ILLINOIS JOURNAL OF MATHEMATICS
Volume 27, Number 4, Winter 1983

THE TRANSFER IN SEGAL'S COHOMOLOGY

BY

A. KOZLOWSKI

1. Introduction

Let R be a commutative ring and let $G^*(X; R)$ denote Segal's cohomology with coefficients in the graded ring $R[x]$ of polynomials in one indeterminate x, of degree one if char $R = 2$ and two otherwise (see [9]).

Recall [3] that every generalized cohomology theory admits a transfer homomorphism for finite coverings such that stable transformations of cohomology theories commute with the transfer. The main purpose of this paper is to compute the transfer for the functor $G^0(X; R)$, where $R = Z/hp$, for coverings of the form

$$X \approx E(Z/p) \times X \rightarrow B(Z/p) \times X.$$

As an application of this computation we show that multiplicative operations (in the sense of Atiyah-Hirzebruch [1]) of classical cohomology with coefficients in Z/p—and in particular the total Steenrod *p-th* power operation (Steenrod square for $p = 2$)—when restricted to units do not extend to operations in Segal's cohomology. This result is surprising, and should be compared with the situation in K-theory localised at a prime q. In this case the Adams operations ψ^k, where k is not divisible by q, are automorphisms of connective K-theory, and their restrictions to multiplicative units extend to operations in the multiplicative (tensor product) cohomology theory described by Segal in [8] (see [6]).

Another application of the computation of the transfer in Segal's cohomology will appear in [4].

The plan of the paper is as follows. In §2 the definition and the main properties of the Kahn-Priddy transfer are recalled. In §3 the transfer in Segal's cohomology is computed. In §4 the impossibility of extending of Atiyah-Hirzebruch operations to transformations of Segal's cohomology is proved. In §5 some implications of this result and some related questions are discussed.

Finally, I wish to thank the Japan Society for the Promotion of Science for financial support under their post-doctoral fellowship scheme.

Received September 9, 1981.

2. The Kahn-Priddy Transfer

We shall assume that all cohomology theories are defined on the category of pointed spaces of the homotopy type of C-W complexes with finite skeleta. However, when the space under consideration is not viewed as being in the "domain" of some cohomology theory we shall only require it to be compactly generated (e.g., spaces representing cohomology theories).

By an infinite loop space, I mean a space V together with a given family of $(n - 1)$ connected spaces $V(n)$ for each $n \geqslant 0$ and weak homotopy equivalences $V(n) \approx \Omega V(n + 1)$, where ΩX is the space of loops on X. There is an obvious corresponding notion of an infinite loop map between two infinite loop spaces.

An infinite loop space determines a connective cohomology theory by the formulas $V^n(X) = [X; V(n)]$ and $V^{-n}(X) = V^0(\Sigma^n X)$ for $n \geqslant 0$ and correspondingly an infinite loop map determines a (stable) transformation of cohomology theories.

Recall from [7] that if V is an infinite loop space then there is an E_∞ operad $C = \{C(n)\}_{n\geqslant 0}$ and an action of C on V, i.e., a family of Dyer-Lashof maps $D_n : C(n) \times_{\Sigma_n} V^n \to V$. Since each $C(n)$ is contractible and Σ_n acts freely on it, $C(n)$ is a model of $E(\Sigma_n)$—the total space of the universal principal bundle $E(\Sigma_n) \to B(\Sigma_n)$. Let S be a subgroup of Σ_n. The Dyer-Lashof maps restrict to maps $D_n^S : E(S) \times_S V^n \to V$.

Let $X \to Y$ be an S-covering, i.e., a fibre bundle with structure group S and fibre $\mathbf{n} = \{1, \ldots, n\}$. Let $\tilde{X} \to Y$ be the associated principal S-bundle. Let

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{u} & E(S) \\ \downarrow & & \downarrow \\ Y & \xrightarrow{v} & B(S) \end{array}$$

be a classifying map of bundles (unique up to a homotopy of maps of bundles). Let $P : \tilde{X} \times \mathbf{n} \to X$ be the principal map (see [12]). We have a S-equivariant map $u \times \mathbf{P} : \tilde{X} \to E(S) \times X^n$, where $\mathbf{P} : \tilde{X} \to X^n$ is the adjoint of P. Hence, by passing to orbit spaces, we obtain a map

$$t : Y \approx \tilde{X}/S \to E(S) \times_S X^n,$$

which is determined uniquely up to homotopy. By slight abuse we shall call t the pre-transfer (cf. [3]). The transfer homomorphism is defined as the composite

$$[X; V] \to [E(S) \times_S X^n; E(S) \times_S V^n] \xrightarrow{D_n^S} [E(S) \times_S X^n; V] \xrightarrow{t} [Y; V].$$

