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ON THE HOMOTOPY TYPE OF DIFFEOMORPHISM GROUPS[1]

BY

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Introduction

Let M be a closed smooth manifold and $Diff_0(M)$ the identity component of the group of C^∞ diffeomorphisms of M. We are concerned here with the way in which the homotopy type of $Diff_0(M)$ depends on the smooth structure of M. Our principal result along these lines states that, if M_1 and M_2 are homeomorphic smooth manifolds, then, for suitable subrings Λ of the rationals Q (obtained from the integers Z by inverting a finite set of primes), $Diff_0(M_1)$ and $Diff_0(M_2)$ have the same Λ-homotopy type. (Recall that two nilpotent spaces X and Y are said to have the same Λ-homotopy type if there is a space W and mappings $X \to W$ and $Y \to W$ inducing isomorphisms

$$\pi_q(X) \otimes \Lambda \simeq \pi_q(W) \otimes \Lambda, \qquad \pi_q(Y) \otimes \Lambda \simeq \pi_q(W) \otimes \Lambda$$

for all $q \geqslant 0$. See [1].) In particular, we define an integer $\nu = \nu(M_1, M_2)$ in Section 1 depending only on bundle data associated to M_1 and M_2 such that the following holds:

THEOREM. *Let M_1 and M_2 be homeomorphic smooth n-manifolds, $n \neq 4$, and let Λ be the subring of Q obtained from Z by inverting $\nu(M_1, M_2)$. Then $Diff_0(M_1)$ and $Diff_0(M_2)$ have the same Λ-homotopy type.*

We prove an analogous result for the (simplicial) group **PL**(M) of PL-homeomorphisms of a PL-manifold (Theorem 1.3). We also prove a similar result regarding the discrete group homology (with coefficients in Λ) of $Diff_0(M_1)$ and $Diff_0(M_2)$ (Theorem 1.2).

Another type of result that we investigate involves the mapping of the diffeomorphism group of a smooth manifold onto its frame bundle. Let M be a smooth closed n-manifold and let $P(M)$ be the frame bundle of M; that is, the principal $GL(n, R)$ bundle associated with the tangent bundle of M. Then $Diff_0(M)$ acts on $P(M)$ and we can define a mapping $\sigma: Diff_0(M)$

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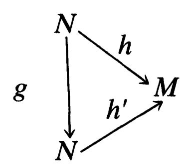
$\to P(M)$ by

$$\sigma(g) = dg(f_0)$$

where f_0 is a fixed frame in $P(M)$ and dg is the differential of g. Our theorem here states that, for suitable subrings of the rationals, the Λ-homotopy type of this mapping does not depend on the smooth structure of M. (See Theorem 1.4 for a precise statement.) This result relates to the work of Schultz [11].

1. Statement of results

Let M be a closed topological n-manifold. A *smoothing* of M is a pair (N, h) where N is a smooth manifold and $h:N \to M$ is a homeomorphism. Two smoothings (N, h) and (N', h') are said to be *equivalent* if there is a diffeomorphism $g:N \to N'$ such that the diagram



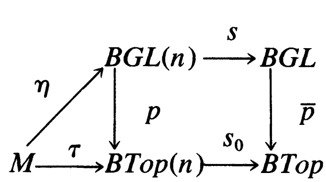
commutes up to topological isotopy. We denote the set of all equivalence classes of smoothings of M by $\mathcal{S}(M)$ and write M_α for the manifold M with the smooth structure defined by $\alpha \in \mathcal{S}(M)$.

Let $Top(n)$ denote the group of homeomorphisms of R^n fixing the origin, $GL(n)$ the subgroup of invertible linear transformations, and Top and GL the limits (in n) of these groups. Suppose that $\tau:M \to BTop(n)$ is a classifying map for the topological tangent bundle of M. Then any smoothing α defines a lift η of τ to $BGL(n)$. If

$$s_0:BTop(n) \to BTop$$

$$s:BGL(n) \to BGL$$

are the stabilization mappings, we have the commutative diagram



If β is a second smoothing for M with lift $\rho:M \to BGL(n)$, then the composites

$$s \circ \eta,\ s \circ \rho:M \to BGL$$

both cover $s_0 \circ \tau:M \to BTop$. Now, $\bar{p}:BGL \to BTop$ can be taken to be a fibration of infinite loop spaces with fibre the infinite loop space Top/GL.

It follows that there is a mapping

$$\Delta = \Delta(\alpha, \beta): M \to Top/GL$$

such that

$$s(\eta(x)) = \Delta(x)s(\rho(x)) \tag{1.1}$$

for all $x \in M$. In fact, $\Delta(\alpha, \beta) = 0$ if and only if α and β represent the same element of $\mathscr{S}(M)$. (For example, see [6] or [7].)

Remark. Using standard techniques, $\bar{p} : BGL \to BTop$ can be taken to be a principal fibration with topological group as fibre. Thus the two sides of equation (1.1) are equal and not just homotopic.

Let $[M, Top/GL]$ denote the group of homotopy classes of mappings $M \to Top/GL$. This group is finite since the homotopy groups of Top/GL are finite. In fact, $\pi_q(Top/GL) \simeq \vartheta_q$, the group of q-homotopy spheres if $q > 4$ and $\pi_q(Top/GL) \simeq \pi_q(Top/PL)$ if $q \leqslant 6$. (See [5].) We denote the order of $[\Delta(\alpha, \beta)]$ in $[M, Top/GL]$ by $\nu(\alpha, \beta)$ and let $\Lambda(\alpha, \beta)$ be the subring of the rational numbers Q obtained from the ring of integers by inverting $\nu(\alpha, \beta)$.

For any smooth structure α on M, let $Diff'(M_\alpha)$ denote the subgroup of $Diff(M_\alpha)$ which maps into the identity component $Top_0(M)$ of $Top(M)$ under the natural mapping. Thus $Diff'(M_\alpha)$ consists of those diffeomorphisms of M_α which are topologically isotopic to the identity. We can now state our main result.

THEOREM 1.1. *Let α and β be smoothings of the closed n-manifold M, $n \neq 4$. Then the classifying spaces $BDiff'(M_\alpha)$ and $BDiff'(M_\beta)$ have the same $\Lambda(\alpha, \beta)$ homotopy type.*

Remark. It follows from Propositions 2.2, 2.3, and 2.4 of Section 2 that $BDiff'(M_\alpha)$ and $BDiff'(M_\beta)$ are nilpotent spaces.

The proof of this theorem is given in Section 2.

COROLLARY 1. *If α and β are smoothings of the n-manifold M, $n \neq 4$, then $BDiff_0(M_\alpha)$ has the same $\Lambda(\alpha, \beta)$ homotopy type as $BDiff_0(M_\beta)$ and $Diff_0(M_\alpha)$ has the same $\Lambda(\alpha, \beta)$ homotopy type as $Diff_0(M_\beta)$.*

The first assertion follows from the observation that $BDiff_0(M_\alpha)$ is the universal covering space of $BDiff'(M_\alpha)$; the second follows from the fact that $Diff_0(M_\alpha)$ has the same homotopy type as the loop space of $BDiff_0(M_\alpha)$.

