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A WEAK INVARIANCE PRINCIPLE FOR HILBERT SPACE VALUED MARTINGALES

BY

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1. Introduction

We start with a probability space $(\Omega, \mathscr{F}, P)$ and a real separable Hilbert space $(H, \langle\cdot, \cdot\rangle)$ with $|x|^2 = \langle x, x\rangle$. Let $\{x_n, \mathscr{F}_n; n \geq 1\}$ be an adapted sequence of H-valued random variables on Ω. We denote by $E_n(\cdot)$ the conditional expectation operator $E(\cdot \mid \mathscr{F}_{n-1})$ and assume that $E_n|x_n|^2$ is well-defined[1] for each $n \geq 1$. We are concerned primarily with martingale differences, i.e., random vectors $\{x_n\}$ with $E_n x_n = 0$ a.s., $n \geq 1$. A useful clock for the partial sums of such vectors is the sequence

$$V_n = \sum_{k=1}^{n} E_k|x_k|^2, \quad n \geq 1. \tag{1.1}$$

We assume that

$$\lim_{n\to\infty} V_n = \infty \quad \text{a.s.} \tag{1.2}$$

and consider the process

$$M_n = \sum_{k\geq 1} x_k \cdot 1\,(V_k \leq n), \quad n \geq 1. \tag{1.3}$$

Let $C_H[0, 1]$ denote the space of continuous H-valued mappings, ξ, on the unit interval, $[0, 1]$, endowed with the supremum norm:

$$\|\xi\|_\infty = \max_{0\leq t\leq 1} |\xi(t)|.$$

Recalling (1.3), we define for each $n \geq 1$ a random element X_n of $C_H[0, 1]$:

$$X_n(t) = \begin{cases} n^{-1/2}M_k & \text{for} \quad t = k/n, k = 0, \ldots, n, \\ \text{linear interpolation in } t \text{ over } \left[\dfrac{k-1}{n}, \dfrac{k}{n}\right], & k = 1, \ldots, n. \end{cases} \tag{1.4}$$

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[1] Given a non-negative random variable w on Ω and a sub-σ-field $\mathscr{G} \subset \mathscr{F}$, the conditional expectation $E(w \mid \mathscr{G})$ is well-defined if the measure $\nu(A) = \int_A w\, dP$, $A \in \mathscr{G}$, is σ-finite.

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The summands $\{x_n\}$ are said to satisfy the central limit theorem (CLT) if the sequence $\{n^{-1/2}M_n\}_{n\geq 1}$ converges in distribution to a mean zero Gaussian law, γ, and more generally, the (weak) invariance principle if the random elements $\{X_n\}_{n\geq 1}$ converge weakly to the corresponding (induced) Wiener measure, W_γ, on $C_H[0, 1]$. For a large class of independent real valued summands the Lindeberg-Feller condition is necessary and sufficient for the CLT (and therefore also for the invariance principle).

The basic result here is a weak invariance principle for H-valued martingale differences, $\{x_n\}$. The formulation of this result rests on the principle that the behavior of

$$n^{-1/2} \sum_{k\geq 1} x_k \cdot 1(V_k \leq n)$$

for large n is determined by that of $n^{-1/2} \sum_{k\geq 1} x_k' \cdot 1(V_k \leq n)$, where x_k' is a truncated version of x_k. One may note the usefulness of this approach by looking at an example provided by McLeish [9]. This example amounts to letting, for $n \geq 1$, independent random variables z_n take the values $\pm 2^{n/2}$ each with probability 2^{-n-1} and the value zero otherwise, and letting $\{\beta_n\}$ be independent mean zero variables taking the values ± 1 independent of the sequence $\{z_n\}$. Then, with $x_n = z_n + \beta_n$ and $S_n = \sum_{k=1}^n x_k$, S_n is a sum of independent random variables each having mean 0 and variance 2, but $S_n/\sqrt{2n} \not\to_w N(0, 1)$. In fact, the Lindeberg-Feller condition fails. Yet $S_n/\sqrt{n} \to N(0, 1)$ (cf. Theorem 1).

