tau$.

Now, given an integer $N > K$, we embed tori $T_{k+1}, \ldots, T_N$ in Int $U(J)$ according to the same standard linking pattern, so that individually they lie in 3-cells but pairwise any two of them (or of the total collection $\{T_1, \ldots, T_N\}$) are linked as before. They are embedded sort of locally parallel

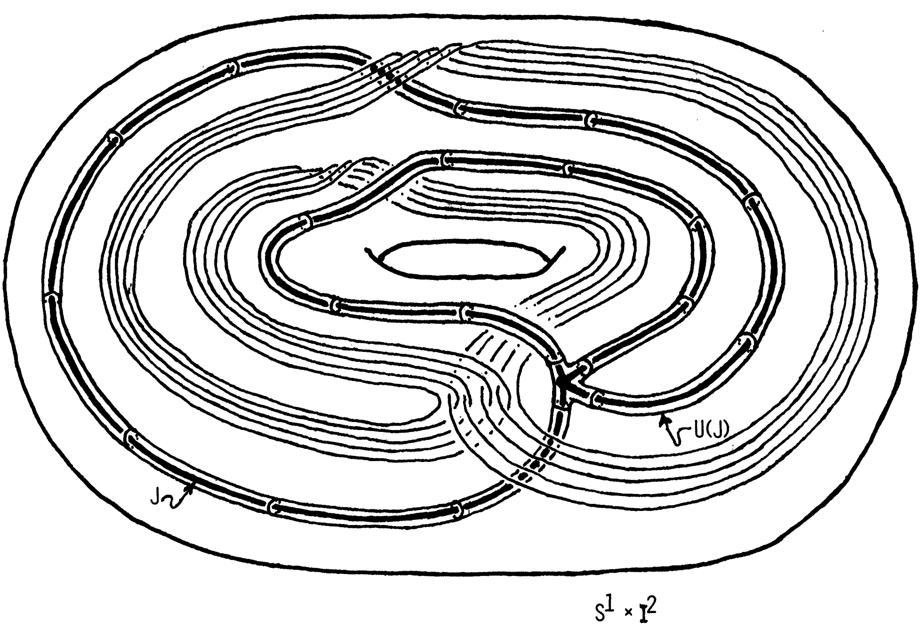


FIGURE 2

to $J$, so that if $\sigma'$ is a 3-cell of $\Delta$-$\{\sigma\}$ with $\sigma' \subset U(J)$, then $\sigma \cap T_i$ is an unknotted spanning cell of $\sigma'$ $(i > K)$. See Figure 3.

Let $\tau \in \Delta - \{\sigma\}$ be a 3-cell, and let $T_j$ be one of these tori. If $\tau \cap T_j = \emptyset$, then, by construction, $\tau \cup T_j$ collapses to $T_j$, so the promised 3-cell in $T$ containing $T_j$ can be modified (fixing $T_j$) to engulf $\tau$. If $\tau \cap T_j = \emptyset$, a similar modification of the 3-cell is possible. Similarly, when $i \in \{1, \ldots, K\}$, $\sigma \cap T_i = \emptyset$; then $\sigma \cup T_i$ is contained in a 3-cell in $T$ as well.

Since the tori are pairwise linked in $T$ like the pairs in Figure 1, Conclusion (3) follows from Proposition 3.5.

