ILLINOIS JOURNAL OF MATHEMATICS
Volume 28, Number 3, Fall 1984

GENERALIZATIONS OF RIESZ POTENTIALS AND L^p ESTIMATES FOR CERTAIN k-PLANE TRANSFORMS

BY

S. W. DRURY[1]

0. Introduction

In this article we consider certain generalizations of the complex Riesz potentials on $\mathbf{R}^n$. For $f \in C_c^\infty(\mathbf{R}^n)$ these are defined by

$$R_z f(x) = \alpha(z) \int |x - y|^{-n+z} f(y)\, d\lambda(y) \tag{1}$$

for $\mathscr{R}z > 0$ and by

$$(R_z f)\hat{}(u) = \alpha(n - z)|u|^{-z}\hat{f}(u) \tag{2}$$

for $\mathscr{R}z < n$ [7, Chapter 5]. Here we have denoted λ the Lebesgue measure on $\mathbf{R}^n$, $\hat{f}$ the Fourier transform of f and α the entire function

$$\alpha(z) = \frac{\pi^{z/2}}{\Gamma(\frac{1}{2}z)}$$

which has no zeros in $\mathscr{R}z > 0$. The definitions agree in $0 < \mathscr{R}z < n$.

The generalizations with which we are concerned are all motivated by the k-plane transform. For f a suitable function defined on $\mathbf{R}^n$ we define the k-plane transform $T_k f$ by

$$T_k f(\Pi) = \int f(x)\, d\lambda_\Pi(x)$$

where Π is an affine k-plane in $\mathbf{R}^n$ and λ_Π is the Lebesgue measure on Π. Thus $T_k f$ is a function on the manifold $M_{n,k}$ of affine k-planes in $\mathbf{R}^n$. In view of [1, Chapter 7, Section 2, Theorem 3] one may construct on $M_{n,k}$ a measure μ invariant under the action of Euclidean motions. Aside from renormalization, μ is unique with this property.

CONJECTURE. *Let*

$$1 \le q \le n + 1, \quad np^{-1} - (n - k)q^{-1} = k$$

Received April 2, 1982.

[1] The research for this article was sponsored by the Natural Sciences and Engineering Research Council of Canada.

(*so that* $1 \le p \le (n+1)(k+1)^{-1}$). *Then* T_k *is a bounded operator:*

$$T_k\colon L^p(\mathbf{R}^n, \lambda) \to L^q(M_{n,k}, \mu).$$

The conjecture is trivially true for $p = 1$, $q = 1$ and is known in the case of the Radon transform [2]. In fact in that article, Oberlin and Stein obtain considerably more delicate estimates. The conjecture is also true in the case $k = 1$ of the x-ray transform at least for $1 \le q < n+1$ [3]. In this article we establish the conjecture for $n \le 2k+1$. For other values of n and k only fragmentary results are known. (*Added in proof.* The conjecture has now been settled affirmatively by M. Christ.)

Our proof makes use of an analytic family of multilinear operators

$$A_z(f_0, \dots, f_n) = \gamma_n(z) \int \left\{ \prod_{k=0}^{n} f_k(x_k) \right\} \Delta^{-n+z}\, d\lambda(x_0), \dots, d\lambda(x_n). \tag{3}$$

Here $f_k \in C_c^\infty(\mathbf{R}^n)$,

$$\Delta = |\det(x_1 - x_0, x_2 - x_0, \dots, x_n - x_0)|$$

and $\gamma_n(z) = \prod_{k=0}^{n-1} \alpha(z-k)$ is an entire function with no zeros in $\mathscr{R}z > n-1$. The integral in (3) converges absolutely for $\mathscr{R}z > n-1$ and we make this definition only for these values of z. In case $n = 1$ we have

$$A_z(f_0, f_1) = \int (R_z f_0) f_1\, d\lambda$$

so that A_z is just a bilinear formulation of the Riesz potential.

It follows from some work of Gelbart [4] that A_z can be continued analytically to the whole complex plane. The connection of A_z with the k-plane transform is simply that

$$A_k(f_0, \dots, f_n) = c_{n,k} \int \left\{ \prod_{j=0}^{n} T_k f_j(\Pi) \right\} d\mu(\Pi) \tag{4}$$

for k an integer $0 \le k \le n$.

THEOREM 1. *Let* $\frac{1}{2}(n-1) \le \mathscr{R}z \le n$ *and* $(n+1)p^{-1} = 1 + \mathscr{R}z$ (*so that* $1 \le p \le 2$). *Then*

$$|A_z(f_0, \dots, f_n)| \le c_{n,z} \prod_{j=0}^{n} \|f_j\|_p.$$

The proof of the conjecture (in case $n \le 2k+1$) follows almost immediately from these facts. We give the details in Section 1.

In Section 2 we introduce generalizations of Riesz potentials on the Grassmann manifold $G_{2k,k}$ and on $M_{2k+1,k}$. We feel that these potentials designated Ω_z and Λ_z respectively are of independent interest. We rely on the work of Gelbart both for the definition of these potentials and for the estimates obtained.

Finally, in Section 3 we relate the potentials Ω_z and Λ_z to k-plane transforms and to A_z, giving a different proof of Theorem 1 in the case n odd.

1. The multilinear forms A_z

We first need to calculate a Jacobian determinant $J_{n,k}$.

LEMMA 1. *We have*

$$d\lambda_{\Pi}(x_0), \ldots, d\lambda_{\Pi}(x_k)\, d\mu)(\Pi) = J_{n,k}\, d\lambda(x_0), \ldots, d\lambda(x_k)$$

where $J_{n,k} = c_{n,k}\Delta^{-(n-k)}$, Δ *is the volume of the k-simplex with vertices* $x_0, \ldots, x_k$ *and* μ *is the invariant measure on* $M_{n,k}$.

Proof. It is clear that $J_{n,k}$ is a Euclidean invariant of the k-simplex with vertices $x_0, x_1, \ldots, x_k$. Unfortunately the action of Euclidean motions on k-simplices has too many orbits ($k > 1$). Hence we make a proof by induction on k. If $k = 0$ or 1 the lemma is obvious. Assume it holds for $k - 1$ and all n simultaneously. Let V be the volume of the $(k - 1)$ simplex with vertices $x_1, x_2, \ldots, x_k$. Let ν be the invariant measure on $M_{n,k-1}$, and for Π a k-plane let ν_{Π} denote the invariant measure on the hyperplanes of Π. Further let μ_{x_0} be the invariant probability measure on the manifold of k-plane passing through the point x_0. By the uniqueness of the invariant measure on the homogeneous space

$$\{(x_0, \Pi);\quad x_0 \in \mathbf{R}^n, \Pi \in M_{n,k}, x_0 \in \Pi\},$$

we have

$$d\mu_{x_0}(\Pi)\, d\lambda(x_0) = d\lambda_{\Pi}(x_0)\, d\mu(\pi) \tag{5}$$

for suitable normalizations of these measures. The orbits of

$$\{(x_0, \Theta);\quad x_0 \in \mathbf{R}^n, \Theta \in M_{n,k-1}\}$$

are parametrized by r, the perpendicular distance from x_0 to Θ. The action of dilations about the point x_0 yields

$$d\nu_{\Pi}(\Theta)\, d\mu_{x_0}(\Pi) = \mathrm{Cr}^{-(n-k)}\, d\nu(\Theta). \tag{6}$$