Our motivation is to formulate an invariance principle that unifies several known theorems for real valued martingales (cf. [2], [4], [9]) and generalizes those results to the case of values in Hilbert space. By using the truncation method we avoid Lindeberg-type conditions. We give an example to illustrate some interesting martingale behavior in our theorem. (See the example in Section 3.)

We mention that the case of values in Banach space has been considered for sums of independent random vectors with identical distributions by Kuelbs [8] and without the assumption of identical distributions by Garling [6]. Also for the Banach space setting Rosinski [11] has studied martingales. Garling and Rosinski employ Lindeberg-type conditions for their results. Kuelbs obtains the invariance principle from the CLT and a very mild condition. This approach has been extended to φ-mixing arrays by Eberlein [5].

We state and prove our invariance principle for sequences $\{x_n\}$ and later (in Section 3) state the corresponding result for dependent arrays.

THEOREM 1. *Let $\{x_n, \mathscr{F}_n;\ n \geq 1\}$ be an adapted sequence of H-valued random variables such that V_n in* (1.1) *is well-defined and satisfies* (1.2). *Suppose that there is a non-decreasing function $\lambda : \mathbf{R}^+ \to \mathbf{R}^+$* satisfying

$$\lim_{u\to\infty} \lambda(u) = \infty \quad \text{and} \quad \lim_{u\to\infty} \lambda(u)/u = 0 \tag{1.5}$$

such that, upon defining

$$x_n' = x_n \cdot 1(|x_n| \leq \sqrt{\lambda(V_n)}) - E_n(x_n \cdot 1(|x_n| \leq \sqrt{\lambda(V_n)}), \tag{1.6}$$

one has[2]

$$\operatorname*{Plim}_{n\to\infty} n^{-1} \sum_{k\geq 1} E_k\langle h, x_k'\rangle^2 \cdot 1(V_k \leq n) = v(h) \tag{1.7}$$

with some number $v(h)$ for each element $h \in H$.

Suppose further that

$$\limsup_{n\to\infty} n^{-1} \sum_{k\geq 1} \sum_{i\geq N} E(\langle f_i, x_k'\rangle^2 \cdot 1(V_k \leq n)) \to 0 \quad as \quad N \to \infty \tag{1.8}$$

where $\{f_i\}_{i\geq 1}$ denotes a complete orthonormal basis for H. Then there is a mean zero Gaussian measure γ on H with $\int_H \langle h, x\rangle^2 \gamma(dx) = v(h)$, $h \in H$, such that, upon defining

$$\begin{aligned} M_k' &= R.H.S.\ of\ (1.3)\ with\ x_k'\ in\ place\ of\ x_k, \\ X_n' &= R.H.S.\ of\ (1.4)\ with\ M_k'\ in\ place\ of\ M_k, \end{aligned} \tag{1.9}$$

one obtains

$$P \circ (X_n')^{-1} \to_w W_\gamma \tag{1.10}$$

with the Wiener measure W_γ on $C_H[0, 1]$ induced by γ. In particular, if

$$\operatorname*{Plim}_{n\to\infty} n^{-1/2} \max_{1\leq k\leq n} \left| \sum_{j\leq k} (x_j - x_j') \cdot 1(V_k \leq n) \right| = 0 \tag{1.11}$$

then

$$P \circ X_n^{-1} \to_w W_\gamma, \tag{1.12}$$

with X_n defined by (1.4).

Remarks (i) Whenever the sequence $\{M_n'/\sqrt{n}\}_{n\geq 1}$, as defined by (1.6) and (1.9), converges in law to a Gaussian measure, condition (1.8) must hold. This assertion is proved at the very end of Section 2.

(ii) When $\{x_n\}$ is an H-valued martingale difference sequence one observes by Doob's submartingale maximal inequality and Lemma 2 of Brown [2] that the classical Lindeberg condition implies (1.11) whenever

$$\operatorname{Plim}_{n\to\infty} V_n/EV_n = 1.$$

Our Theorem 1 thus generalizes Theorem 3 in [2].

(iii) Finally, we give a series condition that implies (1.11) for the case of martingale differences, namely,

$$\sum_{n\geq 1} \lambda(V_n)^{-1/2} |E_n(x_n \cdot 1(|x_n| > \sqrt{\lambda(V_n)}))| < \infty \quad \text{a.s.} \tag{1.13}$$

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[2] Plim denotes limit in probability.