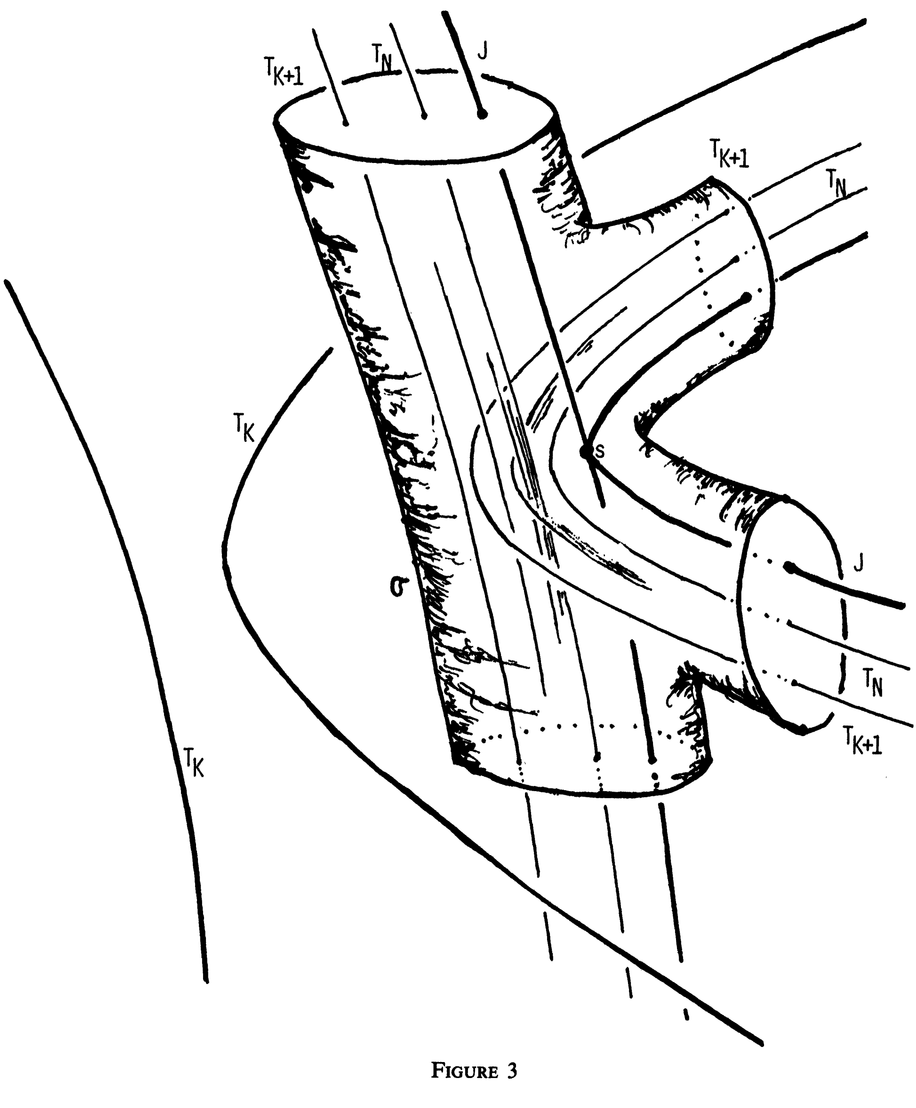


FIGURE 3

DEFINITION 4.2. A *PL* solid torus $T$ in $I^3$ is said to be *standard* if $T \cap \partial I^3$ is a 2-cell.

*Remark* 4.3. When $T$ is a standard torus in $I^3$, one can establish a variation to Lemma 4.1 by carefully connecting $\partial T_1, \ldots, \partial T_N$ to $T \cap \partial I^3$ by regular neighborhoods of arcs, determining a collection of $N$ mutually exclusive standard tori, $\{T_1, \ldots, T_N\}$ satisfying conclusion (1) there, as well as the following:

(2′) for each 3-simplex $\tau \in \Delta - \{\sigma\}$, $\tau \cup T_j$ lies in a *PL* 3-cell $C_j$ in $T$ such that $C_j \cap \partial I^3$ is a 2-cell; and
(3′) for each virtually I-essential map $f$: $H \to T - \partial I^3$ of a disc with holes $H \subset B^2$, $f$ is virtually I-essential with respect to at least $(V - 1)$ of $T_1 - \partial I^3, \ldots, T_N - \partial I^3$.

PROPOSITION 4.4. *Let $n \geq 5$ be an integer, $T$ a standard torus in $I^3$, and $\varepsilon > 0$. Then there exist a positive integer $K$, a cell decomposition $\Delta_\varepsilon$ of $M = Sp^{n-3}(T)$ having mesh less than $\varepsilon$, and a collection of $K$ distinct $n$-cells $\sigma_1, \ldots, \sigma_K \in \Delta_\varepsilon$ such that, for every integer $N \geq K$, $M$ contains a collection of $N$ mutually exclusive $n$-manifolds with boundary, $\{Q_1, \ldots, Q_N\}$, satisfying:*

(0) *Each $Q_j$ is the $(n-3)$-spin of a standard torus in $I^3$.*
(1) *$\sigma_i \cap Q_i = \emptyset$ and $\sigma_i \cup Q_i$ lies interior to an $n$-cell in $M$ whenever $i \in \{1, \ldots, K\}$.*
(2) *$\tau \cup Q_j$ lies in an $n$-cell in $M$ whenever $\tau$ is an $n$-cell of*

$$\Delta_\varepsilon - \{\sigma_1, \ldots, \sigma_K\}$$

*and $j \in \{1, \ldots, N\}$.*
(3) *For each virtually I-essential map $f$: $H \to M$ of a disc with holes $H \subset B^2$ such that $f(H) \cap Sp^{n-3}(\partial I^3) = \emptyset$, $f$ is virtually I-essential with respect to at least $N - 1$ of the manifolds $Q_1, \ldots, Q_N$.*

*Proof.* Find $\eta > 0$ such that, for the map $p$: $I^3 \times S^{n-3} \to S^n$ of Section 3, diam $p(Z) < \varepsilon$ whenever $Z \subset I^3 \times S^{n-3}$ and diam $Z < 2\eta$. Choose a triangulation $\Delta'$ of $S^{n-3}$ having mesh less than $\eta$. Let $K$ be the number of $(n - 3)$-cells in $\Delta'$. Apply Lemma 4.1 (cf. Remark 4.3) with positive number $\eta$ and integer $K$ to obtain a cell decomposition $\Delta$ of $T$ satisfying the conclusions there. Then $\Delta_\varepsilon = p(\Delta \times \Delta')$ is the desired cell decomposition, and $\{p(\sigma \times \tau) \mid \tau \text{ is an } (n-3)\text{-simplex of } \Delta'\}$ forms the desired collection of $K$ special $n$-cells.