Consider now the covering $X \approx E(Z/p) \times X \to B(Z/p) \times X$. This is a principal Z/p-covering, where Z/p is identified with the subgroup of Σ_p consisting of cyclic permutations of p objects. We shall next determine the pre-transfer for this covering. Let

$$\Delta : B(Z/p) \times X \to E(Z/p) \times_{Z/p} X^p$$

be the map induced by the Z/p equivariant map

$$E(Z/p) \times X \to E(Z/p) \times X^p,$$

which takes

$$(e, x) \quad \text{to} \quad (e, \underbrace{x, \ldots, x}_{p}).$$

LEMMA 2.1. *Let*

$$t : B(Z/p) \times X \to E(Z/p) \times_{Z/p} (E(Z/p) \times X)^p$$

be the pre-transfer for the above covering and let

$$s : E(Z/p) \times_{Z/p} (E(Z/p) \times X)^p \to E(Z/p) \times_{Z/p} X^p$$

be the natural homotopy equivalence. Then st is homotopic to Δ.

Proof. Since the covering is a principal one we have only to describe a classifying map and the principal map. The principal map is a map $(E(Z/p) \times X) \times (Z/p) \to E(Z/p) \times X$, where Z/p is the fibre and is identified with the finite set **p** above. It is easy to see that the principal map is given by the action of Z/p on $E(Z/p)$ and hence its adjoint

$$E(Z/p) \times X \to (E(Z/p) \times X)^p$$

is given by $(e, x) \to ((\mathbf{0}e, x), \ldots, ((\mathbf{p} - \mathbf{1})e, x))$, where $\mathbf{n}e$ denotes the action of $\mathbf{n} \in Z/p$ on $e \in E(Z/p)$. As a classifying map for our bundle (covering) we can take the map

$$\begin{array}{ccc} E(Z/p) \times X & \xrightarrow{u} & E(Z/p) \\ \downarrow & & \downarrow \\ B(Z/p) \times X & \xrightarrow{v} & B(Z/p) \end{array}$$

where u and v are the projections onto the first factor. It now follows from the definition that the pre-transfer is induced by the Z/p-equivariant map

$$E(Z/p) \times X \to E(Z/p) \times (E(Z/p) \times X)^p,$$

which takes (e, x) to $(e, (\mathbf{0}e, x), (\mathbf{1}e, x), \ldots, ((\mathbf{p} - \mathbf{1})e, x)$ and this, after the above identification, coincides with the map inducing Δ.

3. Segal's Cohomology

From now on p will always denote an odd prime. In what follows, I shall give complete proofs only for the case of coefficients in $Z/2$. Analogous statements for the odd primes will be often given in brackets.

For a commutative ring R, let $K(R)$ be the graded topological R-module $\{K(R, q)\}_{q\geq 0}$, if char $R = 2$; $\{K(R, 2q)\}_{q\geq 0}$ otherwise. (Here $K(R, n)$ denotes the standard topological abelian group model of Eilenberg-Mac Lane space; see [9].) Let $S(n)$ be the space of graded, n-linear maps $K(R) \times \ldots \times K(R) \to K(R)$. (Strictly speaking we should write $S(n; R)$ for $S(n)$.)

LEMMA 3.1. *$\{S(n)\}_{n\geq 0}$, with obvious structure maps, is an E_∞ operad.*

Proof. See [9]. It only remains to show that Σ_n acts freely on $S(n)$. I shall only consider the case $n = 2$ and coefficients in $Z/2$. The general case is analogous.