Let $\Lambda(n)$ denote the subring of Q obtained by inverting the orders of the groups of homotopy spheres $\pi_q(Top/GL)$, $q \leqslant n$. Then

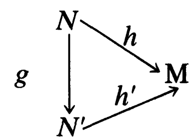
$$\Lambda(\alpha, \beta) \subset \Lambda(n)$$

for any smoothings α, β of the n-manifold M. Thus we have:

COROLLARY 2. *Let α and β be two smoothings of the n-manifold M, $n \neq 4$. Then $BDiff'(M_\alpha)$ and $BDiff'(M_\beta)$ have the same $\Lambda(n)$-homotopy type.*

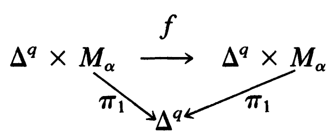
This result was announced in [3].

We now turn to the *PL* (piecewise linear) category. A *PL structure* on a topological n-manifold M is a pair (N, h) where N is a *PL* manifold and $h:N \to M$ is a homeomorphism. We say that two *PL* structures (N, h) and (N', h') on M are equivalent if there is a *PL* homeomorphism $g:N \to N'$ such that the diagram



commutes up to topological isotopy. We denote the set of all equivalence classes of *PL* structures on M by $\mathcal{P}(M)$ and write M_α for the manifold M with *PL* structure defined by $\alpha \in \mathcal{P}(M)$.

For any $\alpha \in \mathcal{P}(M)$, we let $\mathbf{PL}(M_\alpha)$ be the simplicial group whose q-simplices are *PL* homeomorphisms f: $\Delta^q \times M_\alpha \to \Delta^q \times M_\alpha$ such that the diagram



commutes. Here Δ^q is the standard q-simplex and $\pi_1:\Delta^q \times M_\alpha \to \Delta^q$ is projection onto the first factor. (The face and degeneracy mappings are defined in the obvious way.) Let $\mathbf{PL}_0(M_\alpha)$ denote the identity component of $\mathbf{PL}(M_\alpha)$ and $\mathbf{BPL}_0(M_\alpha)$ the simplicial classifying set for $\mathbf{PL}_0(M_\alpha)$.

THEOREM 1.2. *Let α and β be PL structures on the closed n-manifold M, $n \neq 4$. Then $\mathbf{BPL}_0(M_\alpha)$ and $\mathbf{BPL}_0(M_\beta)$ have the same $Z[\frac{1}{2}]$-homotopy type.*

The proof of this theorem is given in Section 3.

COROLLARY. *Let α and β be PL structures on the closed n-manifold M, $n \neq 4$. Then the simplicial groups $\mathbf{PL}_0(M_\alpha)$ and $\mathbf{PL}_0(M_\beta)$ have the same $Z[\frac{1}{2}]$-homotopy type.*

Remark. If M is a compact manifold with boundary, then Theorem 1.1 and its corollaries also hold for the group $Diff'(M_\alpha; \partial M_\alpha)$ of diffeomorphism leaving the boundary fixed. Similarly, Theorem 1.2 holds for the group $\mathbf{PL}(M_\alpha;\, \partial M_\alpha)$ of PL-homeomorphisms leaving the boundary fixed.

Our next result concerns the group $\mathit{Diff}_0^\delta(M_\alpha)$; this is the group $\mathit{Diff}_0(M_\alpha)$ topologized in the discrete topology.

THEOREM 1.3. *Let α and β be smoothings of the closed n-manifold M, $n \neq 4$. Then there is a space Z and maps*

$$B\mathit{Diff}_0^\delta(M_\alpha) \rightarrow Z \leftarrow B\mathit{Diff}_0^\delta(M_\beta)$$

which induce isomorphisms on homology with coefficients in $\Lambda(\alpha, \beta)$.

The proof of this result is similar to the proofs of Theorems 1.1 and 1.2. We sketch it at the end of Section 3.

In order to state our next result, we need the category of spaces over M. A *space over M* is a pair (A, a) where A is a space and $a: A \rightarrow M$ is a continuous mapping. For example, if $\alpha \in \mathcal{S}(M)$ and $P(M_\alpha)$ is the frame bundle of M_α, then $P(M_\alpha)$ is a space over M. A mapping $f:(A, a) \rightarrow (B, b)$ of spaces over M is simply a continuous mapping $f: A \rightarrow B$ such that the diagram

$$\begin{array}{ccc} A & \xrightarrow{\ a\ } & M \\ {\scriptstyle f}\big\downarrow & \nearrow_{b} & \\ B & & \end{array}$$

is commutative. If Λ is a subring of Q, then $f:(A, a) \rightarrow (B, b)$ is called a *Λ-equivalence over M* if the homotopy fibres $F(a)$ and $F(b)$ of a and b respectively are nilpotent spaces and the mapping $F(a) \rightarrow F(b)$ induced by f is a Λ-equivalence.

For any $\alpha \in \mathcal{S}(M)$ define $\sigma: \mathrm{Diff}_0(M_\alpha) \rightarrow P(M_\alpha)$ by

$$\sigma(g) = dg(f_0)$$

where $dg: TM_\alpha \rightarrow TM_\alpha$ as the differential of g and f_0 is a fixed frame in $P(M_\alpha)$.

THEOREM 1.4. *Let M be an n-manifold, $n \neq 4$. Pick $\alpha, \beta \in \mathcal{S}(M)$ and let*

$$\sigma: \mathit{Diff}_0(M_\alpha) \rightarrow P(M_\alpha) \quad \text{and} \quad \sigma': \mathit{Diff}_0(M_\beta) \rightarrow P(M_\beta)$$

be the mappings defined above. Then there are spaces X and Y and a commutative diagram of mappings

$$\begin{array}{ccccc} \mathit{Diff}_0\, M_\alpha & \longrightarrow & X & \longleftarrow & \mathit{Diff}_0\, M_\beta \\ \big\downarrow{\scriptstyle \sigma} & & \big\downarrow{\scriptstyle \gamma} & & \big\downarrow{\scriptstyle \sigma'} \\ P(M_\alpha) & \longrightarrow & Y & \longleftarrow & P(M_\beta) \end{array}$$

such that:

(i) *the mappings in the upper row are $\Lambda(\alpha, \beta)$-equivalences;*
(ii) *Y is a space over M in a natural way;*
(iii) *the mappings in the bottom row are $\Lambda(\alpha, \beta)$ equivalences of spaces over M.*

The proof of this result is given in Section 4.

As an application of Theorem 1.4, we suppose that $M_\alpha = S^n$ and $M_\beta = \Sigma^n$ where S^n is the standard sphere and Σ^n is a homotopy sphere. In this case, $\nu = \nu(\alpha, \beta)$ is simply the order of Σ^n in θ_n. Furthermore, the mapping

$$\sigma_* : \pi_q Diff_0(S^n) \to \pi_q P(S^n)$$

is a split epimorphism since $P(S^n)$ has the homotopy type of $SO(n + 1)$ and $SO(n + 1) \subset Diff_0(S^n)$. We therefore have the following.

COROLLARY. *Let Σ^n be a homotopy n-sphere, $n \neq 4$, and Λ the subring of Q obtained from Z by inverting the order of Σ^n in θ_n. Then*

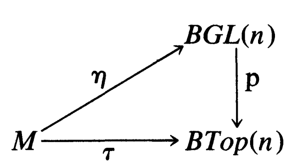
$$\pi_q(Diff_0(\Sigma^n)) \otimes \Lambda \to \pi_q(P(\Sigma^n)) \otimes \Lambda$$

is a split epimorphism.

This corollary is in contrast with results of Schultz [11].