Consider an integer $N \geq K$. There exist $N$ standard tori $T_1, \ldots, T_N$ in $T$ satisfying the conclusion of Lemma 4.1 (Remark 4.3). Set

$$Q_j = Sp^{n-3}(T_j) \quad \text{for } j = 1, \ldots, N.$$

In case $R$ is an $n$-cell of $\Delta_\varepsilon$ and $R \notin \{\sigma_1, \ldots, \sigma_K\}$ if $j \leqslant K$, we modify what results from conclusion (1) or (2′) of Remark 4.3 to concoct a 3-cell $C_j$ in $T$ such that $C_j \cap \partial I^3$ is a 2-cell and Int $C_j$ contains $(\psi(R) - \text{Int } C_j) \cup T_j$. According to Lemma 3.2, $Sp^{n-3}(C_j)$ is an $n$-cell and it contains $R \cup Q_j$, showing that conclusions (1) and (2) here are valid. Finally, conclusion (3) follows immediately from Proposition 3.8 and conclusion (3′) of Remark 4.3.

*Remark* 4.9. Since the spun tori $Q_i$ are regular neighborhoods of ($n$-2)-complexes in $S^n$, it is possible to choose them so that $\tau - \cup Q_i$ is nonempty and connected, for each $n$-cell $\tau \in \Delta_\varepsilon$.

## 5. The Construction

In this section we describe a defining sequence in $S^n$ for a decomposition $G$, based upon a construction similar in many respects to that of [5]. In the next section we will show how to make $G$ be cellular and intrinsically $(n - 1)$-dimensional.

Throughout the rest of this section the body of the text concerns the construction for the case $n \geqslant 5$; whenever a given procedure must be modified for the case $n = 3$, the necessary variations will be listed immediately afterwards, set off by brackets.

*Description of* $\mathcal{M}_1$. Let $T$ denote a standard torus in $I^3$ and $X$ a 3-dimensional $PL$ annulus in Int $I^3$ such that $X \cap T$ is a 2-cell in the boundary of each. Define a set $Z \subset S^n$ as $Sp^{n-3}(X \cup T)$ [as the image of one copy of $X \cup T$ in $Sp^0(I^3) = S^3$] and let $Q(Z)$ denote the closure of $Sp^{n-3}(T)$ minus an interior $PL$ collar on $\partial Sp^{n-3}(T)$ [the closure of that copy of $T$ minus a $PL$ collar on $\partial T$]. Then $Z$ is a compact $n$-manifold with boundary separating $S^n$ and $Q(Z) \subset \text{Int } Z$.

Let $\mathcal{M}_1 = \{Z\}$.

*Inductive Hypothesis* $(j - 1)$. There exist collections $\mathcal{M}_1, \ldots, \mathcal{M}_{j-1}$ and there exist compact, $PL$ $n$-manifolds with boundary $Q(A)$, for each $A \in \cup_{i=1}^{i-1} \mathcal{M}_i$, satisfying the following conditions:

(1) $\mathcal{M}_1, \ldots, \mathcal{M}_{j-1}$ satisfy conditions (1) and (2) of Definition 1.1.

(2) Each $Q(A)$ is contained in Int $A$ and is the $(n - 3)$-spin of a standard torus in $I^3$ [is a $PL$ torus].

(3) For $i = 1, \ldots, j - 1$, $\cup \{A \mid A \in \mathcal{M}_i\} = Z$.

(4) For $A \in \mathcal{M}_i$ where $i \geqslant 2$, diam $(\partial A \cap \text{Pre } A) < 1/i$.

(5) For each virtually I-essential map $F\colon H \to Q(A)$ of a disc with holes $H \subset B^2$ such that $F(H) \cap Sp^{n-3}(\partial I^3) = \emptyset$ [no restriction concerning $F(H)$], where $A \in \mathcal{M}_i$ and $1 \leqslant i < j$, $F$ is virtually $I$-essential with respect to all except possibly one of $\{Q(E) \mid E \in \mathcal{M}_{i+1}$ and Pre $E = A\}$.

(6) For $A \in \mathcal{M}_i$ where $1 < i \leq j - 1$, $A$ is contained in an $n$-cell $D \subset$ Pre $A$.