Finally our induction hypothesis yields both

$$c_{n,k-1}V^{-(n-k+1)}\, d\lambda(x_1), \ldots, d\lambda(x_k) = d\lambda_{\Theta}(x_1), \ldots, d\lambda_{\Theta}(x_k)\, d\nu(\Theta) \tag{7}$$

and, when applied to the hyperplanes of Π,

$$c_{k,k-1}V^{-1}\, d\lambda_{\Pi}(x_1), \ldots, d\lambda_{\Pi}(x_k) = d\lambda_{\Theta}(x_1), \ldots, d\lambda_{\Theta}(x_k)\, d\nu_{\Pi}(\Theta). \tag{8}$$

Now, using (8), (5), (6) and (7) in turn we have

$$\begin{aligned} d\lambda_\Pi(x_0), \dots, d\lambda_\Pi(x_k)\, d\mu(\Pi) &= cV\, d\lambda_\Pi(x_0)\, d\lambda_\Theta(x_1), \dots, d\lambda_\Theta(x_k)\, d\nu_\Pi(\Theta)\, d\mu(\Pi) \\ &= cV\, d\lambda_\Theta(x_1), \dots, d\lambda_\Theta(x_k)\, d\nu_\Pi(\Theta)\, d\mu_{x_0}(\Pi)\, d\lambda(x_0) \\ &= cV\, r^{-(n-k)}\, d\lambda_\Theta(x_1), \dots, d\lambda_\Theta(x_k)\, d\nu(\Theta)\, d\lambda(x_0) \\ &= cV^{-(n-k)} r^{-(n-k)}\, d\lambda(x_0)\, d\lambda(x_1), \dots, d\lambda(x_k). \end{aligned}$$

Since $\Delta = crV$ we have our result.

Next we shall need to review the work of Oberlin and Stein [2]. Let $G_{n,k}$ denote the Grassmann manifold of linear k-planes (i.e., k-planes passing through the origin). It is a compact manifold and possesses an invariant probability measure γ under the action of the orthogonal group. We may view $M_{n,k}$ as a bundle over $G_{n,k}$ in which each fibre consists of a family of mutually parallel k-planes. We follow Solmon [5] in denoting a generic element Π on $M_{n,k}$ by

$$\Pi = (\pi, x) = \pi + x,$$

the translate of $\pi \in G_{n,k}$ by $x \in \pi^\perp$. In this way the fibre over π is realized as the $(n-k)$-dimensional space $\pi^\perp$. We may take

$$d\mu(\pi, x) = d\lambda_{\pi^\perp}(x)\, d\gamma(\pi)$$

since the right hand side is invariant under Euclidean motions.

Oberlin and Stein are concerned with the case $k = n - 1$. Let us denote by S $(= T_{n-1})$ the Radon transform, and by S_z the Radon transform followed by the Riesz potential R_z on the 1-dimensional fibre. Thus

$$Sf(\pi, x) = \int f(x + y)\, d\lambda_\pi(y)$$

and

$$S_z f(\pi, x) = \alpha(z) \int |x - y|^{-1+z} Sf(\pi, y)\, d\lambda_{\pi^\perp}(y)$$

for $\mathscr{R}z > 0$, and

$$S_z f\hat{\ }(\pi, u) = \alpha(1 - z)|u|^{-z} Sf\hat{\ }(\pi, u) \quad (u \in \pi^\perp)$$

for $\mathscr{R}z < 1$ where $\hat{}$ denotes the Fourier transform along the fibre. Since

$$Sf\hat{\ }(\pi, u) = \hat{f}(u),$$

Oberlin and Stein find that for $\mathscr{R}z = -\frac{1}{2}(n-1)$,

$$\|S_z f\|_2 = C_{z,n}\|f\|_2. \tag{9}$$

From this and the trivial estimate

$$\|S_z f\|_\infty \le C_{z,n}\|f\|_1 \quad (\mathscr{R}z = 1)$$

they deduce

$$\|Sf\|_{n+1} \leq C\|f\|_{(n+1)/n}. \tag{10}$$

For $f_k \in C_c^\infty(\mathbf{R}^n)$ $(0 \leq k \leq n)$ let us define $F \in C_c^\infty(M(n, n))$ on the space $M(n, n)$ of $n \times n$ real matrices by

$$F(y_1, \dots, y_n) = \int f_0(x_0)f_1(x_0 + y_1), \dots, f_n(x_0 + y_n)\, d\lambda(x_0).$$

Then for $\mathscr{R}z > n - 1$ we have by (3)

$$A_z(f_0, \dots, f_n) = \gamma_n(z) \int F(Y)\,|\det Y|^{-n+z}\, dY \tag{11}$$

where dY denotes Lebesgue measure on $M(n, n)$.

According to the work of Gelbart [4, Section 4] the locally integrable density

$$\gamma_n(z)\,|\det Y|^{-n+z} \quad (\mathscr{R}z > n - 1)$$

can be continued analytically to the whole complex plane as a distribution Σ_z on $M(n, n)$. Thus we have:

LEMMA 2. *For $f_k \in C_c^\infty(\mathbf{R}^n)$ $(0 \leq k \leq n)$, $A_z(f_0, \dots, f_n)$ can be continued analytically to the whole complex plane. Furthermore for fixed z, A_z is a continuous multilinear form on $C_c^\infty(\mathbf{R}^n)$.*

Proof of Theorem 1. We proceed by induction on n. For $n = 1$ the result is well known [7, Chapter 5]. Assume that the result holds for $n - 1$. Let $f_k \in C_c^\infty(\mathbf{R}^n)$ $(0 \leq k \leq n)$ and assume for the moment that $\mathscr{R}z > n - 1$. Then

$$A_z(f_0, \dots, f_n) = \gamma_n(z) \int \left\{\prod_{k=0}^{n} f_k(x_k)\right\} \Delta^{-n+z}\, d\lambda(x_0), \dots, d\lambda(x_n).$$

Let Π be the hyperplane passing through $x_1, x_2, \dots, x_n$. Then, according to Lemma 1,

$$A_z(f_0, \dots, f_n) = \gamma_n(z) \int \left\{\prod_{k=0}^{n} f_k(x_k)\right\} \Delta^{-n+z}\Delta'\, d\lambda(x_0)\, d\lambda_\Pi(x_1), \dots, d\lambda_\Pi(x_n)\, d\mu(\Pi)$$

where Δ' is the volume of the simplex with vertices $x_1, \dots, x_n$. Now $\Delta = C_n$ $d(x_0, \Pi)\Delta'$ where $d(x_0, \Pi)$ is the perpendicular distance from x_0 to Π so that

$$A_z(f_0, \dots, f_n) = c_n \int g_z(\Pi)h_z(\Pi)\, d\mu(\Pi) \tag{12}$$

where

$$h_z(\Pi) = A_z(f_1|_\Pi, f_2|_\Pi, \dots, f_n|_\Pi)$$

and

$$g_z(\Pi) = \alpha(z - n + 1) \int f_0(x_0)\, d(x_0, \Pi)^{-n+z}\, d\lambda(x_0).$$

An easy calculation shows that $g_z = S_{z-n+1} f_0$.