Let K_n denote $K(Z/2, n)$. We have

$$\mathrm{Bilin}(K_n, K_n; K_{2n}) \approx \mathrm{Maps}(S^n \wedge S^n; K_{2n})$$

(see [9]), where Bilin denotes the space of bilinear maps. Also clearly

$$\mathrm{SymmBilin}(K_n, K_n; K_{2n}) \approx \mathrm{Maps}(S^n \wedge S^n/\Sigma_2; K_{2n}).$$

If $S(2)$ is not Σ_2-free then there is a symmetric bilinear map inducing the cup product, and hence an element of $H^{2n}(S^n \wedge S^n/\Sigma_2; Z/2)$ which is non-zero in $H^{2n}(S^n \wedge S^n; Z/2)$, contradicting the obvious fact that the map $S^n \wedge S^n \to S^n \wedge S^n/\Sigma_2$ has degree 2.

Let R be a commutative ring. Let $G(R) = \bigoplus_{i=1}^{\infty} K(R, i)$, if char $R = 2$ and $\bigoplus_{i=1}^{\infty} K(R; 2i)$ otherwise. It is easy to see that the operad $S = \{S(n)\}_{n\geq 0}$ acts on $G(R)$. Hence by [7], $G(R)$ is an infinite loop space. (In [9], Segal considered the infinite product of Eilenberg-Mac Lane spaces instead of $G(R)$. The two spaces are clearly weakly homotopy equivalent, and the corresponding cohomology theories coincide on the category of C-W complexes. However, the space $G(R)$, being of the homotopy type of a C-W complex (it is, in fact, the infinite product in the homotopy category of C-W complexes), is often more convenient. Therefore I shall take $G(R)$ as the representing space for the zero term of Segal's cohomology $G^0(X; R)$.)

We can now prove our main result.

THEOREM 3.2. (a) *For the covering*

$$X \approx E(Z/2) \times X \to B(Z/2) \times X,$$

the transfer

$$\mathrm{tr}_G : G^0(X; Z/2) \to G^0(B(Z/2) \times X; Z/2)$$

is given by

$$1 + x_1 + \cdots + x_n + \cdots \mapsto 1 + \sum_{i=1}^{\infty} \sum_{k=0}^{i} t^{i-k} \otimes Sq^k(x_i),$$

where $0 \neq t \in H^1(B(Z/2); Z/2)$ and Sq^n is the n-th Steenrod squaring operation.

(b) *For the covering*

$$X \approx E(Z/p) \times X \to B(Z/p) \times X,$$

the transfer

$$\mathrm{tr}_G : G^0(X; Z/p) \to G^0(B(Z/p) \times X; Z/p)$$

is given by

$$1 + x_1 + \cdots + x_n + \cdots \mapsto 1 + \sum_{i=1}^{\infty} \sum_{k=0}^{(p-1)i} w_k \otimes D_k(x_i),$$

where $w_k \in H^k(B(Z/p); Z/p)$ *are generators as in* [11] *and*

$$D_k : H^i(X; Z/p) \to H^{pi-k}(X; Z/p)$$

are related to Steenrod's p-th power operations by the formula

$$P^k(x) = (-1)^r (m!)^q D_{(q-2k)(p-1)}(x),$$

where $x \in H^q(X; Z/p)$, $m = (p - 1)/2$, $r = k + m(q^2 + q)/2$ (*see* [11]).

Proof. We shall only give a proof of (a), the argument in the case (b) being analogous.

First, observe that we can assume that X is a finite C-W complex. For, we have

$$G^0(X; Z/2) = \varprojlim_n G^0(X_n; Z/2),$$

$$G^0(B(Z/2) \times X; Z/2) = \varprojlim_n G^0(B(Z/2) \times X_n; Z/2),$$

where X_n is the n-skeleton of X, since, $G^*(X; Z/2)$ having finite coefficients, the $\lim^1$ terms in the Milnor exact sequence (e.g., [13]) must vanish. The above assertion now follows from the naturality of the statement of Theorem 3.2. Consider the composite homomorphism

$$[X; G] \to [E(\Sigma_2) \times_{\Sigma_2} X^2; E(\Sigma_2) \times_{\Sigma_2} G^2]$$
$$\xrightarrow{D} [E(\Sigma_2) \times_{\Sigma_2} X^2; G] \xrightarrow{\Delta} [B(\Sigma_2) \times X; G], \text{ where } G = G(Z/2).$$