For, observe that

$$P(|x_n| > \sqrt{\lambda(V_n)} \mid \mathscr{F}_{n-1}) \leq \frac{|E_n(x_n \cdot 1(|x_n| > \sqrt{\lambda(V_n)}))|}{\sqrt{\lambda(V_n)}}$$

so that by (1.13) and Lévy's generalization of the Borel-Cantelli lemma,

$$|x_n| > \sqrt{\lambda(V_n)} \quad \text{only finitely often with probability one.} \tag{1.14}$$

Further,

$$n^{-1/2}\max_{k\leq n}\left|\sum_{j\geq 1} E_j(x_j \cdot 1(|x_j| \leq \sqrt{\lambda(V_j)})) \cdot 1(V_j \leq k)\right|$$
$$\leq \left(\frac{\lambda(n)}{n}\right)^{1/2} \sum_{j\geq 1} \lambda(n)^{-1/2}|E_j(x_j \cdot 1(|x_j| \leq \sqrt{\lambda(V_j)}))| \cdot 1(V_j \leq n).$$

Now, using $E_n x_n = 0$ in (1.13) and Kronecker's lemma one verifies that this last sum tends to zero in probability. This together with (1.14) yields (1.11).

2. Proofs

For future reference we recall by (1.6) that

$$|x_n'| \leq 2\sqrt{\lambda(V_n)} \text{ a.s.}, \quad n \geq 1. \tag{2.1}$$

To establish (1.10) we proceed in the usual way. That is, first we find a Gaussian measure γ so that in accordance with (1.10) the finite dimensional distributions of the sequence $\{X_n'\}$ converge properly. Secondly, we demonstrate that this sequence is tight in $C_H[0, 1]$. We summarize these points in Propositions 1 and 2, below.

To prove Proposition 1 we need the following lemma whose proof parallels the lines of Brown [2].

LEMMA. *Let* $0 = t_0 < t_1 < \cdots < t_\nu = 1$ *and let* $h_1, \ldots, h_\nu \in H$ *for some* $\nu \geq 1$. *Then*

$$\lim_{n\to\infty} E \exp\left\{i \sum_{\mu=1}^{\nu} \langle h, X_n'(t_\mu) - X_n'(t_{\mu-1})\rangle\right\} = \exp\left(-\frac{1}{2}\sum_{\mu=1}^{\nu} (t_\mu - t_{\mu-1})v(h_\mu)\right), \tag{2.2}$$

where, for each $h \in H$, $v(h)$ *is the limit appearing in* (1.7).

Proof. We define for all $k, n \geq 1$,

$$y_k = y_k^{(n)} = n^{-1/2} \sum_{\mu=1}^{\nu} \langle h_\mu, x_k' \rangle \cdot 1([nt_{\mu-1}] < V_k \leqslant [nt_\mu]) \tag{2.3}$$

with x_k' and V_k defined by (1.6) and (1.1) respectively. Our first task is to show that the lemma follows upon proving that

$$\lim_{n\to\infty} E \exp\left\{ i \sum_{k\geqslant 1} y_k^{(n)} \right\} = \text{R.H.S. of (2.2).} \tag{2.4}$$

To accomplish this observe that

$$\Delta_n = \sum_{\mu=1}^{\nu} \langle h_\mu, X_n'(t_\mu) - X_n'(t_{\mu-1}) \rangle - \sum_{k\geqslant 1} y_k^{(n)} = \sum_{k\geqslant 1} \Delta_{k,n} \tag{2.5}$$

with

$$\Delta_{k,n} = \sum_{\mu=0}^{\nu-1} n^{-1/2}(nt_\mu - [nt_\mu]) \langle h_\mu, x_k' \rangle \delta_{k,n}^{(\mu)}$$

and

$$\delta_{k,n}^{(\mu)} = 1([nt_\mu] < V_k \leqslant [nt_\mu] + 1).$$

Since the sums $\Sigma_{k=1}^{N} \Delta_{k,n}$, $N \geqslant 1$, form a martingale, which, as we shall now see, is L^2 bounded, we have by (1.1), the definition of $\delta_{k,n}^{(\mu)}$ and (2.1),

$$\begin{aligned} E\,\Delta_n^2 &\leqslant n^{-1} \sum_{\mu=1}^{\nu} \sum_{k\geqslant 1} E_k \langle h_\mu, x_k' \rangle^2 \cdot \delta_{k,n}^{(\mu)} \\ &\leqslant \sum_{\mu=1}^{\nu} |h_\mu|^2 (1 + 2\lambda(n))/n \qquad (2.6) \\ &= o(1) \text{ as } n \to \infty. \end{aligned}$$