In equation (12), A_z, g_z and h_z are defined and analytic on the whole complex plane. By Lemma 2, h_z is a continuous function of compact support on $M_{n,n-1}$. It is easy to see that g_z is locally integrable on $M_{n,n-1}$. It follows that the identity (12) holds for all complex z. Let us take $\mathscr{R}z = \frac{1}{2}(n-1)$. Then by (9),

$$\|g_z\|_2 \le C_{z,n} \|f_0\|_2 \quad (\mathscr{R}z = \tfrac{1}{2}(n-1)). \tag{13}$$

On the other hand, h_z is controlled by the induction hypothesis

$$|h_z(\Pi)| \le C_{z,n} \prod_{k=1}^{n} \{S|f_k|^a(\Pi)\}^{1/a} \tag{14}$$

where $a = 2n/(n+1)$. It follows from (14), (10) and Holder's inequality that

$$\|h_z\|_2 \le C_{z,n} \prod_{k=1}^{n} \|f_k\|_2 \quad (\mathscr{R}z = \tfrac{1}{2}(n-1)). \tag{15}$$

It now follows from (13) and (15) that

$$|A_z(f_0, \ldots, f_n) \le C_{z,n} \prod_{k=0}^{n} \|f_k\|_2 \quad (\mathscr{R}z = \tfrac{1}{2}(n-1)). \tag{16}$$

Combining this with the trivial estimate

$$|A_z(f_0, \ldots, f_n) \le C_{z,n} \prod_{k=0}^{n} \|f_k\|_1 \quad (\mathscr{R}z = n)$$

and the fact that the constants generated by these methods have at worse exponential growth in $\mathscr{I}z$, we have the conclusion of Theorem 1 by routine complex interpolation arguments.

By the same methods and the use of the mixed norm estimates of Oberlin and Stein one may prove the following generalization.

THEOREM 1. *Suppose that* $\frac{1}{2}(n-1) \le \mathscr{R}z \le n$,

$$\sum_{k=0}^{n} p_k^{-1} = 1 + \mathscr{R}z,$$

$$n^{-1}\mathscr{R}z \le p_k^{-1} \le n(n+1)^{-1} + n^{-1}(n+1)^{-1}\mathscr{R}z \quad (0 \le k \le n).$$

Then

$$|A_z(f_0, \ldots, f_n)| \le C_{z,n} \prod_{k=0}^{n} \|f_k\|_{p_k}.$$

We leave the details to the reader.

At this point let us digress to take the Fourier transform of Theorem 1 in the case $p = 2$, $n = 2$. Gelbart [4] has shown that the Fourier transform of Σ_z is locally integrable for $\mathscr{R}z < 1$ and is given explicitly by

$$\hat{\Sigma}_z(Y) = \gamma_n(n - z)|\det Y|^{-z}.$$

This leads to the identity

$$\begin{aligned} (17) \qquad & A_z(f_0, \dots, f_n) \\ &= \gamma_n(n - z) \int \hat{f}_0(-(u_1 + \cdots + u_n))\hat{f}_1(u_1), \dots, \hat{f}_n(u_n) D^{-z} d\lambda(u_1), \dots, d\lambda(u_n) \end{aligned}$$

where $D = |\det(u_1, u_2, \dots, u_n)|$. Incidentally, (17) with $z = 0$ immediately gives (4) with $k = 0$. Specializing now to the case $n = 2$, from Plancherel's theorem we have:

THEOREM 1″. *Let $\phi \in L^2(\mathbf{R}^2)$, $\alpha \in \mathbf{R}$. Then*

$$\phi(u_1 + u_2)|\det(u_1, u_2)|^{-(1/2)+i\alpha} \quad (u_1, u_2 \in \mathbf{R}^2)$$

is an L^2 bounded kernel on $\mathbf{R}^2$.

Our next task is to establish the relation (4) between A_k and T_k. We will do this by induction on n with k fixed. If $k = n$ the relation follows directly from the definition (3) of A_z. Further if $k = n - 1$ then (12) yields

$$A_{n-1}(f_0, \dots, f_n) = c_n \int g_{n-1}(\Pi) h_{n-1}(\Pi)\, d\mu(\Pi)$$

where $g_{n-1} = S_0 f_0 = c_n Sf$ by well known properties of the standard Riesz potential, and

$$\begin{aligned} h_{n-1}(\Pi) &= A_{n-1}(f_1|_\Pi, \dots, f_n|_\Pi) \\ &= c_n \prod_{j=1}^{n} \left\{ \int f_j(x_j)\, d\lambda_\Pi(x_j) \right\} \\ &= c_n \prod_{j=1}^{n} Sf_j(\Pi). \end{aligned}$$

It follows that

$$A_{n-1}(f_0, \dots, f_n) = c_n \int \prod_{j=0}^{n} Sf_j(\Pi)\, d\mu(\Pi)$$

as required. In this way the induction starts.

For the general induction step we assume the result for $n-1$ and prove it for n. We may assume that $k < n-1$. Again by (12) we have

$$A_k(f_0, \dots, f_n) = c_n \int g_k(\Pi) h_k(\Pi)\, d\mu(\Pi) \tag{18}$$

where $g_k = S_{k-n+1} f_0$ and

$$h_k(\Pi) = A_k(f_1|_\Pi, \dots, f_n|_\Pi) = \int \left\{ \prod_{j=1}^{n} T_k f_j(\Theta) \right\} d\nu_\Pi(\Theta) \tag{19}$$

by the induction hypothesis. In (18) and (19), Π denotes a generic hyperplane, Θ a k-plane, μ is the invariant measure on $M_{n,n-1}$ and ν_Π is the invariant measure on the k-planes of Π.

By the uniqueness of invariant measures on homogeneous spaces we have

$$d\nu_\Pi(\Theta)\, d\mu(\Pi) = c_{n,k}\, d\mu_\Theta(\Pi)\, d\nu(\Theta) \tag{20}$$

where ν is the invariant measure on $M_{n,k}$ and μ_Θ is the invariant measure on the manifold of hyperplanes containing the k-plane Θ. The general induction step is an immediate consequence of (18), (19), (20) and the identity

$$\int g_k(\Pi)\, d\mu_\Theta(\Pi) = c_{n,k} T_k f_0(\Theta) \tag{21}$$

which has to be interpreted in the distributional sense since we know only that g_k is locally integrable on $M_{n,n-1}$. We now establish (21) by means of the uniqueness of Fourier transforms.