By Lemma 2.1., this is the transfer for the covering

$$X \to B(\Sigma_2) \times X (= B(Z/2) \times X).$$

As a functor of X, the right hand side is represented by the space $\mathrm{Maps}(B(\Sigma_2); G)$. We can identify $B(\Sigma_2)$ with $S(2)/\Sigma_2$ (Lemma 3.1.). It is easy to verify that the above homomorphism is induced by the map $S : G \to \mathrm{Maps}(S(2)/\Sigma_2; G)$,

$$S(1 + x_1 + \cdots + x_n)(s) = \tilde{s}(1 + x_1 + \cdots + x_n, 1 + \cdots + x_n)$$

where $s \in S(2)/\Sigma_2$ and $\tilde{s} \in S(2)$ projects onto s. Next, observe that S is homotopic to the map $\mathbf{S}$,

$$\mathbf{S}(1 + x_1 + \cdots + x_n)(s) = 1 + \tilde{s}_{1,1}(x_1, x_1) + \cdots + \tilde{s}_{n,n}(x_n, x_n),$$

where $\tilde{s}_{i,j} : K(Z/2; i) \times K(Z/2; j) \to K(Z/2; i + j)$ is the (i, j)-*th* component of the graded, bilinear map

$$\tilde{s} : K(Z/2) \times K(Z/2) \to K(Z/2),$$

which projects onto $s \in S(2)/\Sigma_2$. (There are, of course, two such maps, but the formula is clearly independent of the choice made.) To see this; let $F : I \times S(2) \to S(2)$ be a homotopy contracting $S(2)$ to some $v \in S(2)$, i.e., $F(0, \ ) = 1 : S(2) \to S(2)$ and $F(1, \ ) : S(2) \to S(2)$ takes $S(2)$ to v. By bilinearity, we have

$$\begin{aligned} S(1 + x_1 + \cdots + x_n)(s) &= \tilde{s}(1 + x_1 + \cdots + x_n, 1 + x_1 + \cdots + x_n) \\ &= 1 + \sum_{i\geq 1} \tilde{s}_{i,i}(x_i, x_i) \\ &\qquad + \sum_{i>j\geq 1} ((\tilde{s}_{i,j}(x_i, x_j) + \tilde{s}_{j,i}(x_j, x_i)). \end{aligned}$$

For $\tilde{s} \in S(2)$ let $\tilde{s}' \in S(2)$ be given by $\tilde{s}'_{i,j}(x, y) = \tilde{s}_{j,i}(y, x)$. Then the formula

$$1 + \sum_i \tilde{s}_{i,i}(x_i, x_i) + \sum_{i,j} (F(t, \tilde{s})_{i,j}(x_i, x_j) + F(t, \tilde{s}')_{i,j}(x_i, x_j))$$

gives the required homotopy, since

$$F(1, \tilde{s})_{i,j}(x_i, x_j) = F(1, \tilde{s}')_{i,j}(x_i, x_j) = v_{i,j}(x_i, x_j)$$

and $2x = 0$ in $K(Z/2; n)$.

Now, consider the map

$$S_i : K(Z/2) \to \mathrm{Maps}(S(2)/\Sigma_2; K(Z/2; 2i)), \ S_i(x)(s) = \tilde{s}_{i,i}(x, x).$$

The induced homomorphism $H^i(X; Z/2) \to H^{2i}(B(\Sigma_2) \times X; Z/2)$ is the composite

$$[X; K(Z/2; i)] \xrightarrow{P_i} [S(2) \times_{\Sigma_2} X^2; K(Z/2; 2i)] \to [B(\Sigma_2) \times X; K(Z/2; 2i)],$$

where P_i is defined by means of the map $S(2) \times_{\Sigma_2} K(Z/2; i) \to K(Z/2; 2i)$ taking $(\tilde{s}, x, y)$ to $\tilde{s}_{i,i}(x, y)$. We claim that this P_i coincides with the map denoted by P in [11], p. 100 (after obvious identifications). This follows from the fact that the map P is uniquely characterised by the following three properties

(1) $P(0) = 0$.

(2) Let $j : X^2 \to E(\Sigma_2 \times_{\Sigma_2} X^2$ be defined by $j(x, y) = (s, x, y)$, where

s is some element of $E(\Sigma_2)$. For every $u \in H^i(X; Z/2)$, $j^*P(u)$ is the n-fold cross-product $u \times \cdots \times u$.