2. The Proof of Theorem 1.1

The proof of Theorem 1.1 proceeds in three steps (see [3]). We first recall that the space $BDiff'(M_\alpha)$ is determined by the quotient $Top_0(M)/Diff'(M_\alpha)$ as a $Top_0(M)$-space, where $Top_0(M)$ is the identity component of the space of homeomorphisms of M. (We work largely in the simplicial category, but will supress the fact in this paragraph.) Next, we use an equivariant form of the result of Morlet [10] and Burghelea-Lashof [2] to conclude that $Top_0(M)/Diff'(M_\alpha)$ has the same $Top_0(M)$-homotopy type as the space $\mathscr{L}$ of lifts $\eta : M \to BGL(n)$ of the classifying map $\tau : M \to BTop(n)$ of the topological tangent bundle of M:



Finally, we form the fibrewise localization $p' : E' \to BTop(n)$ relative to $\Lambda(\alpha, \beta)$ of the fibration $BGL(n) \to BTop(n)$ and map $\mathscr{L}$ into the space $\mathscr{L}'$ of lifts $\eta' : M \to E'$ in this localized fibration. Our hypotheses insure that both

$$Top_0(M)/Diff'(M_\alpha) \quad \text{and} \quad Top_0(M)/Diff'(M_\beta)$$

map equivariantly by $\Lambda(\alpha, \beta)$-equivalences into the same component of $\mathscr{L}'$ and the result follows.

The rest of this section provides the details of the above sketch. To avoid technical difficulties we will work from now on either in the category of simplicial sets or (when necessary) in the category of compactly generated spaces.

Let $\mathbf{T}$ be the singular complex of the space of homeomorphisms of M; $\mathbf{T}$ is a simplicial set whose k-simplices correspond to commutative diagrams

$$\begin{array}{ccc} \Delta^k \times M & \xrightarrow{\ h\ } & \Delta^k \times M \\ & {\scriptstyle \pi_1}\searrow \quad \swarrow{\scriptstyle \pi_1} & \\ & \Delta^k & \end{array}$$

in which h is a homeomorphism. Similarly, let $\mathbf{D}_\alpha$ be the smooth singular complex of the space of diffeomorphisms of M_α. Composition gives $\mathbf{T}$ and $\mathbf{D}_\alpha$ the structure of simplicial groups; forgetting smoothness gives a simplicial subgroup inclusion $\mathbf{D}_\alpha \to \mathbf{T}$.

Let $\mathbf{E}$ be a contractible simplicial set on which $\mathbf{T}$ acts freely from the right. Let $\mathbf{T}$ act from the right on the product $\mathbf{E} \times (\mathbf{T}/\mathbf{D}_\alpha)$ by the rule

$$(x, y)g = (xg, g^{-1}y)$$

We will denote the quotient $(\mathbf{E} \times (\mathbf{T}/\mathbf{D}_\alpha))/\mathbf{T}$ by $\mathbf{E} \times_{\mathbf{T}} (\mathbf{T}/\mathbf{D}_\alpha)$.

PROPOSITION 2.1. *The simplicial sets* $\mathbf{E}/\mathbf{D}_\alpha$ *and* $\mathbf{E} \times_{\mathbf{T}} (\mathbf{T}/\mathbf{D}_\alpha)$ *are isomorphic.*

This is a straightforward calculation which we leave to the reader.

Note that the geometric realization $|\mathbf{E}/\mathbf{D}_\alpha|$ has the homotopy type of the classifying space $BDiff(M_\alpha)$. This gives the following.

COROLLARY. *The space* $BDiff(M_\alpha)$ *is determined up to homotopy by the simplicial set* $\mathbf{T}/\mathbf{D}_\alpha$ *and the action of the simplicial group* $\mathbf{T}$ *on this simplicial set.*

Let $\mathbf{T}_0$ be the singular complex of the identity component of $Top(M)$ and let $\mathbf{D}'_\alpha = \mathbf{D}_\alpha \cap \mathbf{T}_0$. Replacing $\mathbf{T}$ and $\mathbf{D}_\alpha$ by $\mathbf{T}_0$ and $\mathbf{D}'_\alpha$ respectively in the above statement gives the following.

PROPOSITION 2.2. *The simplicial sets* $\mathbf{E}/\mathbf{D}'_\alpha$ *and* $\mathbf{E} \times_{T_0} (\mathbf{T}_0/\mathbf{D}'_\alpha)$ *are isomorphic.*

COROLLARY. *The space* $BDiff'(M_\alpha)$ *is determined up to homotopy by the simplicial set* $\mathbf{T}_0/\mathbf{D}'_\alpha$ *and the action of the simplicial group* $\mathbf{T}_0$ *on this simplicial set.*

The geometric realization $|\mathbf{T}|$ is a topological group which acts on M (from the left) and on $|\mathbf{E}|$ (from the right). The natural projection

$$|\mathbf{E}| \times_{|\mathbf{T}|} M \to |\mathbf{E}/\mathbf{T}|$$

is a topological fibre bundle with fiber M. Let

$$\xi : |\mathbf{E}| \times_{|\mathbf{T}|} M \to BTop(n)$$

be a classifying map for the corresponding topological R^n-microbundle of tangents along the fibre. This is the bundle $T_F\mathbf{E}$ associated to $\mathbf{E}$ with fibre $M \times M$ (a homeomorphism of M induces a homeomorphism of $M \times M$ via the diagonal action); the projection of $M \times M$ onto the first factor makes $T_F\mathbf{E}$ into a microbundle over $\mathbf{E}$. The composite

$$\varphi : |\mathbf{E}| \times M \to |\mathbf{E}| \times_{|\mathbf{T}|} M \to BTop(n)$$

is then equivariant with respect to the right action of $|\mathbf{T}|$ on $|\mathbf{E}| \times M$ (and the trivial action on $Btop(n)$) and classifies a topological bundle on $|\mathbf{E}| \times M$ that can be identified in a natural way with the topological tangent bundle of M.

On the other hand, the geometric realization $|\mathbf{D}_\alpha|$ is a topological group which acts on M_α (from the left, by diffeomorphisms) and on $|\mathbf{E}|$ (from the right). The natural projection

$$|\mathbf{E}| \times_{|\mathbf{D}_\alpha|} M \to |\mathbf{E}/\mathbf{D}_\alpha|$$

is a smooth fibre bundle with M_α as the fibre. Let

$$\xi_\alpha : |\mathbf{E}| \times_{|\mathbf{D}_\alpha|} M_\alpha \to BGL(n)$$

classify the corresponding linear bundle of tangents along the fibre, and let

$$\tau_\alpha : |\mathbf{E}| \times M_\alpha \to BGL(n)$$

denote the composite

$$|\mathbf{E}| \times M_\alpha \to |\mathbf{E}| \times_{|\mathbf{D}_\alpha|} M_\alpha \xrightarrow{\xi_\alpha} BGL(n).$$

Then τ_α is equivariant with respect to the right action of $|\mathbf{D}_\alpha|$ on $|\mathbf{E}| \times M_\alpha$ (and the trivial action on $BGL(n)$) and classifies a bundle on $|\mathbf{E}| \times M$ that can be identified in a natural way with the linear tangent bundle of M_α.