Let us write $\Theta = (\theta, x)$ with $\theta \in G_{n,k}$, $x \in \theta^\perp$. Then we fix θ and calculate the Fourier transforms of each side of (21) along the fibre $\theta^\perp$. We have

$$T_k f_0\hat{}(\theta, u) = c_{n,k} \hat{f}_0(u) \quad (u \in \theta^\perp). \tag{22}$$

The left hand side of (21) is more difficult. It can be rewritten as

$$\int g_k(\Pi)\, d\mu_{\theta+x}(\Pi) = \int g_k(\pi + x)\, d\mu_\theta(\pi)$$

where $\pi \in G_{n,n-1}$. If $x = y + y'$, $y \in \pi^\perp$, $y' \in \pi \cap \theta^\perp$ is the orthogonal decomposition of x, we may write $\pi + x = (\pi, y)$ $(y \in \pi^\perp)$. Thus the Fourier transform of the left hand member of (21) is

$$\int g_k(\pi, y) e^{-2\pi i u \cdot (y+y')}\, d\lambda_{\pi^\perp}(y)\, d\lambda_{\pi \cap \theta^\perp}(y')\, d\mu_\theta(\pi), \tag{23}$$

at least in the distributional sense. But $g_k = S_{k-n+1} f_0$ and $k - n + 1 < 1$ so that, by definition of S_z,

$$\hat{g}_k(\pi, u) = c_{n,k} |u|^{-k+n-1} \hat{f}_0(u) \quad (u \in \pi^\perp).$$

Thus (23) becomes

$$c_{n,k}|u|^{-k+n-1}\hat{f}_0(u)\int e^{-2\pi i u\cdot y'}\,d\lambda_{\pi\cap\theta^\perp}(y')\,d\mu_\theta(\pi)$$

and the integral is easily seen to be equal to $c_{n,k}|u|^{-n+k-1}$ in the distributional sense. This completes the proof of (21) and the general induction step.

We now establish the conjecture for the cases outlined in the introduction.

THEOREM 2. *Let*

$$n \le 2k+1, \quad 1 \le q \le n+1, \quad np^{-1}-(n-k)q^{-1}=k$$

(so that $1 \le p \le (n+1)(k+1)^{-1}$). Then T_k is a bounded operator:

$$T_k\colon L^p(\mathbf{R}^n,\lambda)\to L^q(M_{n,k},\mu).$$

Proof. The result is easy for $p=1$, $q=1$:

$$\begin{aligned}\|T_k f\|_1 &= \int\left|\int f(x+y)\,d\lambda_\theta(y)\right|\,d\lambda_{\theta^\perp}(x)\,d\gamma(\theta)\\ &\le \iint |f(x+y)|\,d\lambda_\theta(y)\,d\lambda_{\theta^\perp}(x)\,d\gamma(\theta)\\ &= \|f\|_1.\end{aligned}$$

By the principle of convexity it suffices to establish the result at the other endpoint $p=(n+1)(k+1)^{-1}$, $q=n+1$. By (4) we have

$$\|T_k f\|_{n+1}^{n+1} = c_{n,k}A_k(f,\dots,f) \quad (n+1 \text{ arguments})$$

and, by Theorem 1,

$$|A_k(f,\dots,f)| \le c_{n,k}\|f\|_p^{n+1}$$

where $(n+1)p^{-1}=1+k$ as required.

2. Riesz potentials on $G_{2k,k}$ AND $M_{2k+1,k}$

Let $\pi_1, \pi_2 \in G_{n,k}$. Select an orthonormal basis $e_a^{(j)}$ $(1 \le a \le k)$ for π_j. Let us put

$$A_{a,b} = (e_a^{(1)}, e_b^{(2)})$$

so that A is a $k\times k$ matrix with operator norm ≤ 1. Different choices of basis would yield the matrix UAV with $U, V \in O(k)$. Now define

$$s(\pi_1,\pi_2) = (\det(I-A^tA))^{1/2} \tag{24}$$

an invariant of the two k-planes π_1 and π_2. If $k=1$, $s(\pi_1,\pi_2)$ is just the sine of the angle between π_1 and π_2.

Next let us fix a reference k-plane $\pi_0 \in G_{n,k}$ and define the open subset $\mathscr{U}$ of $G_{n,k}$ by

$$\mathscr{U} = \{\pi;\ \pi \in G_{n,k},\ \pi \cap \pi_0^\perp = \{0\}\}.$$

We observe that the complement $\mathscr{U}^c$ of $\mathscr{U}$ is of codimension one in $G_{n,k}$ and hence is γ-null. From the measure-theoretic viewpoint we may replace $G_{n,k}$ by $\mathscr{U}$. We now parametrize $\mathscr{U}$ in the standard way. For $\pi \in \mathscr{U}$ we denote by ρ_{π,π_0} the restriction to π of the orthogonal projection onto π_0. Since $\pi \in \mathscr{U}$, ρ_{π,π_0} is invertible. Thus

$$u(\pi) = \rho_{\pi,\pi_0^\perp} \circ (\rho_{\pi,\pi_0})^{-1} \in \mathscr{L}(\pi_0, \pi_0^\perp).$$

A little linear algebra shows that $u\colon \mathscr{U} \to \mathscr{L}(\pi_0, \pi_0^\perp)$ is a bijective diffeomorphism.

Select an orthonormal basis $e_1, \ldots, e_n$ of $\mathbf{R}^n$ such that $e_1, \ldots, e_k$ is a basis of π_0. Let $\pi \in \mathscr{U}$ and let $f_1, \ldots, f_k$ be an orthonormal basis of π. Let

$$A_{a,b} = (e_a, f_b), \quad 1 \le a \le k, \quad 1 \le b \le k,$$

$$Q_{a,b} = (e_a, f_b), \quad k + 1 \le a \le n, \quad 1 \le b \le k,$$

so that A is a $k \times k$ matrix, Q an $(n-k) \times k$ matrix such that $A^tA + Q^tQ = I$. Further $u(\pi)$ is represented by the $(n-k) \times k$ matrix $R = QA^{-1}$. The matrix R will be considered as the "coordinate matrix" of the k-plane $\pi \in \mathscr{U}$.