*Description of $\mathcal{M}_j$.* Assuming Inductive Hypothesis $(j - 1)$, in five steps we shall specify $\mathcal{M}_j$ and $Q(A)$ for the sets $A \in \mathcal{M}_j$ so that $\mathcal{M}_1, \ldots, \mathcal{M}_{j-1}, \mathcal{M}_j$ and the collection of associated $n$-manifolds $Q(A)$ fulfill the parallel Inductive Hypothesis $(j)$. Fix $A \in \mathcal{M}_{j-1}$ and $Q = Q(A)$. It suffices to describe the elements of $\mathcal{M}$ contained in $A$, in other words, those $E \in \mathcal{M}_j$ for which $A =$ Pre $E$. Towards that end, we choose a collar neighborhood $\partial A \times [0, 1]$ on $\partial A$ in $A$, disjoint from $Q(A)$. We also choose $\varepsilon_j \in (0, 1/j)$ so small that $2\varepsilon_j$-subsets of $A$ are contained in $n$-cells in $A$. More will be said about this choice of $\varepsilon_j$ in the next section, when we deal with technical requirements leading to the desired intrinsic dimension.

*Step 1. Decomposing $A$ into cells.* Apply Proposition 4.4 [Lemma 4.1 with $K = 1$] to find a cell decomposition $\Delta_Q$ of $Q(A)$ having mesh less than $\varepsilon_j$. Regard $\partial A \times [0, 1]$ as a *PL* collar on $A$, missing $Q(A)$. Find triangulations $\Delta_\partial$ of $\partial A$ and $\Delta_I$ of $[0, 1]$ so that $\Delta_\partial \times \Delta_I$ gives a cell decomposition of $\partial A \times [0, 1]$ having mesh less than $\varepsilon_j$. Extend $\Delta_Q$ and $\Delta_\partial \times \Delta_I$ to a cell decomposition $\Delta_A$ of $A$ also having mesh less than $\varepsilon_j$. Let $N$ be the number of $n$-cells in $\Delta_A$.

*Step 2. Embedding spun tori in $A$.* Using the results of Section 4 we find distinct elements $\sigma_1, \ldots, \sigma_K$ of $\Delta_Q \subset \Delta_A$ ($K = 1$ in case $n = 3$) as well as $N$ pairwise disjoint $(n - 3)$-spins of standard tori [*PL* tori], $R_1, \ldots, R_N$, with the following properties:

(7) Each $R_i$ is contained in Int $Q(A)$.

(8) For each $n$-cell $\sigma \in \Delta_A$, $\sigma - \cup R_i$ is connected.

(9) $\sigma_i \cap R_i = \emptyset$ and $\sigma_i \cup R_i$ lies interior to an $n$-cell in $Q(A)$ whenever $i \in \{1, \ldots, K\}$.

(10) $\tau \cup R_i$ lies in an $n$-cell in $Q(A)$ whenever $\tau$ is an $n$-cell of

$$\Delta_Q - \{\sigma_1, \ldots, \sigma_K\}$$

and $j \in \{1, \ldots, N\}$.

(11) For each virtually I-essential map $f: H \to Q(A)$ of a disc with holes $H \subset B^2$ such that $f(H) \cap Sp^{n-3}(\partial I^3) = \emptyset$ [no restriction concerning $f(H)$], $f$ is virtually I-essential with respect to at least $(N - 1)$ of $\{R_1, \ldots, R_N\}$.

Since each $R_i$ lies in an $n$-cell in Int $Q(A)$, one can use that cell to engulf any $\tau \in \Delta_A - \Delta_Q$ and establish:

(10′) $\tau \cup R_i$ lies in an $n$-cell in $A$ whenever $\tau$ is an $n$-cell of

$$\Delta_A - \{\sigma_1, \ldots, \sigma_K\}$$

and $i \in \{1, \ldots, N\}$.

*Step 3. Connecting the cells to the spun tori.* Let $\sigma_1, \ldots, \sigma_N$ be an enumeration of the $n$-cells of $\Delta_A$. Each $\sigma_i$ will be connected to $R_i$, but there are two separate methods.

*Case* 1. $i \leqslant K$. Join $\sigma_i$ to $R_i$ by the regular neighborhood $N_i$ of an arc, staying within the $n$-cell in Int $Q(A)$ promised in condition (9) above. Do this so that both $N_i \cap \sigma_i$ and $N_i \cap R_i$ are $PL$ $(n - 1)$-cells. Then $\sigma_i \cup N_i \cup R_i$ is a connected $n$-manifold with boundary contained in an $n$-cell in Int $Q(A)$ and, hence, in Int $A$.

*Case* 2. $i > K$. If $\sigma_i \cap R_i = \emptyset$, join $\sigma_i$ to $R_i$ by the regular neighborhood $N_i$ of an arc, as before, this time staying within the $n$-cell in $A$ promised by condition (10′). If $\sigma_i \cap R_i \neq \emptyset$, make a small general position adjustment to $R_i$ so that $\sigma_i \cup R_i$ is a connected $n$-manifold with boundary and regard $N_i$ as the empty set. In either circumstance, $\sigma_i \cup N_i \cup R_i$ is a connected manifold with boundary contained in an $n$-cell in $A$.