Replace the natural map $BGL(n) \to BTop(n)$ with an equivalent Serre fibration. To avoid introducing more notation, we will denote this fibration by $p : BGL(n) \to BTop(n)$. It is clearly possible to choose ξ and ξ_α so that the diagram

$$\begin{array}{ccccc}
\tau_\alpha : |\mathbf{E}| \times M_\alpha & \longrightarrow & |\mathbf{E}| \times_{|D_\alpha|} M_\alpha & \xrightarrow{\xi_\alpha} & BGL(n) \\
\mathrm{id} \downarrow & & \downarrow & & \downarrow p \\
\tau : |\mathbf{E}| \times M & \longrightarrow & |\mathbf{E}| \times_{|T|} M_\alpha & \xrightarrow{\xi} & BTop(n)
\end{array}$$

commutes.

Let $\mathscr{L}$ be the space of maps

$$l : |\mathbf{E}| \times M \to BGL(n)$$

such that the diagram

$$\begin{array}{ccc} & & BGL(n) \\ & \overset{l}{\nearrow} & \downarrow p \\ |\mathbf{E}| \times M & \xrightarrow[\tau]{} & BTop(n) \end{array}$$

commutes. Recall that $|\mathbf{T}|$ acts on the right on $|\mathbf{E}| \times M$ and that, for any x in $|\mathbf{E}| \times M$, g in $|\mathbf{T}|$,

$$\tau(xg) = \tau(x).$$

It follows that composition gives a left action on $|\mathbf{T}|$ on $\mathscr{L}$. Since τ_α is a distinguished point of $\mathscr{L}$ which is fixed by the action of the subgroup $|\mathbf{D}_\alpha|$ of $|\mathbf{T}|$, the action of $|\mathbf{T}|$ on the orbit of τ_α induces a map $|\mathbf{T}/\mathbf{D}_\alpha| \to \mathscr{L}$. The adjoint of this is a $\mathbf{T}$-equivariant simplicial map $\mathbf{T}/\mathbf{D}_\alpha \to \mathbf{S}(\mathscr{L})$. (Here $\mathbf{S}(\mathscr{L})$ is the singular complex of $\mathscr{L}$.)

PROPOSITION 2.3. *The* $\mathbf{T}$*-equivariant simplicial map* $\mathbf{T}/\mathbf{D}_\alpha \to \mathbf{S}(\mathscr{L})$ *defined above induces a homotopy equivalence between* $\mathbf{T}_0/\mathbf{D}'_\alpha$ *and the component of* $\mathbf{S}(\mathscr{L})$ containing τ_α.

Proof. This follows directly from [2]. Let $\underline{\mathbf{R}}$ be the simplicial group of germs of topological microbundle automorphisms of the tangent bundle of M which cover the identity map of M and $\overline{\mathbf{R}}$ the simplicial group of such automorphisms that cover arbitrary homeomorphisms of M. The simplicial groups $\overline{\mathbf{R}}_\alpha$ and $\underline{\mathbf{R}}_\alpha$ are defined similarly, but the bundle automorphisms in question are required to be linear automorphisms of the smooth tangent bundle of M_α. (In the notation of [2], page 11, these would be written

$$\underline{\mathbf{R}}^t(M, M),\ \overline{\mathbf{R}}^t(M, M),\ \ \underline{\mathbf{R}}^d(M_\alpha, M_\alpha), \text{ and } \overline{\mathbf{R}}^d(M_\alpha, M_\alpha)$$

respectively.)

In [2], Burghelea and Lashof consider the diagram of simplicial groups

$$\begin{array}{ccccc} \mathbf{D}_\alpha & \longrightarrow & \overline{\mathbf{R}}_\alpha & \longleftarrow & \underline{\mathbf{R}}_\alpha \\ \downarrow & & \downarrow & & \downarrow \\ \mathbf{T} & \longrightarrow & \overline{\mathbf{R}} & \longleftarrow & \underline{\mathbf{R}} \end{array}$$

in which the vertical maps are subgroup inclusions, and prove that the induced coset diagram

$$\mathbf{T}/\mathbf{D}_\alpha \longrightarrow \overline{\mathbf{R}}/\overline{\mathbf{R}}_\alpha \longleftarrow \underline{\mathbf{R}}/\underline{\mathbf{R}}_\alpha$$

is a diagram of IHE mappings. (See Proposition 1.4 and 4.3 of [2]; an IHE mapping is injective on the set of components and a homotopy equivalence on each component.)

Since $\mathbf{T}$ is a simplicial subgroup of $\overline{\mathbf{R}}$, we can assume without loss of generality that the simplicial set $\mathbf{E}$ chosen above is a contractible simplicial set on which $\overline{\mathbf{R}}$ acts freely. From the definitions it is clear that the topological groups $|\overline{\mathbf{R}}|$ and $|\overline{\mathbf{R}}_\alpha|$ act on M and M_α respectively and that there is a natural topological microbundle over $|\mathbf{E}| \times_{|\overline{\mathbf{R}}|} M$ which, when pulled back to $|\mathbf{E}| \times_{|\overline{\mathbf{R}}_\alpha|} M_\alpha$, has a distinguished linear structure. Classifying these bundles gives a commutative diagram

$$\begin{array}{ccc} |\mathbf{E}| \times_{|\overline{\mathbf{R}}_\alpha|} M_\alpha & \longrightarrow & BGL(n) \\ \downarrow & & \downarrow \\ |\mathbf{E}| \times_{|\overline{\mathbf{R}}|} M & \longrightarrow & BTop(n) \end{array}$$

and subsequently, by the process described above, a map

$$\overline{\mathbf{R}}/\overline{\mathbf{R}}_\alpha \to \mathbf{S}(\mathscr{L}).$$

Proceeding in the same way with $\overline{\mathbf{R}}$ and $\overline{\mathbf{R}}_\alpha$ replaced by $\underline{\mathbf{R}}$ and $\underline{\mathbf{R}}^\alpha$ gives a map

$$\underline{\mathbf{R}}/\underline{\mathbf{R}}_\alpha \to \mathbf{S}(\mathscr{L}).$$

It is straightforward to fit these maps into a commutative diagram

$$\begin{array}{ccccc} \mathbf{T}/\mathbf{D}_\alpha & \to & \overline{\mathbf{R}}/\overline{\mathbf{R}}_\alpha & \leftarrow & \underline{\mathbf{R}}/\underline{\mathbf{R}}_\alpha \\ & \searrow & \downarrow & \swarrow & \\ & & \mathbf{S}(\mathscr{L}) & & \end{array}$$

The map $\underline{\mathbf{R}}/\underline{\mathbf{R}}_\alpha \to \mathbf{S}(\mathscr{L})$ is a homotopy equivalence by the argument given on the bottom of page 29 of [2] and Proposition 2.3 follows.

We now need the notion of the fibrewise localization of a fibration. (The book of Bousfield-Kan [1] should serve as a general reference here.) Let $p: E \to B$ be a fibration with the nilpotent space X as fibre and let Λ be any subring of rationals. We can then form the fibration $p': \tilde{\Lambda}_\infty E \to B$, the fibrewise localization of $p: E \to B$ relative to Λ. (See [1], page 40 for example.) This fibration has fibre $\Lambda_\infty X$ where

$$\pi_q(X, x_0) \otimes \Lambda \simeq \pi_q(\Lambda_\infty X, x_0') \qquad (2.1)$$

for all $q > 0$. In fact, there is a natural fibre preserving mapping $\varphi: E \to \tilde{\Lambda}_\infty E$ which induces the isomorphism (2.1) as fibres.