LEMMA 3. *We have* $d\gamma(R) = c(\det(I + R^tR))^{-n/2}\, dR$ *where* dR *denotes Lebesgue measure on the space* $M(n-k, k)$ *of* $(n-k) \times k$ *matrices.*

Proof. Let ϕ be a nice rapidly decreasing positive function on $\mathbf{R}^+$. The measure

$$d\theta\begin{pmatrix} A \\ \hdashline Q \end{pmatrix} = \phi(\operatorname{tr}(A^tA + Q^tQ))\, dA\, dQ$$

on the space $M(n, k)$ is invariant under left multiplication by $O(n)$. It follows that the image measure $\check{\kappa}(\theta)$ under κ,

$$\kappa\begin{pmatrix} A \\ \hdashline Q \end{pmatrix} = QA^{-1},$$

is a constant multiple of γ. Putting $Q = RA$ yields

$$d\theta = \phi(\operatorname{tr}(A^tA + A^tR^tRA))\,|\det A|^{n-k}\, dA\, dR.$$

Finally integrating out with respect to A yields the conclusion of the lemma.

Now let $n = 2k + 1$, let σ_{π_0} be the invariant measure on the sphere of unit vectors $u = (0, \ldots, 0, u_{k+1}, \ldots, u_n)$ in $\pi_0^\perp$. Let γ_u be the invariant measure on the Grassmann of k-planes in $u^\perp$.

LEMMA 4. $d\gamma(\pi) = cs(\pi_0, \pi)\, d\gamma_u(\pi)\, d\sigma_{\pi_0}(u)$.

Proof. Clearly $\pi \in u^\perp$ if and only if $uR = 0$. It follows from Lemma 3 that

$$d\gamma_u(R) = c(\det (I + R^tR))^{-(n-1)/2}\, d\alpha_u(R) \tag{25}$$

where α_u is Lebesgue measure on the space of $(n-k) \times k$ matrices R such that $uR = 0$. Up to choice of sign, u can be recovered from R by

$$u = \pm \|\Lambda^k R\|^{-1} \Lambda^k R$$

and it follows that

$$d\alpha_u(R)\, d\sigma_{\pi 0}(u) = j(R)\, dR$$

for some jacobian $j(R)$. By invariance,

$$j(URA) = j(R) \quad \text{for all } U \in O(k+1), \quad A \in SL(k, \mathbf{R}).$$

A little linear algebra shows that outside the null set on which $\det (R^tR) = 0$, the orbits are parametrized by $\det (R^tR)$. Thus j is a function of $\det (R^tR)$ alone. The action of dilations on R now yields $j(R) = (\det (R^tR))^{-1/2}$. Combining this with (25) and Lemma 3 we have

$$d\gamma_u(R)\, d\sigma_{\pi 0}(u) = (\det (I + R^tR))^{1/2} (\det R^tR)^{-1/2}\, d\gamma(R). \tag{26}$$

Finally $A^tR^tRA = Q^tQ = I - A^tA$ and $A^t(I + R^tR)A = I$ so that

$$\det (I + R^tR)^{-1/2} \det (R^tR)^{1/2} = \det (I - A^tA)^{1/2}.$$

Thus the lemma follows from (26) and (24).

LEMMA 5. *We have $d\gamma(\pi_1)\, d\gamma(\pi_2) = cs(\pi_1, \pi_2)\, d\gamma_u(\pi_1)\, d\gamma_u(\pi_2)\, d\sigma(u)$ where σ is the invariant measure on the sphere in $\mathbf{R}^n$.*

Proof. The manifold $\{(\pi, u);\ \pi \in G_{n,k},\ u \in \mathbf{R}^n,\ \|u\| = 1,\ u \in \pi^\perp\}$ is clearly a homogeneous space of $O(n)$. The probability measures

$$d\sigma_\pi(u)\, d\gamma(\pi) \quad \text{and} \quad d\gamma_u(\pi)\, d\sigma(u)$$

are both invariant on this homogeneous space. Thus by [1, Chapter 7, Théorème 3] they must coincide. The result now follows from Lemma 4.

LEMMA 6. *Let $\pi_1, \pi_2 \in \mathscr{U}$. Then*

$$s(\pi_1, \pi_2) = |\det (R_1 - R_2)| (\det (I + R_1^tR_1))^{-1/2} (\det (I + R_2^tR_2))^{-1/2}.$$

Remark. In case $k = 1$ this is just the difference formula for sine in the form

$$|\sin (\theta_1 - \theta_2)| = |\tan \theta_1 - \tan \theta_2| (1 + \tan^2 \theta_1)^{-1/2} (1 + \tan^2 \theta_2)^{-1/2}.$$

Proof. Let A_j, Q_j and R_j be the matrices relating to π_j. Let $f_a^{(j)}$ $(1 \leq a \leq k)$ be an orthonormal basis of π_j and let $A_{a,b} = (f_a^{(1)}, f_b^{(2)})$. Then

$$A = A_1^t A_2 + Q_1^t Q_2 = A_1^t(I + R_1^t R_2)A_2,$$
$$A^t A = A_2^t(I + R_2^t R_1)A_1 A_1^t(I + R_1^t R_2)A_2.$$

But $A_j A_j^t = (I + R_j^t R_j)^{-1}$ $j = 1, 2$ so that

$$\begin{aligned}
&\det(I - A^t A)\\
&= \det(I - (I + R_2^t R_2)^{-1}(I + R_2^t R_1)(I + R_1^t R_1)^{-1}(I + R_1^t R_2))\\
&= (\det(I + R_2^t R_2))^{-1} \det(I + R_2^t R_2 - (I + R_2^t R_1)(I + R_1^t R_1)^{-1}(I + R_1^t R_2)).
\end{aligned}$$

But

$$\begin{aligned}
I + R_2^t R_2 &- (I + R_2^t R_1)(I + R_1^t R_1)^{-1}(I + R_1^t R_2)\\
&= I + R_2^t R_2 - (I + R_1^t R_1 + (R_2 - R_1)^t R_1)\\
&\quad \times (I + R_1^t R_1)^{-1}(I + R_1^t R_1 + R_1^t(R_2 - R_1))\\
&= (R_2 - R_1)^t(I - R_1(I + R_1^t R_1)^{-1} R_1^t)(R_2 - R_1)\\
&= (R_2 - R_1)^t(I + R_1 R_1^t)^{-1}(R_2 - R_1).
\end{aligned}$$

Since $\det(I + R_1 R_1^t) = \det(I + R_1^t R_1)$, the result now follows.

At this point let us again recall the distribution of Gelbart, this time on the space of $k \times k$ matrices $M(k, k)$. It is designated Σ_z and is defined as the locally integrable density

$$\gamma_k(z)\,|\det R\,|^{-k+z}$$

for $\mathscr{R}z > k - 1$ and can be continued analytically to the whole complex plane. Furthermore Gelbart shows that the Fourier transform $\hat{\Sigma}_z$ is given by the locally integrable function

$$\hat{\Sigma}_z(S) = \gamma_k(k - z)\,|\det S\,|^{-z}$$

for $\mathscr{R}z < 1$. In particular if $\mathscr{R}z = 0$, $\hat{\Sigma}_z$ is a constant multiple of a unitary convolver on L^2. One has the estimate

$$\|\Sigma_{i\gamma}\| \leq c_1 e^{c_2|\gamma|} \quad (\gamma \text{ real})$$

on the L^2 convolver form.