The arcs named above can be threaded through $A$ so that the thickenings $N_i$ are pairwise disjoint. Special care must be taken in the collar. The $n$-cells of condition (10′) must be arranged to either avoid the interior of the collar or to run rather directly through its [0, 1] factor, so that if

$$\sigma_i = \tau \times [t_k, t_{k+1}] \subset \partial A \times [0, 1] \quad \text{where } \tau \times [t_k, t_{k+1}] \in \Delta_\partial \times \Delta_I$$

then

$$N_i \cap (\partial A \times [0, t_{k+1}]) \subset \operatorname{Int} \tau \times \{t_{k+1}\},$$

and so that if $\sigma_i \subset A - (\partial A \times [0, 1])$, then $N_i \cap (\partial A \times [0, 1]) = \emptyset$.

*Step 4. Defining the elements of $\mathcal{M}_j$ in $A$.* For $i = 1, \ldots, N$, let $A_i$ be the closure of $\{(\sigma_i \cup N_i \cup R_i) - \cup_{k \neq i} (R_k \cup N_k)\}$. These are the elements of $\mathcal{M}_j$ in $A$. Note that $A = \cup_{i=1}^N \operatorname{Pre} A_i$.

*Step 5. Defining the corresponding $Q(A_i)$.* For $i = 1, \ldots, N$, let $Q(A_i)$ be the closure of $\{R_i$ minus a small $PL$ interior collar on $\partial R_i\}$.

This completes the inductive description of the defining sequence

$$\mathcal{S} = \{\mathcal{M}_1, \ldots, \mathcal{M}_{j-1}, \mathcal{M}_j, \ldots\}.$$

It is a relatively elementary matter to verify that $\mathcal{M}_j$, defined in this way, satisfies the six conditions of Inductive Hypothesis $(j)$.

## 6. Verifying Properties of the Example

Here we impose two minor additional requirements upon the construction of the defining sequence given in Section 5, to ensure that the resulting decomposition $G$ is cellular and intrinsically $(n$-$1)$-dimensional. The first of these requirements pertains to the internal structure of the defining sequence; the second simply involves a bit of epsilonics.

*Cellularity.* Since each $A \in \cup \{\mathcal{M}_j \mid j = 2, 3, \ldots\}$ is contained in an $n$-cell in Pre $A$, $A$ contracts in Pre $A$. By Theorem 1.4, each $g \in G$ is cell-like.

*Definition* 6.1. A compact subset $C$ of $S^n$ satisfies *McMillan's Cellularity Criterion* if every neighborhood $U$ of $C$ contains a neighborhood $V$ of $C$ such that each loop in $V - C$ is null homotopic in $U - C$.

Let $g \in G$ and let $\mathcal{B} = \cup \{\partial A \mid A \in \cup \mathcal{M}_j\}$. To prove that $g$ is cellular, we shall consider two cases. In both cases we shall assume that $g$ consists of more than one point.

*Case* 1. $g \cap \mathcal{B} = \emptyset$. According to Theorem 1.4, $g = \cap_j st(x, \mathcal{M}_j)$ where $x$ represents any point of $g$. Given a neighborhood $U$ of $x$, choose an integer $j$ large enough that $st(x, \mathcal{M}_{j-1}) \subset U$. Because $x \notin \mathcal{B}$, $st(x, \mathcal{M}_j)$ consists of a single $A \in \mathcal{M}_j$, and condition (6) of Inductive Hypothesis ($j$) implies the existence of an $n$-cell $D$ and $A \subset D \subset$ Pre $A$. In particular, $g \cap \partial A = 0$, so we have

$$g \subset \operatorname{Int} D \subset D \subset \operatorname{Pre} A = st(x, \mathcal{M}_{j-1}) \subset U,$$

which shows that $g$ is cellular.

*Case* 2. $g \cap \mathcal{B} \neq \emptyset$. In this case we must take extra care about permissible intersections of elements from different stages $\mathcal{M}_j$ of the defining sequence. We employ the following concept from [5].

DEFINITION 6.2. A defining sequence $\mathcal{S} = \{\mathcal{M}_1, \mathcal{M}_2, \ldots\}$ (in $S^n$) is *regular* provided that the following hold:

(a) Whenever $A \supset A'$ where $A$ and $A'$ are distinct elements of $\cup \mathcal{M}_j$, then $\partial A \cap \partial A'$ is a (possibly empty) $PL$ $(n-1)$-manifold with (possibly empty) boundary $\partial(A, A')$.