Let Y be any space and $g: Y \to B$ a continuous mapping. We then have the diagram

$$\begin{array}{ccccc} & & E & \xrightarrow{\varphi} & \tilde{\Lambda}_\infty E \\ & & \downarrow p & & \downarrow p' \\ Y & \xrightarrow[g]{} & B & \xrightarrow[\mathrm{id}]{} & B \end{array} .$$

Let $\mathscr{L} = \mathscr{L}(p; g)$ be the space of lifts of g over p and $\mathscr{L}' = \mathscr{L}(p', g)$ the space of lifts of g over p'. Then $\mathscr{L}$ and $\mathscr{L}'$ are nilpotent spaces and we have the following:

PROPOSITION 2.4. *For every map $f \in \mathcal{L}$, φ induces isomorphisms*

$$\Lambda \otimes \pi_i(\mathcal{L}, f) \to \pi_i(\mathcal{L}', \varphi f)$$

This result is an analogue of Proposition 5.1 of [1], p. 141. The proof of Proposition 2.4 follows the same lines as the proof of this proposition and is left to the interested reader.

Now let α and β be smoothings of M and let $\Lambda = \Lambda(\alpha, \beta)$ be as in Theorem 1.1. Let $\tau_\alpha, \tau_\beta : ETop_0(M) \times M \to BGL(n)$ be lifts of

$$\tau : ETop_0(M) \times M \to BTop(n)$$

defined by α, β respectively as in the definition of $\psi : \mathbf{T}/\mathbf{D}_\alpha \to \mathbf{L}$ given earlier. We then have the following diagram:

$$\begin{array}{ccccc}
 & \overset{\tau_\alpha,\, \tau_\beta}{\nearrow} & \mathrm{BGL}(n) & \xrightarrow{\varphi} & \bar{\Lambda}_\infty BGL(n) \\
 & & \downarrow p & & \downarrow p' \\
ETop_0(M) \times M & \xrightarrow[\tau]{} & \mathrm{BTop(n)} & \xrightarrow[\mathrm{id}]{} & BTop(n)
\end{array}$$

PROPOSITION 2.5. *The lifts $\varphi \circ \tau_\alpha$ and $\varphi \circ \tau_\beta$ of τ over p' are homotopic (as lifts).*

Before giving the proof of this proposition, we complete the proof of Theorem 1.1.

Let $\mathbf{L}'$ be the singular complex of the space $\mathcal{L}'$ of lifts of τ over p'. The simplicial group $\mathbf{T}$ acts on $\mathbf{L}'$ (in the same way that it acts on $\mathbf{L}$) and the mapping $\varphi_\# : \mathbf{L} \to \mathbf{L}'$ induced by φ is $\mathbf{T}$-equivariant. Let $\mathbf{L}_\alpha$ and $\mathbf{L}_\beta$ be the components of $\mathbf{L}$ containing τ_α and τ_β respectively and $\psi : \mathbf{T}/\mathbf{D}'_\alpha \to \mathbf{L}_\alpha$, $\psi' : \mathbf{T}/\mathbf{D}'_\beta \to \mathbf{L}_\beta$ the homotopy equivalences which exist by Proposition 2.4.

Consider now the composites

$$\theta : \mathbf{T}/\mathbf{D}'_\alpha \xrightarrow{\psi} \mathbf{L}_\alpha \xrightarrow{\varphi_\#} \mathbf{L}',$$

$$\theta' : \mathbf{T}/\mathbf{D}'_\beta \xrightarrow{\psi'} \mathbf{L}_\beta \xrightarrow{\varphi_\#} \mathbf{L}'$$

These mappings are $\mathbf{T}$-equivariant and both map into the same component $\mathbf{L}'_0$ of $\mathbf{L}'$ since $\varphi \circ \tau_\alpha$ and $\varphi \circ \tau_\beta$ are homotopic lifts (by Proposition 2.5). Thus, according to Proposition 2.4,

$$\theta_\# : \pi_q(\mathbf{T}/\mathbf{D}'_\alpha) \otimes \Lambda \to \pi_q(\mathbf{L}_0), \quad \text{and} \quad \theta'_\# : \pi_q(\mathbf{T}/\mathbf{D}'_\beta) \otimes \Lambda \to \pi_q(\mathbf{L}_0)$$

are isomorphisms for $q \geqslant 0$.

Finally, let $SBDiff'(M_\alpha)$ and $SBDiff'(M_\beta)$ be the singular complexes of $BDiff'(M_\alpha)$ and $BDiff'(M_\beta)$. Then

$$SBDiff'(M_\alpha) = \mathbf{S}(ETop_0(M)/Diff'(M_\alpha)) \simeq \mathbf{ET}/\mathbf{D}'_\alpha,$$

$$SBDiff'(M_\beta) = \mathbf{S}(ETop_0(M)/Diff'(M_\beta)) \simeq \mathbf{ET}/\mathbf{D}'_\beta$$

since the action of $Diff'(M_\alpha)$ and $Diff'(M_\beta)$ on $ETop_0(M)$ is a principal action. Using Proposition 2.2, the 5-lemma, and the results above, we see that the composites

$$\mathbf{ET}/\mathbf{D}'_\alpha \simeq \mathbf{ET} \times_{\mathbf{T}} (\mathbf{T}/\mathbf{D}_\alpha) \to \mathbf{ET} \times_{\mathbf{T}} \mathbf{L}_\alpha \to \mathbf{ET} \times_{\mathbf{T}} \mathbf{L}'_0,$$

$$\mathbf{ET}/\mathbf{D}'_\beta \simeq \mathbf{ET} \times_{\mathbf{T}} (\mathbf{T}/\mathbf{D}_\beta) \to \mathbf{ET} \times_{\mathbf{T}} \mathbf{L}_\beta \to \mathbf{ET} \times_{\mathbf{T}} \mathbf{L}'_0$$

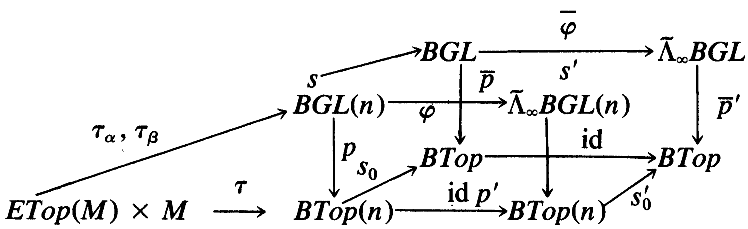
induce isomorphisms

$$\pi_q(\mathbf{ET}/\mathbf{D}'_\alpha) \otimes \Lambda \simeq \pi_q(\mathbf{ET} \times_{\mathbf{T}} \mathbf{L}'_0),$$

$$\pi_q(\mathbf{ET}/\mathbf{D}'_\beta) \otimes \Lambda \simeq \pi_q(\mathbf{ET} \times_{\mathbf{T}} \mathbf{L}'_0)$$

for $q \geqslant 0$. This completes the proof of Theorem 1.1.