For $\mathscr{R}z > k - 1$ we may define a distribution Ω_z on $G_{2k,k} \times G_{2k,k}$ by

$$d\Omega_z(\pi_1, \pi_2) = \gamma_k(z)\, s(\pi_1, \pi_2)^{-k+z}\, d\gamma(\pi_1)\, d\gamma(\pi_2).$$

LEMMA 7. (a) *The distribution Ω_z can be continued analytically for all complex z.*

(b) *Let $0 \leq \mathscr{R}z \leq k$, $2kp^{-1} = k + \mathscr{R}z$. Then*

$$\left|\int f_1(\pi_1) f_2(\pi_2)\, d\Omega_z(\pi_1, \pi_2)\right| \leq c_{k,z}\|f_1\|_p \|f_2\|_p.$$

Furthermore $c_{k,z}$ increases at most exponentially in $\mathscr{I}z$.

Proof. It is easy to write

$$G_{2k,k} \times G_{2k,k} = \bigcup_{l=1}^{L} \mathscr{U}_l \times \mathscr{U}_l$$

where $\mathscr{U}_l$ $(1 \le l \le L)$ are the open sets determined by finitely many reference planes $\pi_1, \ldots, \pi_L$. Part (a) now follows from the corresponding fact for Σ_z by Lemma 6 and a standard resolution of unity argument.

For (b), the case $p = 1$, $\mathscr{R}z = k$ is trivial since Ω_z is a bounded function. By routine complex interpolation arguments it suffices to prove the result for $p = 2$, $\mathscr{R}z = 0$.

For this it suffices to work with one reference plane π_0. Let $f_1, f_2 \in C_c^\infty(\mathscr{U})$. Then by Lemmas 3 and 6 we have

$$\begin{aligned}
&\left| \int f_1(\pi_1) f_2(\pi_2)\, d\Omega_z(\pi_1, \pi_2) \right| \\
&\qquad = |c(z)| \left| \int \left| \tilde{f}_1(R_1)\tilde{f}_2(R_2) \right| \det (R_1 - R_2) \right|^{-k+z} \\
&\qquad\qquad \times (\det (I + R_1^t R_1) \det (I + R_2^t R_2))^{-1/2(k+z)}\, dR_1\, dR_2 \Bigg| \\
&\qquad \le c_1 e^{c_2|\gamma|} \|f_1\|_2 \|f_2\|_2
\end{aligned}$$

in case $z = i\gamma$ (γ real) since

$$\|f_j\|_2^2 = c \int |\tilde{f}_j(R)|^2 (\det (I + R^t R))^{-k}\, dR \quad (j = 1, 2)$$

and we use the L^2 estimate on $\Sigma_{i\gamma}$. Part (b) now follows since $C_c^\infty(\mathscr{U})$ is dense in $L^2(G_{2k,k})$.

For $\mathscr{R}z > k$ we define the distribution Λ_z on $M_{2k+1,k} \times M_{2k+1,k}$ by

$$d\Lambda_z(\Pi_1, \Pi_2) = \gamma_{k+1}(z) \Delta(\Pi_1, \Pi_2)^{z-k-1}\, d\mu(\Pi_1)\, d\mu(\Pi_2)$$

where

$$\Delta(\Pi_1, \Pi_2) = \delta(\Pi_1, \Pi_2) s(\pi_1, \pi_2),$$

$$\Pi_j = (\pi_j, x_j), \quad \pi_j \in G_{2k+1,k}, \quad x_j \in \pi_j^\perp$$

and $\delta(\Pi_1, \Pi_2)$ is the orthogonal distance between the k-planes Π_1 and Π_2.

LEMMA 8. *Λ_z can be continued analytically as a tempered distribution for all complex z.*

Proof. Let G be in the Schwartz class of $M_{2k+1,k} \times M_{2k+1,k}$. For $\pi_1, \pi_2 \in G_{2k+1,k}$ and $u \in \pi_1^\perp \cap \pi_2^\perp$ we define the Schwartz class function $\tilde{G}$ by

$$\tilde{G}(u, \pi_1, \pi_2) = \int G(\pi_1, x_1; \pi_2, x_2) e^{-2\pi i u(x_1 - x_2)}\, d\lambda_{\pi_1^\perp}(x_1)\, d\lambda_{\pi_2^\perp}(x_2).$$

We may view $\delta(\Pi_1, \Pi_2)$ as the length of the orthogonal projection of $x_1 - x_2$ onto $\pi_1^\perp \cap \pi_2^\perp$. Using this fact, the relation $d\mu(\Pi_j) = d\mu_{\pi_j{}^\perp}(x_j)\, d\gamma(\pi_j)$ and the standard theory of Euclidean Fourier transforms and Reisz potentials [7] we have

$$\int G\, d\Lambda_z \tag{27}$$

$$= c(z) \int |u|^{k-z} s(\pi_1, \pi_2)^{-k-1+z} \tilde{G}(u, \pi_1, \pi_2)\, d\lambda_{\pi_1{}^\perp \cap \pi_2{}^\perp}(u)\, d\lambda(\pi_1)\, d\gamma(\pi_2)$$

where $c(z) = \gamma_1(k + 1 - z)\gamma_k(z)$. Certainly (27) holds in the range $k + 1 > \mathscr{R}z > k$.

It is important to realise that the function $\tilde{G}(u, \pi_1, \pi_2)$ is left invariant under a change of origin in $\mathbf{R}^n$. That is $\tilde{G}$ is intrinsic to the bundle $M_{2k+1,k}$. We now wish to make a change of viewpoint. We regard u as a point of linear Euclidean space $\mathbf{R}^n$ and π_1, π_2 as k-planes in the $2k$-dimensional space $u^\perp$. By Lemma 5 we find

$$d\lambda_{\pi_1{}^\perp \cap \pi_2{}^\perp}(u)\, d\gamma(\pi_1)\, d\gamma(\pi_2) = c\,|u|^{-2k} s(\pi_1, \pi_2)\, d\gamma_u(\pi_1)\, d\gamma_u(\pi_2)\, d\lambda(u)$$

and the right hand side of (27) becomes

$$c_1(z) \int |u|^{-k-z} s(\pi_1, \pi_2)^{-k+z} \tilde{G}(u, \pi_1, \pi_2)\, d\gamma_u(\pi_1)\, d\gamma_u(\pi_2)\, d\lambda(u) \tag{28}$$

which makes sense for $k + 1 > \mathscr{R}z > k - 1$. Thus (28) extends the definition of Λ_z into $\mathscr{R}z > k - 1$. To extend it further we denote by $\Omega_{u,z}$ the distribution Ω_z taken relative to the k-planes of $u^\perp$. We may then rewrite (28) as

$$\gamma_1(k + 1 - z) \int |u|^{-k-z} \tilde{G}(u, \pi_1, \pi_2)\, d\Omega_{u,z}(\pi_1, \pi_2)\, d\lambda(u) \tag{29}$$

which is valid for $\mathscr{R}z < k + 1$. Thus (29) extends the definition of Λ_z to the whole complex plane.