(3) For every $f : X \to Y$ the diagram

$$\begin{array}{ccc} H^i(Y; Z/2) & \xrightarrow{P} & H^{2i}(E(\Sigma_2) \times_{\Sigma_2} Y^2; Z/2) \\ \downarrow f^* & & \downarrow f^* \\ H^i(X; Z/2) & \xrightarrow{P} & H^{2i}(E(\Sigma_2) \times_{\Sigma_2} X^2; Z/2) \end{array}$$

is commutative. (See Remark 2.4. p. 101 and Chapter VIII of [11].)

It is easy to verify that our P_i satisfies the above conditions and hence coincides with Steenrod's P. The conclusion of the theorem now follows from Definitions 3.2. and 6.1 of [11].

4. Atiyah-Hirzebruch Transformations

Let $A(Z/2)$ (resp. $A(Z/p)$) be the group of all natural automorphisms of the ring $\Pi_{i\geq 0} H^i(X; Z/2)$ (resp. $\Pi_{i\geq 0} H^{2i}(X; Z/p)$). Let

$$x \in H^1(RP^\infty; Z/2) \quad (\text{resp. } H^2(CP^\infty; Z/p))$$

be the non-zero element (a generator). In [1], it is shown that the map $f \to f(x)$ is a group isomorphism between $A(Z/2)$ (resp. $A(Z/p)$) and the group of formal power series of the form

$$x + a_1x^2 + \cdots + a_nx^{2^n} + \cdots \quad (\text{resp. } x + a_1x^p + \cdots + a_nx^{p^n} + \cdots),$$

where $a_n \in Z/2$ (resp. Z/p) and the group operation is given by substituting one power series into another. (It is easy to show that this is a commutative group isomorphic to the "usual" multiplicative group of formal power series of the form $1 + \Sigma_{i=1}^{\infty} a_it^i$.) It is also shown in [1], that each element of $A(Z/2)$ (resp. $A(Z/p)$) determines a stable operation in classical cohomology. It is natural to expect that restrictions of elements of $A(Z/2)$ (resp. $A(Z/p)$) to $G^0(X; Z/2)$ (resp. $G^0(X; Z/p)$) extend to stable transformations of Segal's cohomology. However, we have the following:

THEOREM 4.1. *The only element of $A(Z/2)$ (resp. $A(Z/p)$) whose restriction to $G^0(X; Z/2)$ (resp. $G^0(X; Z/p)$) extends to an operation in $G^*(X; Z/2)$ (resp. $G^*(X; Z/p)$) is the identity element.*

Proof. Let $x \in H^1(RP^\infty; Z/2)$ be non-zero, and suppose

$$f(x) = x + a_1x^2 + \cdots + a_nx^n + \cdots, \quad f \in A(Z/2).$$

Consider the covering $RP^\infty \to B(\Sigma_2) \times RP^\infty$. Then

$$1 + x \in G^0(RP^\infty; Z/2)$$

and we have, from Theorem 3.1,

$$tr_G f(1 + x) = \mathrm{tr}_G\left(1 + \sum_{i\geq 1} a_i x^{2^i}\right) = 1 + \sum_{i\geq 1} a_i t^{2^i} x^{2^i} + \sum_{i\geq 1} a_i x^{2^{i+1}},$$

where $t \in H^1(B(\Sigma_2); Z/2)$ is the non-zero element. On the other hand,

$$\begin{aligned} f(\mathrm{tr}_G(1 + x)) &= f(1 + tx + x^2) \\ &= 1 + f(t)f(x) + f(x)^2 \\ &= 1 + \left(\sum_i a_i t^{2^i}\right)\left(\sum_j a_j x^{2^j}\right) + \sum_i a_i x^{2^{i+1}}. \end{aligned}$$

Hence, $\mathrm{tr}_G(f(1 + x)) - f(\mathrm{tr}_G(1 + x)) = \Sigma_{i\neq j} a_i a_j t^{2^i} x^{2^j} \neq 0$, unless $a_i = 0$ for all $i \geq 1$. Hence if f is not the identity it does not commute with the transfer, and therefore does not extend to an operation in $G^*(X; Z/2)$.

The proof for an odd prime p is analogous.