(b) If $A_1 \supset A_2 \supset \ldots \supset A_k$ are all distinct elements of $\cup \mathcal{M}_j$, then

$$\dim[\partial(A_1, A_2) \cap \ldots \cap \partial(A_{k-1}, A_k)] \leq n - k.$$

Exactly as in [5, Section 5] we modify the construction outlined in Section 5 slightly so that the defining sequence $\mathcal{S} = \{\mathcal{M}_1, \mathcal{M}_2, \ldots\}$ is regular.

LEMMA 6.3. *For each $x \in \mathcal{B}$ there exists a positive integer $J(x)$ such that, for each $i = 1, 2, \ldots$, the cardinality of*

$$\{A \in \mathcal{M}_i \mid A \subset st(x, \mathcal{M}_i)\}$$

*is less than $J(x)$.*

*Proof.* If not, it would be possible to find an infinite sequence $A_1 \supset A_2$

$\supset \ldots$ of distinct elements from $\cup_i \mathcal{M}_i$ such that $x \in \partial(A_k, A_{k+1})$ for every $k \geq 1$, in contradiction to (b) of Definition 6.2.

Returning to the matter of cellularity in the case at hand, we will show that the cell-like set $g$ satisfies McMillan's Cellularity Criterion. Then Theorem 1 of [16] (or Theorem 1′ of [16], when $n = 3$) will imply that $g$ is cellular.

Let $x$ denote the unique point of $g \cap \mathcal{B}$. (That there is only one such point results from Theorem 1.4 and condition (3) of Definition 1.1.) It is a consequence of Lemma 6.3 that the cardinality of

$$\{A \in \mathcal{M}_i \mid A \subset st(x, \mathcal{M}_i)\}$$

stabilizes at some $j(x)$, in the sense that for $i \geq j(x)$ each $A \in \mathcal{M}_i$ in $st(x, \mathcal{M}_i)$ contains exactly one element $A' \in \mathcal{M}_{i+1}$. In addition, by Theorem 1.4, $g = \cap_j st(x, \mathcal{M}_j)$.

Consider a neighborhood $U$ of $g$. The cell-likeness of $g$ guarantees another neighborhood $V$ of $G$ that is contractible in $U$. Let

$$h : \partial B^2 \to V - g$$

be given. Choose $k > j(x)$ such that

$$st(x, \mathcal{M}_k) \subset V \quad \text{and} \quad h(\partial B^2) \subset V - st(x, \mathcal{M}_k).$$

Enumerate the elements $A_1, A_2, \ldots, A_s$ of $\mathcal{M}_{k+1}$ contained in $st(x, \mathcal{M}_{k+1})$. By condition (6) of Inductive Hypothesis $(j)$, for $i = 1, 2, \ldots, s$ there exists an $n$-cell $D_i$ such that $A_i \subset D_i \subset \text{Pre } A_i$, and by our choice of $k > j(x)$, which yields distinct sets Pre $A_1$, Pre $A_2$, $\ldots$, Pre $A_s$, we see that $D_i \cap D_j \subset \partial D_i \cap \partial D_j$ whenever $i \neq j$.

The contractibility of $V$ in $U$ implies that the given map $h : \partial B^2 \to V$ extends to a map $h : B^2 \to U$, which then can be adjusted so that $x \notin h(B^2)$. The resulting image(s) can be cut off on the sets $(\partial D_i - x)$, one after another, to excise all points of Int $D_i$ from that and later images. This eventually gives a new map $\tilde{h} : B^2 \to U$ for which $\tilde{h} \mid \partial B^2 = h \mid \partial B^2$ and $\tilde{h}(B^2)$ misses $(\{x\} \cup \cup_{i=1}^s \text{Int } D_i)$. As a result, $\tilde{h}(B^2) \cap g = \emptyset$, revealing that $g$ satisfies McMillan's Cellularity Criterion.

*Intrinsic dimension.* We go back to the Description of $\mathcal{M}_j$ in the construction of Section 5 to further restrict the choice of $\varepsilon_j > 0$ there. Assume, as before, that $\mathcal{M}_{j-1}$ has been prescribed and that

$$\mathcal{M}_{j-1} = \{M_1, \ldots, M_m\}.$$

Let

$$\mathcal{P} = \{P \mid P \text{ is a union of some elements of } \mathcal{M}_{j-1} \text{ and } C\ell(S^n - P) \text{ has at least two components}\}.$$

Since $\cup_i M_i = Z$ and $Z$ separates $S^n$, $\mathcal{P} \neq \emptyset$. Associate with each $P \in \mathcal{P}$ a number $\delta(P) > 0$ as in Proposition 2.5. Now choose $\varepsilon_j > 0$ as in Section

5 but so that, in addition,

$$\varepsilon_j < \min\{\delta(P)/2m \mid P \in \mathcal{P}\}.$$

Let $\Delta_i$ denote the cell decomposition of $M_i$ described in Step 1 of Section 5.