LEMMA 9. *Let* $0 \le \mathscr{R}z \le k + 1$, $2(k + 1)p^{-1} = k + 1 + \mathscr{R}z$. *Then*

$$\left| \int g_1(\Pi_1)\overline{g_2(\Pi_2)}\, d\Lambda_z(\Pi_1, \Pi_2) \right| \le c_{k,z} \|g_1\|_p \|g_2\|_p.$$

Proof. The case $p = 1$, $\mathscr{R}z = k + 1$ is trivial since Λ_z is a bounded function. By routine complex interpolation arguments we need only prove the result in case $p = 2$, $\mathscr{R}z = 0$. Let us put

$$G(\pi_1, x_1; \pi_2, x_2) = g_1(\pi_1, x_1)\overline{g_2(\pi_2, x_2)}.$$

Then

$$\tilde{G}(u, \pi_1, \pi_2) = \hat{g}_1(\pi_1, u)\overline{\hat{g}_2(\pi_2, u)}$$

so that, by (29),

$$\int g_1 \otimes \bar{g}_2 \, d\Lambda_z = \gamma_0(k+1-z) \int |u|^{-k-z} \hat{g}_1(\pi_1, u) \overline{\hat{g}_2(\pi_2, u)} \, d\Omega_{u,z}(\pi_1, \pi_2) \, d\lambda(u).$$

This yields

$$\left| \int g_1 \otimes \bar{g}_2 \, d\Lambda_z \right| \le c_z \|g_1\|_2 \|g_2\|_2 \quad (\mathscr{R}z = 0),$$

using Lemma 7(b) and the fact that

$$\|g_j\|_2^2 = c_k \int |u|^{-k} | \hat{g}_j(\pi, u)|^2 \, d\gamma_u(\pi) \, d\lambda(u).$$

This completes the proof.

3. Applications of Λ_z

In this chapter we relate Λ_z to the k-plane transform in $2k+1$ dimensions and give different proofs of special cases of Theorems 1 and 2.

We start out by giving a new proof of Theorem 2 in case $n = 2k+1$. As already pointed out we need only establish the result at the difficult endpoint $p = 2$, $q = 2k+2$. Towards this we calculate T_k^* the formal adjoint of T_k. We do this by means of the Fourier transform. For $\pi \in G_{2k+1,k}$, $u \in \pi^\perp$, let

$$\dot{g}\hat{\ }(\pi, u) = \int e^{-2\pi i u \cdot x} g(\pi, x) \, d\lambda_{\pi^\perp}(x).$$

That is, for g defined on $M_{2k+1,k}$ we find $\dot{g}\hat{\ }$ by taking the Fourier transform along each fibre. Then

$$(T_f f)\check{\ }(\pi, u) = \int e^{-2\pi i u \cdot x} f(x+y) \, d\lambda_\pi(y) \, d\lambda_{\pi^\perp}(x),$$

and since $u \cdot x = u \cdot (x+y)$ for $u \in \pi^\perp$, we have

$$(T_k f)\check{\ }(\pi, u) = \hat{f}(u).$$

Now, by Plancherel's Theorem,

$$\begin{aligned}\int T_k f(\pi, x) \overline{g(\pi, x)} \, d\lambda_{\pi^\perp}(x) \, d\gamma(\pi) &= \int T_k f\dot{\ }\hat{\ }(\pi, u) \overline{\hat{g}(\pi, u)} \, d\lambda_{\pi^\perp}(u) \, d\gamma(\pi) \\ &= \int \hat{f}(u) \overline{\hat{g}(\pi, u)} \, d\lambda_{\pi^\perp}(u) \, d\gamma(\pi) \\ &= c \int \hat{f}(u) \overline{\hat{g}(\pi, u)} \, |u|^{-k} \, d\gamma_u(\pi) \, d\lambda(u).\end{aligned}$$

It follows that

$$(T_k^* g)\hat{}(u) = c\,|u|^{-k} \int \hat{g}(\pi, u)\, d\gamma_u(\pi).$$

Again by Plancherel's Theorem we have

$$\int T_k^* g_1(x)\overline{T_k^* g_2(x)}\, d\lambda(x) = \int (T_k^* g_1)\hat{}(u)\overline{(T_k^* g_2)\hat{}(u)}\, d\lambda(u)$$
$$= c \int |u|^{-2k} \dot{g}_1\hat{}(\pi_1, u)\overline{\dot{g}_2\hat{}(\pi_2, u)}\, d\gamma_u(\pi_1)\, d\gamma_u(\pi_2)\, d\lambda(u).$$

If $G(\pi_1, x_1; \pi_2, x_2) = g_1(\pi_1, x_1)\overline{g_2(\pi_2, x_2)}$ then

$$\tilde{G}(u, \pi_1, \pi_2) = \dot{g}_1\hat{}(\pi_1, u)\overline{\dot{g}_2\hat{}(\pi_2, u)}$$

so by (28) we have

$$\int T_k^* g_1(x)\overline{T_k^* g_2(x)}\, d\lambda(x) = c \int g_1 \otimes \bar{g}_2\, d\Lambda_k$$

as required.

An application of Lemma 9 now yields

$$\int |T_k^* g(x)|^2\, d\lambda(x) \le c\|g\|^2_{(2k+2)/(2k+1)}.$$

Hence T_k^* is bounded as a map from $L^{(2k+2)/(2k+1)}(M_{2k+1,k}) \to L^2(\mathbf{R}^{2k+1})$. It follows by duality that T_k is bounded,

$$T_k\colon L^2(\mathbf{R}^{2k+1}) \to L^{2k+2}(M_{2k+1,k}), \tag{30}$$

as required.

Finally we use (30) together with Lemma 9 to give a new proof of Theorem 1 in the case n odd. For this let n be odd and define k by $n = 2k + 1$. Again we need only establish the difficult estimate (cf. (16))

$$|A_z(f_0, \ldots, f_n)| \le c_{z,n} \prod_{j=0}^{n} \|f_j\|_2 \quad (\mathcal{R}z = k). \tag{31}$$

Towards this we need to establish a lemma which gives insight into the geometrical meaning of the invariant $\Delta(\Pi_1, \Pi_2)$ of a pair of k-planes Π_1, Π_2. Let $x_0, \ldots, x_{2k+1}$ be $2k + 2$ generic points of $\mathbf{R}^{2k+1}$. Let Δ denote the volume of the simplex having these points as vertices. Let Π_1 be the k-plane passing through $x_0, x_1, \ldots, x_k$ and Π_2 the k-plane passing through $x_{k+1}, \ldots, x_{2k+1}$. Let Δ_1 and Δ_2 be the volumes of the corresponding simplexes in Π_1 and Π_2 respectively.