Thus, by (2.5), (2.6) and the inequality

$$E|\exp(ia) - \exp(ib)| \leqslant E^{1/2}(a-b)^2,$$

valid for real random variables a and b, it suffices to establish (2.4). We use the method of Brown to do this. For the sake of completeness we include the proof.

Put

$$m_N = \sum_{k=1}^{N} y_k, \quad R_N = \sum_{k=1}^{N} E_k y_k^2, \tag{2.7}$$

and

$$\sigma^2 = \sum_{\mu=1}^{\nu} (t_\mu - t_{\mu-1}) v(h_\mu).$$

We estimate (while suppressing the dependence on n) that

$$\left|E\exp(im_\infty)-\exp\left(-\frac{1}{2}\sigma^2\right)\right|$$

$$\leq E\left|\left(1-\exp\left\{\frac{1}{2}(R_\infty-\sigma^2)\right\}\right)\cdot\exp(im_\infty)\right|$$

$$+\left|E\left(1-\exp\left\{im_\infty+\frac{1}{2}R_\infty\right\}\right)\right|\cdot\exp\left(-\frac{1}{2}\sigma^2\right) \tag{2.8}$$

$$=\mathrm{I}+\mathrm{II}\quad\text{(say)}.$$

By (2.3), (2.7), (1.7) and (1.1), $|R_\infty-\sigma^2|$ is bounded almost surely by a constant and tends to zero in probability as $n\to\infty$. Hence

$$\lim_{n\to\infty}\mathrm{I}=0. \tag{2.9}$$

To estimate II we break the expectation term therein into a telescoping sum, as follows. Let

$$Z_k=\left\{\exp\left(im_{k-1}+\frac{1}{2}R_k\right)\right\}\times\left\{\exp(iy_k)-\exp\left(-\frac{1}{2}E_ky_k^2\right)\right\}. \tag{2.10}$$

Then

$$\mathrm{II}\cdot\exp\left(\frac{1}{2}\sigma^2\right)=\left|E\sum_{k=1}^{\infty}Z_k\right|=\left|E\sum_{k=1}^{\infty}E_k(Z_k)\right|. \tag{2.11}$$

Here the last equality is justified since the partial sums

$$\sum_{k=1}^{N}Z_k=1-\exp\left(im_N+\frac{1}{2}R_N\right),\quad N\geq 1,$$

are uniformly bounded.

Next, employing the expansions

$$e^{i\alpha}=1+i\alpha-\frac{\alpha^2}{2}+O(\alpha^3),\quad\alpha\to 0,\quad\text{and} \tag{2.12}$$
$$e^{-u}=1-u+O(u^2),\quad u\to 0,$$

one calculates, by (2.1), (2.3), (2.10) and (2.12), that

$$E_k(Z_k)=\exp\left(im_{k-1}+\frac{1}{2}R_k\right)\times 0\left(\sqrt{\lambda(n)/n}\right)\times E_ky_k^2.$$

Thus, by (1.1), (2.3) and (1.5),

$$
\begin{aligned}
\left| E \sum_{k \geq 1} E_k(Z_k) \right| &= O\left((\sqrt{\lambda(n)/n} \right) \cdot E\left(\sum_k E_k y_k^2 \right) \\
&= O\left(\sqrt{\lambda(n)/n} \right) \cdot O(1) \qquad (2.13) \\
&= o(1) \quad \text{as } n \to \infty.
\end{aligned}
$$

Finally, by (2.8), (2.9), (2.11) and (2.13) we have (2.4). Whence the lemma is proved. ■

