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# COERCIVE INEQUALITIES FOR INDEFINITE FORMS OVER NONSMOOTH PLANAR REGIONS

BY

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## Introduction

In two previous papers [3], [4] the author derived sufficient conditions for the coerciveness of certain integrodifferential forms over complex-valued functions defined on certain regions in the plane; roughly, these regions are allowed to have edges and corners, but not spikes. These and all other results known to the author involving nonsmooth regions (except for Gårding's inequality [2], where a maximum number of homogeneous boundary conditions are assumed) required that the form be formally positive (semidefinite). This paper extends some of these results to a wider class of forms, which now are allowed to be algebraically indefinite.

In the first part of this paper a result is stated and proved for functions defined on a sector and vanishing on its boundary; in the second part a result is stated, and the method of proof indicated, in the case of certain types of bounded regions with nonsmooth boundaries.

## 1. Constant Coefficients and Sectors

DEFINITIONS. For $n$ a positive integer, let $\alpha = (\alpha_1, \dots, \alpha_n)$ be a multi-index with $|\alpha| = \alpha_1 + \cdots + \alpha_n$. Set

$$D^\alpha u(x) = D_1^{\alpha 1} \cdots D_n^{\alpha n} u(x) \quad \text{where } D_j u(x) = (1/i)\partial u/\partial x_j.$$

For $m$ a non-negative integer and $\Omega$ a domain in $\mathbf{R}^n$, we set

$$|u|_m^2 = \sum_{|\alpha|=m} \int_\Omega |D^\alpha u(x)|^2 dx \quad \text{and} \quad \|u\|_m^2 = \sum_{p=0}^m |u|_p^2.$$

Let $C_0^\infty(\overline{\Omega})$ be the set of all complex-valued functions defined and continuous and of compact support in $\overline{\Omega}$, the closure of $\Omega$, and infinitely differentiable in $\Omega$.

Let $C_{0,1}^\infty(\overline{\Omega})$ be the subset of those functions in $C_0^\infty(\overline{\Omega})$ which vanish on $\partial\Omega$, the boundary of $\Omega$; $H_m(\overline{\Omega})$ the completion of $C_0^\infty(\overline{\Omega})$ in the $\| \ \|_m$-norm; and $H_{m,1}(\overline{\Omega})$ the completion of $C_{0,1}^\infty(\overline{\Omega})$ in the $\| \ \|_m$-norm.

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A vector in $\mathbf{R}^2$ will sometimes be designated by $x = (x_1, x_2)$ and sometimes by $(x, y)$, as convenience dictates; we will likewise have two notations for the integrodifferential form in the theorem.

THEOREM 1. *Let $\Omega$ be a sector in $\mathbf{R}^2$, and let*

$$\begin{aligned} Q[u] &= \sum_{\substack{|\alpha|=2 \\ |\beta|=2}} a_{\alpha\beta} D^\alpha \bar{u} D^\beta u \\ &= a|D_x^2 u|^2 + 2b|D_x D_y u|^2 + c|D_y^2 u| + d_1 D_x^2 \bar{u} D_x D_y u \\ &\quad + d_2 D_x D_y \bar{u} D_x^2 u + e_1 D_x^2 \bar{u} D_y^2 u + e_2 D_y^2 \bar{u} D_x^2 u \\ &\quad + f_1 D_x D_y \bar{u} D_y^2 u + f_2 D_y^2 \bar{u} D_x D_y u \end{aligned}$$

*where the coefficients indicated are all complex constants. Suppose*

(1) $\operatorname{Re} a > 0$ *and*

(2) $\delta_1 < \operatorname{Re} b - \dfrac{|d|^2}{2 \operatorname{Re} a}$

*where $d = \frac{1}{2}(\bar{d}_1 + d_2)$, $e = \frac{1}{2}(\bar{e}_1 + e_2)$, $f = \frac{1}{2}(\bar{f}_1 + f_2)$, and $\delta_1$ is the smallest simple root of the cubic polynomial (in $\delta$)*

$$\det Q_\delta = \det \begin{pmatrix} \operatorname{Re} a & \bar{d} & \bar{e} + \delta \\ d & 2 \operatorname{Re} b - 2\delta & \bar{f} \\ e + \delta & f & \operatorname{Re} c \end{pmatrix}.$$

*Then there exists $K > 0$ such that for all $u \in H_{2,1}(\bar{\Omega})$,*

$$|u|_2^2 \leqslant K \iint_\Omega \operatorname{Re} Q[u] \, dx\, dy. \tag{3}$$

*Remark.* Consider the "diagonal" case $d = e = f = 0$. If $\operatorname{Re} a > 0$ and $-\sqrt{\operatorname{Re} a \cdot \operatorname{Re} c} < \operatorname{Re} b < 0$, this furnishes an example of an algebraically indefinite coercive form, since a short computation will show that (2) is satisfied.

*Remarks about proof.* 1. Since every function in $H_{m,1}(\bar{\Omega})$ may be approximated in the $\| \quad \|_m$-norm by functions in $C_{0,1}^\infty(\bar{\Omega})$, it will suffice to prove the theorems for functions in $C_{0,1}^\infty(\bar{\Omega})$.

2. In our proofs we shall assume that $\Omega$ is the first quadrant of the $(x, y)$-plane. If $\Omega$ is any sector other than a half-plane, we can reduce the proofs to the previous case by a nonsingular coordinate transformation. If $\Omega$ is a half-plane, we can use a similar (but simpler) proof; this would be a special case of the results of Figueiredo [1] for regions with smooth boundaries.