We now prove Proposition 2.5. Consider the commutative diagram



Note first of all that $\bar{\Lambda}_\infty BGL \to \bar{\Lambda}_\infty BTop$ is a fibration of infinite loop spaces with fibre $\Lambda_\infty(Top/GL)$. It follows that

$$(\bar{\varphi} s \tau_\alpha(x)) = \bar{\Delta}(x)(\bar{\varphi} s \tau_\beta(x))$$

where $\bar{\Delta}: M \to \Lambda_\infty(Top/GL)$ is the composite of Δ (defined in Section 1) and the restriction of $\bar{\varphi}$ to Top/GL. However, $\bar{\varphi}|(Top/GL)$ defines an isomorphism

$$\bar{\varphi}_{\#}: [M, Top/GL] \otimes \Lambda \to [M, \Lambda_\infty(Top/GL)].$$

It follows that $\bar{\Delta} \simeq 0$ so $\bar{\varphi} s \tau_\alpha$ and $\bar{\varphi} s \tau_\beta$ are fibrewise homotopic. Since the diagram above is commutative, we see that $s'\varphi\tau_\alpha$ and $s'\varphi\tau_\beta$ are fibrewise homotopic. Now the fibre of $\bar{p}'$ is $\Lambda_\infty(Top/GL)$ and the fibre of p' is $\Lambda_\infty(Top(n)/GL(n))$. Furthermore,

$$\pi_q(Top/GL,\ Top(n)/GL(n)) = 0$$

for $q \leqslant n + 2$, $n \geqslant 5$. (See [5], Essay 4.) Thus

$$\pi_q(\Lambda_\infty(Top/GL),\ \Lambda_\infty(Top(n),\ GL(n))) = 0$$

for $q \leqslant n + 2$, $n \geqslant 5$ and it follows that $\varphi\tau_\alpha$ and $\varphi\tau_\beta$ are fibrewise homotopic. This proves Proposition 2.5.

3. The proof of Theorems 1.2 and 1.3

The proof of Theorem 1.2 follows the same lines as that of Theorem 1.1. We give an outline of it here providing details only when the difference between the two proofs is significant. This occurs in the material preceeding the statement of Lemma 3.2.

We begin by stating an analogue of Proposition 2.1. Note that we can consider $\mathbf{PL}_\alpha = \mathbf{PL}(M_\alpha)$ as a subgroup of $\mathbf{T}$ (the singular complex of $Top(M)$) so $\mathbf{PL}_\alpha$ acts on $\mathbf{T}/\mathbf{PL}_\alpha$ and on $\mathbf{ET}$.

PROPOSITION 3.1. *The simplicial sets* $\mathbf{ET}/\mathbf{PL}_\alpha$ *and* $\mathbf{ET} \times_{\mathbf{T}}(\mathbf{T}/\mathbf{PL}_\alpha)$ *are isomorphic.*

Note that, as a consequence of Lemma 3.1, the simplicial set $\mathbf{ET}/\mathbf{PL}_\alpha$ is determined by $\mathbf{T}/\mathbf{PL}_\alpha$ together with the action of $\mathbf{T}$ on $\mathbf{T}/\mathbf{PL}_\alpha$.

Let $\mathbf{PL}_n$ be the simplicial group whose q-simplices are PL-isomorphism germs $\Sigma^n(\Delta^q) \to \Sigma^n(\Delta^q)$ where $\Sigma^n(\Delta^q)$ is the trivial R^n microbundle over Δ^q. Thus, a q-simplex of $\mathbf{PL}_n$ is a germ of a PL-homeomorphism $h:\Delta^q \times R^n \to \Delta^q \times R^n$ with $h(t, 0) = (t, 0)$ for all $t \in \Delta^q$ and $\pi_1 h = h$ where $\pi_1:\Delta^q \times R^n \to \Delta^q$ is projection onto the first factor.

Let τ_α denote the tangent PL-microbundle of M_α and $\mathbf{S}(M)$ the singular complex of M. Following Milnor [9], Section 5, we associate to τ_α a simplicial principal $\mathbf{PL}_n$ bundle $\mathbf{E}_\alpha$ over $\mathbf{S}(M)$ as follows. A q-simplex of $\mathbf{E}_\alpha$ is a pair (f, F) where $f \in \mathbf{S}(M)$ and $F:\Sigma^n(\Delta^q) \to f^*\tau_\alpha$ is a PL-homomorphism germ; the map $\pi:E_\alpha \to \mathbf{S}(M)$ is given by $\pi(f, F) = f$. Equivalently, F is the germ at $\Delta^q \times 0$ of a PL-embedding $F':\Delta^q \times R^n \to M \times M$ taking $\Delta^q \times 0$ into the diagonal such that

$$\begin{array}{ccc} \Delta^q \times R^n & \xrightarrow{F'} & M \times M \\ \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \pi_1} \\ \Delta^q & \xrightarrow{f} & M \end{array}$$

commutes. The group $\mathbf{PL}_n$ acts on $\mathbf{E}_\alpha$ by

$$(f, F)h = (f, F \circ h^{-1}).$$

In addition, the group $\mathbf{PL}_\alpha = \mathbf{PL}(M_\alpha)$ acts on $\mathbf{S}(M)$ by

$$(gf)(t) = g^{-1}(t, f(t))$$

and on $\mathbf{E}_\alpha$ by

$$g(f, F) = (gf, gF)$$

where gF is the germ at $\Delta^q \times 0$ of the mapping $gF':\Delta^q \times R^n \to M \times M$;

$$(gF')(t, v) = (g^{-1}(t, f(t)), g^{-1}(t, F_2(t, v)))$$

where $F'(t, v) = (f(t), F_2(t, v))$. It is immediate that π is PL-equivariant and that the action of $\mathbf{PL}_\alpha$ on $\mathbf{E}_\alpha$ commutes with the action of $\mathbf{PL}_n$.

Let $\mathbf{T}_n$ be the simplicial group of germs of homeomorphisms $(R^n, 0) \to (R^n, 0)$. Then just as above, we can construct a simplicial principal bundle $\pi:\mathbf{E} \to \mathbf{S}(M)$ from the topological tangent bundle of M with group $\mathbf{T}_n$.

Again as above, the simplicial group **T** acts on both **E** and **S**(M) and π is **T**-equivariant. Finally, there is a commutative diagram

$$\begin{array}{ccc} \mathbf{E}_\alpha & \xrightarrow{G} & \mathbf{E} \\ & {\scriptstyle\pi}\searrow \quad \swarrow{\scriptstyle\pi} & \\ & \mathbf{S}(M) & \end{array}$$

where G is $\mathbf{PL}_\alpha$ equivariant.

Let $\mathbf{BPL}_n$ be a classifying simplicial set for $\mathbf{PL}_n$, $\mathbf{BT}_n$ a classifying simplicial set for $\mathbf{T}_n$ and $p:\mathbf{BPL}_n \to \mathbf{BT}_n$ the simplicial fibre bundle with fibre $\mathbf{T}_n/\mathbf{PL}_n$. We can then choose a classifying map

$$\tau' : \mathbf{ET} \times_{\mathbf{T}} \mathbf{S}(M) \to \mathbf{BT}_n$$

for the simplicial $\mathbf{T}_n$-bundle $\mathbf{ET} \times_{\mathbf{T}} \mathbf{E} \to \mathbf{ET} \times_{\mathbf{T}} \mathbf{S}(M)$ and a classifying map

$$\eta' : \mathbf{ET} \times_{\mathbf{PL}_\alpha} \mathbf{S}(M) \to \mathbf{BPL}_n$$

for the simplicial $\mathbf{PL}_n$ bundle $\mathbf{ET} \times_{\mathbf{PL}_\alpha} \mathbf{E}_\alpha \to \mathbf{ET} \times_{\mathbf{PL}_\alpha} \mathbf{S}(M)$ such that