We are now ready to show that the finite dimensional distributions associated with the sequence $\{X_n'\}_{n \geq 1}$ converge properly. To formulate this result we use the multiple evaluation functional, $e_{t_1, \ldots, t_k} : C_H[0, 1] \to H^k$, defined by

$$\xi \to (\xi(t_1), \ldots, \xi(t_k)).$$

PROPOSITION 1. *There exists a mean zero Gaussian measure γ on H with*

$$\int_H \langle h, x \rangle^2 \gamma(dx) = v(h), \quad h \in H,$$

such that for any $0 \leq t_1 < \ldots < t_k \leq 1$ and $k \geq 1$,

$$P \circ (e_{t_1, \ldots, t_k} \circ X_n')^{-1} \to_w W_\gamma \circ e_{t_1, \ldots, t_k}^{-1} \quad \textit{as } n \to \infty.$$

Proof. By the lemma, it suffices to show that the sequence of measures

$$\left\{ P \circ (e_{t_1, \ldots, t_k} \circ X_n')^{-1} \right\}_{n \geq 1}$$

is tight. Moreover, since a set of probability measures on H^k is tight if and only if each of the k sets of marginals is, we need only verify that the sequence

$$\{P \circ (e_t \circ X_n')^{-1}\}_{n \geq 1}$$

is tight for any fixed $t \in [0, 1]$.

To do this it is enough to have for any positive numbers ε and η finitely many balls A_i of radius η in B such that,

$$P(X_n'(t) \in \cup A_i) > 1 - \varepsilon \quad \text{for all } n \geq 1. \tag{2.14}$$

(For, in a metric space, a set A has compact closure $\bar{A}$ if and only if A is totally bounded and $\bar{A}$ is complete.) To verify condition (2.14) we introduce the projection maps $\pi_N : H \to H$, defined for each $N \geq 1$ by $\pi_N(x) = \sum_{i=1}^N \langle f_i, x \rangle f_i$. Here $\{f_i\}_{i \geq 1}$ is the complete orthonormal basis for H appearing in (1.8). But, since

$$|X_n'(t) - \pi_N X_n'(t)| \leq 3n^{-1/2} \max_{1 \leq k \leq n} |M_k' - \pi_N M_k'|,$$

from the submartingale maximal inequality and (1.8) we get

$$\begin{aligned} E|X'_n(t) - \pi_N X'_n(t)|^2 &\leq 4 \cdot 9E|M'_n - \pi_N M'_n|^2 \\ &= 36 \cdot n^{-1} E \sum_{i>N} \sum_{k\geq 1} E\langle f_i, x'_k\rangle^2 \cdot 1(V_k \leq n)) \\ &\leq \varepsilon(N), \end{aligned}$$

for $n \geq n(N)$, where $\varepsilon(N) \to 0$ as $N \to \infty$. Therefore, since the lemma implies that

$$\left\{P \circ (\pi_{N_0} \circ e_t \circ X'_n)^{-1}\right\}_{n\geq 1}$$

is tight for any finite N_0, by a standard argument one obtains finitely many balls A_i of radius η such that $P(X'_n(t) \in \cup A_i) > 1 - \varepsilon/2$, for all $n \geq n(N_0)$. Finally, by the separability of H one can augment this collection of balls $\{A_i\}$ by finitely many balls of radius η so that (2.14) holds. ∎

We now provide the only remaining ingredient needed to establish (1.10).

PROPOSITION 2. *The sequence of random elements* $\{X'_n\}_{n\geq 1}$ *is tight in* $C_H[0, 1]$.

Proof. By Proposition 1 (and its proof) it is sufficient to have, for each $\varepsilon > 0$,

$$\lim_{h\to 0} \limsup_{n\to\infty} P(\sup_{|s-t|<h} |X'_n(t) - X'_n(s)| > \varepsilon) = 0. \tag{2.15}$$

In turn, for (2.15), it is enough to show that for any $h > 0$ and $s \in [0, 1 - h]$,

$$E \max_{s\leq t\leq s+h} |X'_n(t) - X'_n(s)|^4 \leq C(h \cdot \lambda(n)/n + h^2), \tag{2.16}$$

with some absolute constant C. But, since

$$\max_{s\leq t\leq s+h} |X'_n(t) - X'_n(s)| \leq 3 \max_{[sn]<k\leq[(s+h)n]+1} n^{-1/2}|M'_k - M'_{[sn]}|,$$