The following lemma is used in the proof; it is here that we make use of the boundary condition on u.

LEMMA 1. *Let $\Omega$ be the first quadrant in the $(x, y)$-plane, and let $u$ be a complex-valued function in $C^{\infty}_{0,1}(\overline{\Omega})$. If we define*

$$R[u] = D_x^2\bar{u}D_y^2u - D_xD_y\bar{u}D_xD_yu$$

*then*

$$\iint_\Omega R[u]dxdy = \iint_\Omega \overline{R}[u]dxdy = 0.$$

*Proof of Lemma 1.* We use the identity

$$R[u] = D_x(D_x\bar{u}D_y^2u) - D_y(D_x\bar{u}D_xD_yu). \tag{4}$$

(Details of proof are given in [3, p. 829].)

*Proof of Theorem 1.* Now suppose there exists a real constant $\delta$ such that the Hermitian form

$$Q_\delta[u] = \operatorname{Re} Q[u] + \delta R[u] + \delta\overline{R}[u]$$

is positive definite with respect to $(D_x^2u, D_xD_yu, D_y^2u)$. If $u \in C^{\infty}_{0,1}(\overline{\Omega})$, we can then conclude from Lemma 1 that

$$\iint_\Omega \operatorname{Re} Q[u]dxdy = \iint_\Omega Q_\delta[u]dxdy$$

is a positive linear combination of

$$\iint_\Omega |D_x^2u|^2dxdy, \iint_\Omega |D_xD_yu|^2dxdy, \iint_\Omega |D_y^2u|^2dxdy.$$

This implies conclusion (3) of Theorem 1. Hence in order to prove Theorem 1 in the special case, it now suffices to show that hypotheses (1) and (2) assure the existence of a real $\delta$ that makes $Q_\delta[u]$ positive definite.

$Q_\delta[u]$ is a Hermitian form with the following matrix:

$$\begin{array}{c|ccc} & D_x^2u & D_xD_yu & D_y^2u \\ \hline D_x^2\bar{u} & \operatorname{Re} a & \bar{d} & \bar{e} + \delta \\ D_xD_y\bar{u} & d & 2\operatorname{Re} b - 2\delta & \bar{f} \\ D_y^2\bar{u} & e + \delta & f & \operatorname{Re} c \end{array} \tag{5}$$

We know that this form is positive definite if and only if the three principal minors of the above matrix are all positive. We now check to see if we can satisfy these conditions with an appropriate choice of $\delta$:

(6) Re $a > 0$,

(7) Re $a(2 \text{ Re } b - 2\delta) - |d|^2 > 0$ if and only if $\delta < \text{Re } b - \dfrac{|d|^2}{2 \text{ Re } a}$,

(8) det $Q_\delta > 0$ where $Q_\delta$ is the $3 \times 3$ matrix given in (5).

Since det $Q_\delta$ is a cubic polynomial in $\delta$ with real coefficients, and leading term $2\delta^3$, a moment's reflection will show that hypotheses (1) and (2) guarantee the existence of the $\delta$ specified in (7) and (8).

This concludes the proof of Theorem 1 for the first quadrant.

## 2. Variable Coefficients and Bounded Regions

DEFINITIONS. If $\Omega$ is a domain in $\mathbf{R}^2$, let $C_t^m(\Omega)$ be the class of all complex-valued functions defined and continuous on $\overline{\Omega}$, $m$ times continuously differentiable on $\Omega$, and such that all derivatives of order less than $t$ vanish on $\partial\Omega$, the boundary of $\Omega$.

Let $\Omega_1$ be an open sector with vertex at $x_0$, and let $B_\varepsilon$ be the open disk of radius $\varepsilon$ about $x_0$. We will say that a domain $\Omega^*$ is of *special type* if it is of the form $\Omega_1 \cap B_\varepsilon$.

Let $\Omega$ be a bounded domain in $\mathbf{R}^2$. We say that $\Omega$ is of *class* $P^s$ if every point $x_0$ of $\partial\Omega$ has a neighborhood $\mathcal{N}$ such that the closure of $\mathcal{N} \cap \Omega$ can be mapped homeomorphically onto the closure of a domain $\Omega^*$ of special type, with $\overline{\mathcal{N}} \cap \partial\Omega$ mapped onto $\partial\Omega_1$, and such that the map and its inverse are $s$ times continuously differentiable. (This class contains polygons as well as domains of class $C^s$.)

THEOREM 2. *Let $\Omega$ be a domain of class $P^2$, and let*

$$Q[u] = \sum_{\substack{0 \leq |\alpha| \leq 2 \\ 0 \leq |\beta| \leq 2}} a_{\alpha\beta}(x) D^\alpha \bar{u} D^\beta u$$

*where the coefficients $a_{\alpha\beta}(x)$ are complex-valued functions of $x$; the highest-order coefficients (those for $|\alpha| = |\beta| = 2$) are continuous and all other coefficients are bounded. If the highest-order coefficients satisfy hypotheses* (1) *and* (2) *of Theorem* 1 *for all $x \in \overline{\Omega}$, then there exists $K > 0$ such that*

$$\|u\|_2^2(\Omega) \leq K \left[ \iint_\Omega \text{Re } Q[u] dxdy + \|u\|_0^2(\Omega) \right]. \tag{9}$$

*Indication of proof.* The proof uses Theorem 1 and more-or-less standard "patching" techniques. A sketch of such a proof is given in [4, p. 122].

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