(3.1)
$$\begin{array}{ccc} \mathbf{ET} \times_{\mathbf{PL}_\alpha} \mathbf{S}(M) & \xrightarrow{\eta'} & \mathbf{BPL}_n \\ \downarrow & & \downarrow p \\ \mathbf{ET} \times_{\mathbf{T}} \mathbf{S}(M) & \xrightarrow{\tau'} & \mathbf{BT}_n \end{array}$$

commutes. Define $\eta:\mathbf{ET} \times \mathbf{S}(M) \to \mathbf{BPL}_n$ to be the composite

$$\mathbf{ET} \times \mathbf{S}(M) \to \mathbf{ET} \times_{\mathbf{PL}_\alpha} \mathbf{S}(M) \to \mathbf{BPL}_n$$

and $\tau:\mathbf{ET} \times \mathbf{S}(M) \to \mathbf{BT}_n$ to be the composite

$$\mathbf{ET} \times \mathbf{S}(M) \to \mathbf{ET} \times_{\mathbf{T}} \mathbf{S}(M) \to \mathbf{BT}_n.$$

Then η classifies "*PL*-tangents along fibres" in the trivial bundle $\mathbf{ET} \times \mathbf{S}(M) \to \mathbf{ET}$ and τ classifies "topological tangents along fibres" in this same trivial bundle. Furthermore, η is a lift of τ (from diagram (3.1)),

(3.2) $$\eta((x,f)g) = \eta(x,f)$$

for $g \in \mathbf{PL}_\alpha$, and

(3.3) $$\tau((x,f)g) = \tau(x,f)$$

for $g \in \mathbf{T}$.

Now let **L** be the simplicial set of lifts of τ into $\mathbf{BPL}_n$. Then **T** acts on **L** by

$$(g\nu)(x,f) = \nu((x,f)g)$$

Note that $g\nu \in \mathbf{L}$ by (3.3). Define $\psi:\mathbf{T}/\mathbf{PL}_\alpha \to \mathbf{L}$ by $\psi([g]) = g\eta$. This is well defined by (3.2) and clearly **T** equivariant.

PROPOSITION 3.2. *The mapping* ψ *is a homotopy equivalence of* $\mathbf{T}/\mathbf{PL}_\alpha$ *onto the component of* $\mathbf{L}$ *containing* η.

The proof of this proposition follows from the results of [2] in essentially the same way that Proposition 2.3 did. We leave the details to the reader.

Let $\Lambda = Z[\frac{1}{2}]$ and $p':\tilde{\Lambda}_\infty\mathbf{BPL}_n \to \mathbf{BT}_n$ the fibrewise localization of the fibration $p:\mathbf{BPL}_n \to \mathbf{BT}_n$. Thus we have a commutative diagram

$$\begin{array}{ccc} \mathbf{BPL}_n & \xrightarrow{\varphi} & \tilde{\Lambda}_\infty\mathbf{BPL}_n \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p'} \\ \mathbf{BT}_n & \xrightarrow{\text{id}} & \mathbf{BT}_n \end{array}.$$

Now, for $j \leqslant n$, $n \geqslant 5$,

$$\begin{aligned} \pi_j(\mathbf{T}_n/\mathbf{PL}_n) &= 0 \quad \text{for } j \neq 3 \\ &\simeq Z_2 \quad \text{for } j = 3 \end{aligned}$$

(see [5], Essay 4) so the fibre of p' is n-connected. It follows that the simplicial set $\mathbf{L}'$ of lifts of τ into $\tilde{\Lambda}_\infty\mathbf{BPL}_n$ is connected and Theorem 1.2 now follows from Proposition 2.4.

We now give a rough sketch of the proof of Theorem 1.3, ignoring technical details.

Let F_α be the homotopy fibre of the mapping

$$BDiff_0^\delta(M_\alpha) \to BDiff_0(M_\alpha)$$

Then F_α can be thought of as $Diff_0(M_\alpha)/Diff_0^\delta(M_\alpha)$ so that $BDiff_0^\delta(M_\alpha)$ is determined up to homotopy by the space F_α together with the action of $Diff_0(M_\alpha)$ on this space (just as in Proposition 2.1).

Now let $B\Gamma_n$ be the classifying space for "Haefliger Structures" [4] and let $\mathscr{L}_\alpha$ be the space of lifts $l:E \times M \to B\Gamma_n$ of $\tau_\alpha:E \times M \to BGL(n)$:

$$\begin{array}{ccc} & & B\Gamma_n \\ & {\scriptstyle l}\nearrow & \downarrow{\scriptstyle \pi} \\ E \times M & \xrightarrow{\tau_\alpha} & BGL(n) \end{array}$$

(We write E for $ETop_0(M)$ here.) It follows from the results of [12] (see also [8]) that there is an equivariant mapping $\psi:F_\alpha \to \mathscr{L}_\alpha$ inducing an isomorphism on integral homology. (Note that $\mathscr{L}_\alpha$ is connected since the fibre of $\pi:B\Gamma_n \to BGL_n$ is $(n + 1)$-connected; see [4].) Let $\mathscr{L}_1$ be the space of lifts $l_1:E \times M \to BGL(n)$ of the classifying map $\tau:E \times M \to BTop(n)$ for the bundle of "topological tangents along fibres" and $\mathscr{L}_2$ the space of lifts $l_2:E \times M \to B\Gamma_n$ of $\tau:E \times M \to BTop(n)$;

$$\begin{array}{ccc} & & B\Gamma_n \\ & \overset{l_2}{\nearrow} & \downarrow \\ & \overset{l_1}{\nearrow} & BGL(n) \\ & & \downarrow \\ E \times M & \xrightarrow{\ \tau\ } & BTop(n) \end{array}.$$

We then have a fibration $\mathscr{L}_2 \to \mathscr{L}_1$ with fibre $\mathscr{L}_\alpha$. Theorem 1.3 now follows from the techniques of Section 2 and the result of [12] referred to above.

4. The proof of Theorem 1.4

Let $\mathscr{L}$ be the space of lifts of $\tau: ETop_0(M) \times M \to BTop(n)$ into $BGL(n)$. We begin by constructing a bundle over the space $ETop_0(M) \times_{Top_0(M)} \mathscr{L}$ with the property that the fibre over a point $[x, \nu]$ is the total space of the $GL(n)$-bundle defined by $\nu: ETop_0(M) \times M \to BGL(n)$.