(2.16) will follow from (1.5) and Burkholder's square function inequality [3, Theorem 3.2] with $p = 4$, upon proving that

$$E\left(\left(\sum_{k\geq 1} |x'_k|^2 \cdot \delta_k^{(n)}\right)^2\right) \leq C(hn \cdot \lambda(n) + h^2n^2)$$

with $\delta_k^{(n)} = 1([sn] < V_k \leq [(s + h)n] + 1)$ and an absolute constant C.
Finally, by (1.1) and (2.1),

$$E\left(\left(\sum_{k\geq 1}|x_k'|^2\cdot\delta_k^{(n)}\right)^2\right)=E\sum_{k\geq 1}\left(|x_k'|^4+2\sum_{1\leq j<k}|x_j'|^2\delta_j^{(n)}|x_k'|^2\right)\cdot\delta_k^{(n)}$$

$$\leq E\sum_{k\geq 1}E_k|x_k'|\delta_k^{(n)}\left(4\cdot\lambda(n)+2\sum_{j\geq 1}|x_j'|^2\delta_j^{(n)}\right)$$

$$\leq C(hn\cdot\lambda(n)+h^2n^2).\ \blacksquare$$

Since Propositions 1 and 2 imply (1.10), to finish the proof of Theorem 1 we need only verify that the condition $M_n'/\sqrt{n}\to_w\gamma'$, for some Gaussian measure γ' on H, implies (1.8). But we just saw in the proof of Proposition 2 that

$$\sup_{n\geq 1}E(|M_n'/\sqrt{n}|^4)<\infty,$$

so that the sequence $\{|M_n'/\sqrt{n}|^2\}_{n\geq 1}$ is uniformly integrable. Thus, as $n\to\infty$,

$$E|(I-\pi_N)M_n'/\sqrt{n}|^2\to\int_H(I-\pi_N)^2(x)\gamma'(dx)$$

and this last limit must tend to zero as $N\to\infty$. Whence, (1.8) holds. $\blacksquare$

3. Dependent Arrays

For the sake of reference we reformulate Theorem 1 now for a doubly infinite array

$$\{x_{j,n};\ j\geq 1,\ n\geq 1\}$$

of H-valued random variables. Afterwards we present an example to illustrate this theorem.

We denote the conditional expectation operator $E(\cdot\ |x_{1,n},\ \ldots,\ x_{j-1,n})$ by $E_j^{(n)}(\cdot)$. We then put

$$V_k^{(n)}=\sum_{j=1}^{k}E_j^{(n)}(|x_{j,n}|^2),\quad k\geq 1,\ n\geq 1,$$

and assume that $V_k^{(n)}$ is well defined for all k and n and that

$$\lim_{k\to\infty}V_k^{(n)}\geq 1\quad\text{a.s.},\quad n\geq 1.$$

We set

$$M_k^{(n)}=\sum_{j\geq 1}x_{j,n}\cdot 1(V_j^{(n)}\leq k/n),\quad k=0,\ \ldots,\ n,\tag{3.1}$$

and define, for each n, a random element $X_n\in C_H[0,1]$ by the R.H.S. of (1.4) with $n^{-1/2}M_k$ replaced by $M_k^{(n)}$.

THEOREM 1′. *Suppose there exists* $\lambda_n \downarrow 0$ *such that with*

$$x'_{j,n} = x_{j,n} \cdot 1(|x_{j,n}| \leq \lambda_n) - E_j(x_{j,n} \cdot 1(|x_{j,n}| \leq \lambda_n)) \tag{3.2}$$

one has

$$\operatorname*{Plim}_{n\to\infty} \sum_{k\geq 1} E_k^{(n)}\langle h, x'_{k,n}\rangle^2 \cdot 1(V_k^{(n)} \leq t) = v(h, t) \tag{3.3}$$

with some number $v(h, t)$ *for all* $h \in H$ *and* $t \in [0, 1]$.

Assume that

$$\limsup_{n\to\infty} \sum_{k\geq 1} \sum_{i\geq N} E(\langle f_i, x'_{k,n}\rangle^2 \cdot 1(V_k^{(n)} \leq 1)) \to 0 \quad \text{as } N \to \infty \tag{3.4}$$

where $\{f_i\}$ *denotes a complete orthonormal basis for* H.