LEMMA 6.4. *Let $C_1, \ldots, C_e$ denote the components of* $\mathrm{C}\ell(S^n - P)$ *where $P \in \mathcal{P}$. Let $\tilde{P}$ be formed from $P$ by removing at most two distinct n-cells of $\Delta_i$ from each of those $M_i$ in $P$. Then* $\mathrm{C}\ell(S^n - \tilde{P})$ *has distinct components $\tilde{C}_1, \ldots, \tilde{C}_e$ with $C_k \subset \tilde{C}_k$ $(k = 1, \ldots, e)$.*

Proof. This follows immediately from Proposition 2.5 because the sum of the diameters of the removed cells is less than $2m\varepsilon_j \leqslant \delta(P)$.

LEMMA 6.5. *Let $\sigma_1, \ldots, \sigma_r$ be n-cells removed from $P$ to form $\tilde{P}$ as in Lemma 6.4; let $A_1, \ldots, A_r$ be the elements of $\mathcal{M}_j$ associated with the cells $\sigma_1, \ldots, \sigma_r$, respectively; and let $\hat{P} = P - \bigcup_{i=1}^{r} (\sigma_i \cup A_i)$. Then* $\mathrm{C}\ell(S^n - \hat{P})$ *has distinct components $\hat{C}_1, \ldots, \hat{C}_e$ with $C_k \subset \hat{C}_k (k = 1, \ldots, e)$.*

*Proof.* Let $\hat{C}_k$ denote the component of $\mathrm{C}\ell(S^n - \hat{P})$ containing $C_k$. It suffices to show that $\hat{C}_k \neq \hat{C}_q$ whenever $k \neq q$.

Suppose the contrary. Then there exists a path $\beta$ in $\bigcup_{i=1}^{r} (\sigma_i \cup A_i)$ joining some $C_k$ to some $C_q$ where $k \neq q$. By Lemma 6.4 there is no such path in $\bigcup_{i=1}^{r} \sigma_i$. Hence, there must exist $M \in \mathcal{M}_{j-1}$ and two sets $A_{i(1)}, A_{i(2)} \in \mathcal{M}_j$ with

$$A_{i(1)} \cup A_{i(2)} \subset M$$

such that $\beta \cap (\sigma_{i(1)} \cup A_{i(1)} \cup \sigma_{i(2)} \cup A_{i(2)})$ joins two points of $\partial M$ not joined by any path in $\sigma_{i(1)} \cup \sigma_{i(2)}$. This implies that $\sigma_{i(1)}$ and $\sigma_{i(2)}$ intersect $\partial M$, that $A_{i(1)}$ and $A_{i(2)}$ then contain $\sigma_{i(1)}$ and $\sigma_{i(2)}$, respectively, and that $\sigma_{i(1)} \cap \sigma_{i(2)} = \emptyset$ (see Section 5, Step 3). The construction of the $A_i$'s guarantees that $A_{i(1)} \cap A_{i(2)} = \emptyset$, essentially because, (in the notation of Section 5)

$$A_{i(1)} = \sigma_{i(1)} \cup N_{i(1)} \cup R_{i(1)}, \quad A_{i(2)} = \sigma_{i(2)} \cup N_{i(2)} \cup R_{i(2)},$$

and

$$N_{i(2)} \cup R_{i(1)} \, (N_{i(2)} \cup R_{i(2)})$$

was formed disjoint from $A_{i(2)} \, (A_{i(1)})$. This impossibility completes the proof.

Thicken $Z$ to another *PL* $n$-manifold $\tilde{Z}$ such that Int $\tilde{Z} \supset Z$ and

$$\tilde{Z} - Z \cong \partial\tilde{Z} \times [0, 1].$$

Since some meridional disk of $T$ misses $X$, there exist disjoint embeddings

$$\lambda_1, \lambda_2 : B^2 \to \operatorname{Int} \psi(\tilde{Z}) \cap (\operatorname{Int} I^3 - X)$$

such that $\lambda_e(\partial B^2) \cap \psi(Z) = \emptyset$ and $\lambda_e(B^2) \cap T$ is a meridional disk of $T$ ($e = 1, 2$). As in Lemma 3.7, these embeddings give rise to disjoint embeddings

$$\Lambda_1, \Lambda_2 : B^2 \to \operatorname{Int} \tilde{Z}$$

such that $\psi\Lambda_e = \lambda_e$; a crucial aspect of $\Lambda_e$ is the elementary consequence that $\Lambda_e \mid \partial B^2$ is not homotopically trivial in $\tilde{Z} - Q(Z)$ ($e = 1, 2$).

Let $W$ and $Y$ denote two components of $S^n - Z$. Choose points $w \in \pi(W)$ and $y \in \pi(Y)$ not in $\pi(\Lambda_1(B^2) \cup \Lambda_2(B^2))$.