Let W be the space

$$W = ETop_0(M) \times \mathscr{L} \times ETop_0(M) \times M$$

and define $\theta: W \to BGL(n)$ by

$$\theta(x, \nu, y, z) = \nu(y, z).$$

Let E_1 be the total space of the bundle over W induced by θ from the universal bundle $q: EGL(n) \to BGL(n)$:

$$E_1 = \{(w, v) \in W \times EGL(n): \theta(w) = q(v)\}.$$

If $\rho_1: E_1 \to ETop_0(M) \times \mathscr{L}$ is given by

$$\rho_1(x, \nu, y, z, v) = (x, \nu)$$

then $\rho_1: E_1 \to ETop_0(M) \times \mathscr{L}$ is a fibre bundle and the fibre over the point (x, ν) is the total space of $\nu^* EGL(n)$. Now $Top_0(M)$ acts on $ETop_0(M) \times \mathscr{L}$ (diagonally) and on E_1 by

$$(x, \nu, y, z, v)g = (xg, g^{-1}\nu, yg, g^{-1}z, v).$$

Since ρ_1 is clearly equivariant, we have a fibre bundle

$$\rho: E = E_1/Top_0(M) \to B = ETop_0(M) \times_{Top_0(M)} \mathscr{L}.$$

Suppose that α and β are smoothings of the manifold M, $\Lambda = \Lambda(\alpha, \beta)$ is the subring of the rational numbers defined in Section 1, and $\mathscr{L}'$ the space of lifts of τ into the total space $\tilde{\Lambda}_\infty BGL(n)$ of the fibrewise localization of the fibre bundle $BGL(n) \to BTop(n)$. (See Section 2.) Let W' be the space

$$W' = ETop_0(M) \times \mathscr{L}' \times ETop_0(M) \times M$$

and define $\theta': W' \to \tilde{\Lambda}_\infty BGL(n)$ by $\theta'(x, \nu', y, z) = \nu'(y, z)$. If E_1' is the total space of the bundle over W' induced by θ' from the path fibration $P' \to \tilde{\Lambda}_\infty BGL(n)$, then, just as above, we have a fibration

$$\rho' : E' = E_1'/Top_0(M) \to B' = ETop_0(M) \times_{Top_0(M)} \mathcal{L}'$$

Furthermore, we have a commutative diagram

$$\begin{array}{ccc} E & \longrightarrow & E' \\ \rho \downarrow & & \downarrow \rho' . \\ B & \longrightarrow & B' \end{array} \tag{4.1}$$

This follows from the fact that the natural mapping of $\mathcal{L}$ into $\mathcal{L}'$ is $Top_0(M)$ equivariant and the fact that the mapping $BGL(n) \to \tilde{\Lambda}_\infty BGL(n)$ can be covered by a fibre preserving map $EGL(n) \to P'$.

We now need to describe the smooth frame bundle of any smoothing of M in a particularly convenient way.

Let α be any smoothing of M and $\tau_\alpha : ETop_0(M) \times M \to BGL(n)$ a lift of τ defined by α satisfying

$$\tau_\alpha(yg, g^{-1}z) = \tau_\alpha(y, z) \tag{4.2}$$

for any $g \in Diff'(M_\alpha)$. (See Section 2.) If P_α is the total space of the bundle $\tau_\alpha{}^{\#} EGL(n)$,

$$P_\alpha = \{(y, z, v) \in ETop_0(M) \times M \times EGL(n) : \tau_\alpha(y, z) = qv\},$$

then $Diff'(M_\alpha)$ acts on P_α by $(y, z, v)g = (yg, g^{-1}z, v)$ (using (4.2)). Thus, we can form the bundle

$$E_\alpha = ETop_0(M) \times_{Diff'(M_\alpha)} P_\alpha \to BDiff'(M_\alpha) = ETop_0(M)/Diff'(M_\alpha).$$

LEMMA 4.1. *Let α be any smoothing of M, $\mathcal{L}_0'$ the component of $\mathcal{L}'$ containing the image of τ_α* under the mapping $\mathcal{L} \to \mathcal{L}'$, *and E_0' the part of the bundle $E' \to B'$ over $B_0' = ETop_0(M) \times_{Top_0(M)} \mathcal{L}_0'$. Then there are mappings*

$$\tilde{h} : E_\alpha \to E_0', \quad h : BDiff'(M_\alpha) \to B_0'$$

such that

$$\begin{array}{ccc} E_\alpha & \xrightarrow{\tilde{h}} & E_0' \\ \downarrow & & \downarrow \\ BDiff'(M_\alpha) & \xrightarrow{h} & B_0' \end{array} \tag{4.3}$$

is commutative.

Proof. Let $\mathcal{L}_\alpha$ be the component of $\mathcal{L}$ containing τ_α and E_0 the part of the bundle $E \to B$ over $B_0 = ETop_0(M) \times_{Top_0(M)} \mathcal{L}_\alpha$. We define mappings

$$\tilde{h}_1 : E_\alpha \to E_0, \quad h_1 : BDiff(M_\alpha) \to B_0$$

such that

$$\begin{array}{ccc} E_\alpha & \xrightarrow{\tilde{h}_1} & E_0 \\ \downarrow & & \downarrow \\ BDiff(M_\alpha) & \xrightarrow{h_1} & B_0 \end{array}$$

is commutative. Lemma 4.1 will then follow from diagram (4.1).

The mappings $\tilde{h}_1$ and h_1 are defined by

$$\tilde{h}_1([x, y, z, v]) = [x, \tau_\alpha, y, z, v], \quad h_1([x]) = [x, \tau_\alpha].$$

The diagram (4.3) clearly commutes; we need only show that these mappings are well defined.

Given $g \in Diff'(M_\alpha)$, then

$$\begin{aligned} \tilde{h}_1([xg, yg, g^{-1}z, v]) &= [xg, \tau_\alpha, yg, g^{-1}z, v] \\ &= [xg, g^{-1}\tau_\alpha, yg, g^{-1}z, v] \quad \text{since} \quad g^{-1}\tau_\alpha = \tau_\alpha \text{ by (4.2)} \\ &= [x, \tau_\alpha, y, z, v] \\ &= \tilde{h}_1[x, y, z, v]. \end{aligned}$$

The same reasoning shows h_1 is well defined and Lemma 4.1 is proved.

Remark. If $\tau_\beta : Etop_0(M) \times M \to BGL(n)$ is a lift of τ defined by β satisfying $g\tau_\beta = \tau_\beta$ for $g \in Diff'(M_\beta)$, then according to Lemma 2.4, the image of τ_β under the natural mapping $\mathcal{L} \to \mathcal{L}'$ is contained in $\mathcal{L}_0$. Thus just as above, we can construct mappings

$$\tilde{k} : E_\beta \to E_0', \quad k : BDiff'(M_\beta) \to B_0'$$

such that

$$\begin{array}{ccc} E_\beta & \xrightarrow{\tilde{k}} & B_0' \\ \downarrow & & \downarrow \\ BDiff(M_\beta) & \xrightarrow{k} & B_0' \end{array}$$

is commutative.

Consider now homotopy fibre F_α of the inclusion $P_\alpha \subset E_\alpha$ and let $j : F_\alpha \to P_\alpha$ be the inclusion. Up to homotopy, the inclusion $P_\alpha \subset E_\alpha$ is the fibration

$$ETop_0(M) \times E_\alpha \to ETop_0(M) \times_{Diff'(M_\alpha)} E_\alpha$$

whose fibre is $Diff'(M_\alpha)$. It follows that the mapping

$$\sigma : Diff'(M_\alpha) \to P_\alpha$$

is, up to homotopy, the mapping $j: F_\alpha \to P_\alpha$. Similarly for

$$\sigma': Diff'(M_\beta) \to P_\beta.$$

Now let X be the homotopy fibre of the inclusion $Y \subset E_0'$ of the fibre Y of the bundle $E_0' \to B_0'$. We then have from the discussion above a commutative diagram of mappings

$$\begin{array}{ccccc} Diff'(M_\alpha) & \longrightarrow & X & \longleftarrow & Diff'(M_\beta) \\ \sigma \downarrow & & \downarrow & & \downarrow \sigma' \\ P_\alpha & \longrightarrow & Y & \longleftarrow & P_\beta \end{array}$$

The fact that this diagram has the properties stated in Theorem 1.4 is now a straightforward consequence of the discussion above and is left to the reader.

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