LEMMA 10. $\Delta(x_0, \ldots, x_{2k+1}) = c_k \Delta_1 \Delta_2\, \Delta(\Pi_1, \Pi_2)$.

Proof. Let $\Pi_j = (\pi_j, \xi_j)$ with $\xi_j \in \pi_j^\perp$ $(j = 1, 2)$. Let e_0 be a unit vector in $\pi_1^\perp \cap \pi_2^\perp$. We define

$$\begin{aligned} y_l^{(1)} &= x_l - x_0, & l &= 1, \dots, k, \\ y_l^{(2)} &= x_{l+k} - x_{2k+1}, & l &= 1, \dots, k, \\ y &= x_{2k+1} - x_0. \end{aligned}$$

Then

$$\begin{aligned} \Delta &\sim \det (y_1^{(1)}, \dots, y_k^{(1)}, y_1^{(1)} + y, \dots, y_k^{(2)} + y, y) \\ &= \det (y_1^{(1)}, \dots, y_k^{(1)}, y_1^{(2)}, \dots, y_k^{(2)}, y) \\ &= \pm y \cdot e_0 \det (y_1^{(1)}, \dots, y_k^{(1)}, y_1^{(2)}, \dots, y_k^{(2)}) \end{aligned}$$

where in this last determinant the $y_l^{(j)}$ are considered to be vectors in the $2k$-dimensional space $e_0^\perp$. Clearly $|y \cdot e_0| = \delta(\Pi_1, \Pi_2)$. Now let $e_l^{(j)}$ $(l = 1, \dots, k)$ be an orthonormal basis of π_j $(j = 1, 2)$. It is easy to see that

$$\det (y_1^{(1)}, \dots, y_k^{(1)}, y_1^{(2)}, \dots, y_k^{(2)}) \sim \pm \Delta_1 \Delta_2 \det (e_1^{(1)}, \dots, e_k^{(1)}, e_1^{(2)}, \dots, e_k^{(2)}).$$

Finally taking π_1 as reference plane and using the notations of Section 2 we have

$$\det (e_1^{(1)}, \dots, e_k^{(1)}, e_1^{(2)}, \dots, e_k^{(2)}) = \det \left(\begin{array}{c|c} I & A \\ \hline O & Q \end{array} \right) = \det Q.$$

But $|\det Q| = (\det Q^t Q)^{1/2} = \det (I - A^t A)^{1/2} = s(\pi_1, \pi_2)$. Combining these facts we have

$$\Delta \sim \delta(\Pi_1, \Pi_2) \Delta_1 \Delta_2 \, s(\pi_1, \pi_2) = c_k \Delta_1 \Delta_2 \, \Delta(\Pi_1, \Pi_2)$$

as required.

We return now to the problem at hand—that of establishing (31). By Lemma 1, we have

$$\begin{aligned} d\lambda(x_0), \dots, d\lambda(x_k) &= c_k \Delta_1^{(k+1)} \, d\lambda_{\Pi_1}(x_0), \dots, d\lambda_{\Pi_1}(x_k) \, d\mu(\Pi_1), \\ d\lambda(x_{k+1}), \dots, d\lambda(x_n) &= c_k \Delta_2^{(k+1)} \, d\lambda_{\Pi_2}(x_{k+1}), \dots, d\lambda_{\Pi_2}(x_n) \, d\mu(\Pi_2). \end{aligned}$$

Thus, from the definition of A_z in (3) and by Lemma 10, we have

$$A_z(f_0, \dots, f_n) = c_k \gamma_n(z) \int h_z^{(1)}(\Pi_1) h_z^{(2)}(\Pi_2) \Delta(\Pi_1, \Pi_2)^{-n+z} \, d\mu(\Pi_1) \, d\mu(\Pi_2)$$

for $\mathscr{R}z > n - 1$, where

$$\begin{aligned} h_z^{(1)}(\Pi_1) &= \int \Delta_1^{-k+z} \prod_{j=0}^{k} f_j(x_j) \, d\lambda_{\Pi_1}(x_0), \dots, d\lambda_{\Pi_1}(x_k), \\ h_z^{(2)}(\Pi_2) &= \int \Delta_2^{-k+z} \prod_{j=k+1}^{n} f_j(x_j) \, d\lambda_{\Pi_2}(x_{k+1}), \dots, d\lambda_{\Pi_2}(x_n). \end{aligned}$$

By the definition of Λ_{z-k} and the principle of analytic continuation we now have

$$A_z(f_0, \ldots, f_n) = c_k \gamma_k(z) \int h_z^{(1)}(\Pi_1) h_z^{(2)}(\Pi_2)\, d\Lambda_{z-k}(\Pi_1, \Pi_2)$$

which is valid for $\mathscr{R}z > k - 1$.

Now let $\mathscr{R}z = k$. Then, by Lemma 9,

$$|A_z(f_0, \ldots, f_n)| \le c_k \|h_z^{(1)}\|_2 \|h_2^{(2)}\|_2. \tag{32}$$

Again, for $\mathscr{R}z = k$ we have

$$\|h_z^{(1)}\|_2^2 \le \int \prod_{j=0}^{k} (T_k |f_j|)^2(\Pi_1)\, d\mu(\Pi_1).$$

But by (30),

$$\|(T_k |f_j|)^2\|_{k+1} \le c_k \|f_j\|_2^2$$

which leads to

$$\|h_z^{(1)}\|_2 \le c_k \prod_{j=0}^{k} \|f_j\|_2.$$

This together with a similar estimate for $h_z^{(2)}$ and (32) now gives

$$|A_z(f_0, \ldots, f_n)| \le c_k \prod_{j=0}^{n} \|f_j\|_2$$

as required.

REFERENCES

1. N. BOURBAKI, *Intégration*, Livre VI, Hermann, Paris, 1963.
2. D. M. OBERLIN and E. M. STEIN, *Mapping properties of the Radon transform*, Indiana J. Math., vol. 31 (1982), pp. 641–650.
3. S. W. DRURY, *L^p estimates for the X-ray transform*, Illinois J. Math., vol. 27 (1983), pp. 125–129.
4. S. S. GELBART, *Fourier analysis on matrix space*, Mem. Amer. Math. Soc. No. 108, 1971.
5. D. C. SOLMON, *A note on k-plane integral transforms*, J. Math. Anal. Appl., vol. 71 (1979), pp. 351–358.
6. E. M. STEIN, *Some problems in harmonic analysis suggested by symmetric spaces and semisimple groups*, Actes du Congrès International des Mathematiciens (1970), tome 1, Gauthier-Villars, Paris, 1971, pp. 173–189.
7. ———, *Singular integrals and differentiability properties of functions*, Princeton University Press, Princeton, N.J., 1970.

McGILL UNIVERSITY
MONTREAL, CANADA