According to [1, Lemma 4.2] or [15, Lemma 2.3], every map $f : B^2 \to S^n/G$ has an "approximate lift" $F : B^2 \to S^n$ such that $\pi F$ is close to $f$. Choose $\gamma_0 > 0$ such that

$$\rho(\{w, y\}, \pi(\Lambda_1(B^2) \cup \Lambda_2(B^2)) > \gamma_0$$

and that any map $f_e : B^2 \to S^n/G$ within $\gamma_0$ of $\pi\Lambda_e$ has an approximate lift $F_e : B^2 \to \tilde{Z}$ such that $F_e(\partial B^2) \subset \tilde{Z} - Q(Z)$ and $F_e \mid \partial B^2$ is not null homotopic there ($e = 1, 2$).

LEMMA 6.6. *If $f_1$ and $f_2$ are maps of $B^2$ into $S^n/G$ such that $\rho(f_e, \pi\Lambda_e) < \gamma_0$ for $e = 1, 2$, then $f_1(B^2) \cap f_2(B^2)$ separates* w *from* y *in $S^n/G$.*

*Proof.* Suppose the contrary. Then there exists an arc $\alpha$ from $w$ to $y$ in

$$S^n - (f_1(B^2) \cap f_2(B^2)).$$

Let $F_1, F_2 : B^2 \to S^n$ be approximate lifts of $f_1, f_2$, respectively, such that, for $e = 1, 2$,

(i) $\pi F_1(B^2) \cap \pi F_2(B^2) \cap \alpha = \emptyset$,
(ii) $\rho(\pi F_e, \pi\Lambda_e) < \gamma_0$, and
(iii) $F_e$ is a *PL* map in general position with respect to $Sp^{n-3}(\partial I^3)$ and to all of the sets $Q(A)$, $A \in \cup \mathcal{M}_j$.

We will show that $\pi F_1(B^2) \cap \pi F_2(B^2)$ separates $w$ from $y$, which is impossible in light of (i) above.

It is sufficient to show that at each stage $j$, the union of those elements of $\mathcal{M}_j$ intersecting both $F_1(B^2)$ and $F_2(B^2)$ separates $\pi^{-1}(w)$ from $\pi^{-1}(y)$. This is almost too transparent at stage 1, because $F_e(B^2) \subset \tilde{Z}$ and $F_e \mid \partial B^2$ is not homotopically trivial in $\tilde{Z}$ − Q($Z$), which indicates that both $F_1(B^2)$ and $F_2(B^2)$ meet $Z$, the only element of $\mathcal{M}_1$. To begin an inductive argument, it is necessary to show that, for $e = 1, 2$, $B^2$ contains disks with holes $H_e$ such that $F_e \mid H_e$: $H_e \to Q(Z)$ is virtually I-essential. In order to do this, check that any loop in $\partial(Z - \operatorname{Int} Q(Z))$ null homotopic in $Z - \operatorname{Int} Q(Z)$ must be null homotopic in its boundary, and then apply the argument of Proposition 3.5.

Inductively assume that the union $Z_{j-1}$ ($j > 1$) of all those $A \in \mathcal{M}_{j-1}$ for which both $F_1 \mid H_1$ and $F_2 \mid H_2$ are virtually I-essential with respect to $Q(A)$ separates $\pi^{-1}(w)$ from $\pi^{-1}(y)$. General position features of the maps $F_e$ yield that $F_e(B^2) \cap Sp^{n-3}(\partial I^3) = \emptyset$ if $n \geq 5$. Thus, by condition (5) of Inductive Hypotheses ($j$), all but possibly two of these $A' \in \mathcal{M}_j$ with Pre $A' = A$ have the property that both $F_1 \mid H_1$ and $F_2 \mid H_2$ fail to be virtually I-essential with respect to $Q(A)$. Lemma 6.5 then shows that the corresponding set $Z_j$ separates $\pi^{-1}(w)$ from $\pi^{-1}(y)$, which finishes the proof.

Now we can assemble the various pieces and set forth the main result.

THEOREM 6.7. *For $n = 3$ and $n \geq 5$ there exists a (regular) defining sequence for a cellular usc decomposition $G$ of $S^n$ having intrinsic dimension $(n - 1)$.*

*Proof.* The (regular) defining sequence $\mathcal{M}_1, \mathcal{M}_2, \ldots$ described in Section 5 (as modified) leads to a cellular usc decomposition $G$. Properties of defining sequences, as the term is used here, quickly reveal that $\dim(S^n/G) \leq n$. Therefore, the combination of Lemma 6.6 and Proposition 2.4 shows that $G$ has intrinsic dimension $(n - 1)$.

COROLLARY 6.8. *If $G$ is the decomposition of $S^n$ ($n \geq 5$) named in Theorem 6.7, then $(S^n/G) \times E^1$ is homeomorphic to $S^n \times E^1$.*

Proof. See Corollary 5.9 of [7].

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