Then there exists a family $\{\gamma_t\}_{t\in[0,1]}$ *of mean zero Gaussian measures on* H *with*

$$\int_H \langle h, x\rangle^2 \gamma_t(dx) = v(h, t), \quad h \in H, t \in [0, 1].$$

Let us denote by Γ *the Gaussian measure on* $C_H[0, 1]$ *whose realization is an* H*-valued separable process* $\{\tilde{W}(t); t \in [0, 1]\}$ *with independent increments and one dimensional distributions* $P \circ \tilde{W}(t)^{-1} = \gamma_t$. *Then, defining* $M_k^{(n)\prime}$ *by the R.H.S. of* (3.1) *with* $x'_{j,n}$ *in place of* $x_{j,n}$ *and* X'_n *by the R.H.S. of* (1.4) *with* $M_k^{(n)\prime}$ *in place of* $n^{-1/2}M_k$, *we have*

$$P \circ (X'_n)^{-1} \to_w \Gamma \quad \text{as } n \to \infty. \tag{3.5}$$

In particular, if

$$\operatorname*{Plim}_{n\to\infty} \max_{k\geq 1} \left| \sum_{j=1}^{k} (x_{j,n} - x'_{j,n}) \cdot 1(V_j \leq k/n) \right| = 0 \tag{3.6}$$

then

$$P \circ (X_n)^{-1} \to_w \Gamma \quad \text{as } n \to \infty. \tag{3.7}$$

Remarks. (i) In case $E_j^{(n)}x_{j,n} = 0$ for all $j \geq 1$ and $n \geq 1$, the condition

$$\operatorname*{Plim}_{n\to\infty} \sum_{j\geq 1} |E_j(x_{j,n} \cdot 1(|x_{j,n}| > \lambda))| = 0, \quad \lambda > 0,$$

implies (3.6). This is seen by using Lemma (3.5) of Dvoretsky [4] in place of Lévy's Borel Cantelli lemma in the argument that (1.13) implies (1.11).

(ii) Theorem 1′ generalizes Corollary (3.8) of [9] and improves Theorem C of [7].

Example. Let $x_{j,n}$ be defined for $j = 1, 2, \ldots, n$ and $n \geq 1$ as follows. For $j = 2^k, \ldots, 2^{k+1} - 1$ and $k \geq 0$, define $l_{j,n} = -1 + j2^{-k}$, $r_{j,n} = l_{j,n} + 2^{-k}(1 - (k + 1)2^{-n})$ and put

$$z_{j,n} = (\log n)^{-2/3}(1 + j2^{-n})(-1)^{[j/2]} \cdot I(l_{j,n}, r_{j,n})$$
$$-2^n(\log n)^{-2/3}(1 + j2^{-n})(-1)^{[j/2]}(1 - (k + 1)2^{-n}) \cdot I(r_{j,n}, r_{j,n} + 2^{-n-k}),$$

where $I(a, b)$ denotes the indicator function for a subinterval (a, b) of the Lebesgue unit interval and $[c]$ denotes the integer part of a real number c. Also, for each n, take independent variables $\{\beta_{j,n}\}_{1 \le j \le n}$ with $\beta_{j,n}$ taking each of the values $\pm 1/\sqrt{n}$ with probability $\frac{1}{2}$ and independent of the $\{z_{j,n}\}$. Let $x_{j,n} = z_{j,n} + \beta_{j,n}$. Then the array $\{x_{j,n}\}$ satisfies the hypotheses of Theorem 1′ and also (3.6) with $\lambda_n = (\log n)^{-1/2}$ in (3.2), so (3.7) holds. But,

$$\sum_{j=1}^{n} |E_j x_{j,n} \cdot 1(|x_{j,n}| \le c_{j,n})| \to \infty$$

in probability for any constants $\{c_{j,n}\}$ bounded strictly away from 0 and ∞. Thus, Theorem 1′ is not contained in Theorem (3.6) of [9] even when norming by $\Sigma\, x_{j,n}^2$ is equivalent to norming by $V_j^{(n)}$.

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