The cohomology theory $G^*(X; Z/p)$ (for all prime p) does, however, admit stable operations. Let $W(Z/p)$ denote the group of formal power series of the form $1 + a_1 t + \cdots + a_n t^n + \cdots$, with the operation of usual power series multiplication, where $a_i \in Z/p$. For each $f \in W(Z/p)$ let $G(f) : G^0(X; Z/p) \to G^0(X; Z/p)$ be the natural homomorphism defined by means of the "splitting principle"; i.e., the value of $G(f)$ on an element of the form $1 + x_1 + \cdots + x_n$ is defined by writing it formally as a product $(1 + u_1) \cdots (1 + u_n)$, where each u_i has degree 1, and expressing the symmetric series $f(1 + u_1) \cdots f(1 + u_n)$ in terms of the x_i's. One can show that $G(f)$ can be extended to an operation in $G^*(X; Z/p)$. In fact $W(Z/p)$ has a ring structure such that $f \to G(f)$ is a ring homomorphism from $W(Z/p)$ to the ring of operations in $G^*(X; Z/p)$. For details see [5].

5. Some Further Remarks

We have seen that restrictions of Atiyah-Hirzebruch automorphisms to the multiplicative group, and in particular the total Steenrod squaring operation (which corresponds to the power series $x + x^2$), do not extend to operations in Segal's cohomology or, on space level, cannot be represented by infinite loop maps. In other words, we have examples of H-self homotopy equivalences which are not E_∞-self homotopy equivalences. This can be re-interpreted in the following way: let $f : G \to G$ be such a map (not the identity). Since f is a H-homotopy equivalence it follows from [2], that there is an infinite loop structure on G, denoted by G_f, whose underlying H-structure coincides with that of G, and such that $f : G \to G_f$ is an equivalence of infinite loop spaces. The two cohomology theories will have the same zero term—the multiplicative group of classical cohomology, but different transfer homomorphisms; in fact clearly $\mathrm{tr}_{G_f} = f\,\mathrm{tr}_G f^{-1}$.

In [10], Snaith showed that the total Stiefel-Whitney class

$$w : \tilde{KO}(X) \to G^0(X; Z/2)$$

and the total Chern class

$$c : \tilde{K}(X) \to G^0(X; Z)$$

do not extend to stable transformations of respective connective cohomology theories. However, Snaith's argument depends on a property of Segal's transfer (Lemma 3.2. of [10]) and does not apply to cohomology theories of the form G_f when f is given by an infinite power series. Equivalently, Snaith's proof does not show that there are no infinite loop maps $BO \to G(Z/2)$ and $BU \to G(Z)$. However, we have

PROPOSITION 5.1. *There are no infinite loop maps $BO \to G(Z/2)$ and $BU \to G(Z/p)$ for all prime p.*

Proof. Observe that by the splitting principle, any natural homomorphism $\tilde{KO}(X) \to G^0(X; Z/2)$ must be of the form $G(f)w$, where w is the total Stiefel-Whitney class and $f \in W(Z/2)$. Consider again the covering $RP^\infty \to B(\Sigma_2) \times RP^\infty$. Suppose $G(f)w$ is an infinite loop map. Then

$$\begin{aligned} \mathrm{tr}_G(G(f)(w(\tilde{x}))) &= G(f)(\mathrm{tr}_G(w(\tilde{x})) \quad \text{(since } G(f) \text{ is an infinite loop map)} \\ &= G(f)(w(\mathrm{tr}_{\tilde{K}}(\tilde{x}), \end{aligned}$$

where $\tilde{x} = [H] - [1] \in \tilde{K}(RP^\infty)$ and H is the Hopf line bundle. Hence $G(f)(\mathrm{tr}_G(w(\tilde{x}))/w(\mathrm{tr}_{\tilde{K}}(\tilde{x})) = 1$ and hence $\mathrm{tr}_G(w(\tilde{x}))/w(\mathrm{tr}_{\tilde{K}}(\tilde{x})) = 1$. But this is false, as can be shown by a simple computation using Theorem 3.1. The proof of the second claim is analogous.

Finally, consider again the map $S : G(Z/2) \to \mathrm{Maps}(B(\Sigma_2); G(Z/2))$ of the proof of Theorem 3.1. The operad S which acts on $G(Z/2)$ also acts on $\mathrm{Maps}(B(\Sigma_2); G(Z/2))$, hence its identity component is an infinite loop space by [7]. The zero-*th* term of the corresponding cohomology theory is the group of multiplicative units $\tilde{G}^0(X; Z/2)$ of the ring

$$\prod_{i \geq 0} H^i(X; H^*(B(Z/2); Z/2))$$

of the form $1 + y_1 + \cdots + y_i + \cdots$ where $y_i \in H^i(X; H^*(B(Z/2); Z/2))$. As was shown in the proof of Theorem 3.2, the homomorphism

$$S : G^0(X; Z/2) \to \tilde{G}^0(X; Z/2)$$

is given by $1 + \Sigma_i x_i \mapsto 1 + \Sigma_i \Sigma_k t^{i-k} \otimes Sq^k(x_i)$. (Note that if in this formula we replace t by 1, we obtain the total Steenrod square.) We have:

PROPOSITION 5.2. *S is an infinite loop map.*

Proof. By Theorem 3.1., S induces the transfer for the covering

$$X \to B(Z/2) \times X.$$

By [3], there are transfers

$$S^{(i)} : G^i(X; Z/2) \to G^i(B(Z/2) \times X; Z/2) = \tilde{G}^i(X; Z/2),$$

with $S = S^{(0)}$, such that the following diagram is commutative:

$$\begin{tikzcd}
G^{(i+1)}(\Sigma X; Z/2) \arrow[r, "\Sigma S^{(i+1)}"] & G^{(i+1)}(\Sigma(B(Z/2) \times X); Z/2) \arrow[r, phantom, "="] & \tilde{G}^{(i+1)}(\Sigma X; Z/2) \\
G^{(i)}(X; Z/2) \arrow[u, phantom, "\wr"] \arrow[r, "S^{(i)}"] & G^{(i)}(B(Z/2) \times X; Z/2) \arrow[r, phantom, "="] & \tilde{G}^{(i)}(X; Z/2) \arrow[u, phantom, "\wr"]
\end{tikzcd}$$

This proves our assertion.

It can now be easily shown that the identity component of the space $\mathrm{Maps}(B(Z/2); G(Z/2))$ is equivalent as an H-space to $G(H^*(B(Z/2); Z/2))$, but is not equivalent to it as an infinite loop space. For if it were, we could show using the above that the total Steenrod square is an infinite loop map, contradicting Theorem 4.1.

References

1. M. F. Atiyah and F. Hirzebruch, *Cohomologie-Operationen und characteristische klassen,* Math. Zeitschr., vol. 77 (1961), pp. 187–199.
2. J. M. Boardman and R. M. Vogt, *Homotopy invariant algebraic structures on topological spaces,* Lecture Notes in Mathematics, no. 347, Springer, New York, 1973.
3. D. S. Kahn and S. B. Priddy, *Applications of the transfer to stable homotopy theory,* Bull. Amer. Math. Soc., vol. 78 (1972), pp. 981–987.
4. A. Kozlowski, *Integral "Riemann-Roch" theorems,* to appear.
5. ———, *Operations in Segal's cohomology,* Math. Proc. Cambridge Philos. Soc., to appear.
6. I. Madsen, V. P. Snaith and J. Tornehave, *Infinite loop maps in geometric topology,* Math. Proc. Cambridge Philos. Soc., vol. 81 (1977), pp. 399–430.
7. J. P. May, *The geometry of iterated loop spaces,* Lecture Notes in Mathematics, no. 271, Springer, New York, 1972.
8. G. B. Segal, *Categories and cohomology theories,* Topology, vol. 13 (1974), pp. 293–312.
9. ———, *The multiplicative group of classical cohomology,* Quarterly J. Math., vol. 26 (1975), pp. 289–293.
10. V. P. Snaith, *The total Chern and Stiefel-Whitney classes are not infinite loop maps,* Illinois J. Math. vol. 21 (1977), pp. 300–303.
11. N. E. Steenrod and D. B. Epstein, *Cohomology operations,* Ann. of Math. Studies, vol. 50, Princeton, N.J., 1962.
12. N. E. Steenrod, *The topology of fibre bundles,* Princeton Univ. Press, Princeton, N.J., 1951.
13. G. W. Whitehead, *Elements of homotopy theory,* Springer-Verlag, New York, 1978.

Tsukuba University
Ibaraki, Japan