THE DISTRIBUTION OF VALUES OF AN INNER FUNCTION

BY

PATRICK AHERN

The purpose of this note is to show that there is a theory of the distribution of values for an inner function that is analogous to some parts of the value distribution theory for meromorphic functions.

1. A bounded holomorphic in the unit disc \( U \) whose radial limits have modulus 1 almost everywhere is called an inner function, see [7], Chapter 17, for details on the structure of inner functions. If \( \phi \) is an inner function and \( \alpha \in U \), then \( \phi_\alpha \) will denote the inner function \( (\phi - \alpha)/(1 - \overline{\alpha}\phi) \). We let \( n(r, \alpha) \) denote the number of zeros of \( \phi_\alpha \) whose moduli are at most \( r \) and define

\[
v(r, \alpha) = \int_r^1 \frac{n(t, \alpha)}{t} dt.
\]

Following the notation of O. Frostman [4], we let \( \delta(\alpha) \) be the total mass of the singular measure, \( \sigma_\alpha \), associated to the inner function \( \phi_\alpha \), and

\[
L(r, \alpha) = -\frac{1}{2\pi} \int_0^{2\pi} \log |\phi_\alpha(re^{i\theta})| d\theta.
\]

The quantity

\[
\frac{1}{2\pi} \int_0^{2\pi} (1 - |\phi(re^{i\theta})|^2) d\theta
\]

will be denoted by \( \Delta(r) \). It is a simple consequence of Jensen's formula that

\[
L(r, \alpha) = v(r, \alpha) + \delta(\alpha).
\]

We may say that \( v(r, \alpha) \) is a measure of the number of zeros of \( \phi_\alpha \) in

\[
\{z: r < |z| < 1\}.
\]

Since \( \phi_\alpha \) has a radial limit equal to 0 almost everywhere with respect to \( \sigma_\alpha \), we may say that \( \delta(\alpha) \) measures the number of zeros of \( \phi_\alpha \) on the unit circle. In other words \( L(r, \alpha) \) measures the number of zeros of \( \phi_\alpha \) in

\[
\{z: r < |z| \leq 1\}.
\]

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Next we recall some notions from potential theory. If $\mu$ is a positive Borel measure of total mass 1 in $U$ then define

$$\hat{\mu}(z) = \int \log \left| \frac{1 - \overline{z} \xi}{\xi - z} \right| \, d\mu(\xi) \quad \text{and} \quad V_{\mu} = \sup_{z \in U} \hat{\mu}(z).$$

If $K \subseteq U$ is compact we let $V_K = \inf \{ V_{\mu} : \text{supp } \mu \subseteq K \}$, and if $E \subseteq U$ we let $V_E = \inf \{ V_K : K \subseteq E, \, K \text{ compact} \}$. The inner capacity of $E$ is defined to be $\gamma(E) = e^{-V_E}$. The set function $\gamma$ is monotone and we have the subadditivity property: if $E = \bigcup_{n=1}^{\infty} E_n$, then

$$\frac{1}{V_E} \leq \sum_{n=1}^{\infty} \frac{1}{V_{E_n}}.$$

We refer to [8], Chapter III, for details.

In this note it is shown that the distribution of values of $\phi$ is determined by the quantity $\Delta(r)$ in the following sense:

(i) \[ 0 < \lim_{r \to 1} \frac{L(r, \alpha)}{\Delta(r)} \quad \text{for all } \alpha \in U, \]

and

(ii) \[ \lim_{r \to 1} \frac{L(r, \alpha)}{\Delta(r)} < \infty \quad \text{for all } \alpha \in U, \]

with the exception of a set of capacity 0.

We may say that the exceptional values of $\phi$ are the ones that are taken too often. This can happen in two ways, either $\delta(\alpha) \neq 0$, or $\delta(\alpha) = 0$ and

$$\lim_{r \to 1} \frac{\nu(\alpha, \alpha)}{\Delta(r)} = \infty.$$ 

We show by example that the second possibility can occur. Since

$$\lim_{r \to 1} \Delta(r) = 0,$$

one consequence of having

$$\lim_{r \to 1} \frac{L(r, \alpha)}{\Delta(r)} < \infty$$

with the exception of a set of capacity 0, is that $\delta(\alpha) = 0$ with the exception of a set of capacity 0. This is, of course, the well known theorem of Frostman [4].
It is probably not true that, for every inner function $\phi$, we have
\[ \lim_{r \to 1} \frac{L(r, \alpha)}{\Delta(r)} < \infty, \]
with the exception of a set of capacity 0, but we can find no counterexample. We can show that our results are close to sharp, in that
\[ \lim_{r \to 1} \frac{L(r, \alpha)}{\Delta(r) \lambda(\Delta(r))} = 0 \]
with the exception of a set of capacity 0, if $\lambda$ is any positive decreasing function on $(0, 1)$ such that
\[ \int_0^1 \frac{dt}{t \lambda(t)} < \infty. \]
Finally we show that this result can be somewhat improved if we allow a slightly larger exceptional set. To do this we need to develop a relation between capacity and some Hausdorff-like set functions that may not have been observed before.

2. The notion that $L(r, \alpha)$ is in some way dominated from above by $\Delta(r)$ is suggested by the proof of Theorem 4.3 of [3]. The proof of part (ii) of the following theorem is a modification of that proof.

**Theorem 1.**

(i) \[ \frac{L(r, \alpha)}{\Delta(r)} \geq \frac{1 - |\alpha|}{4} \text{ for all } \alpha \in U \text{ and all } r, 0 < r < 1. \]

(ii) If $0 < \rho < 1$, and if $\mu$ is a distribution of the unit mass on $\{z: |z| \leq \rho\}$, then
\[ \int L(r, \alpha) \, d\mu(\alpha) \leq \frac{V_{\mu}}{\rho(1 - \rho)} \Delta(r). \]

**Proof.** To prove (i) we start with the inequality $1 - x \leq -\log x$, valid for $0 < x < 1$. We obtain
\[ 1 - |\phi_r(re^{i\theta})|^2 \leq -2 \log |\phi_r(re^{i\theta})|. \]
We also have the following identity (see [6], for example):
\[ 1 - |\phi_r(re^{i\theta})|^2 = \frac{1 - |\alpha|^2}{|1 - \overline{\alpha}\phi(re^{i\theta})|^2} (1 - |\phi(re^{i\theta})|^2). \]
We may conclude that
\[
1 - |\phi(re^{i\theta})|^2 \leq \frac{(1 + |\alpha|)^2}{1 - |\alpha|^2} (1 - |\phi_a(re^{i\theta})|^2)
\]
\[
\leq \frac{2}{1 - |\alpha|} (-2 \log |\phi_a(re^{i\theta})|)
\]
\[
= \frac{-4}{1 - |\alpha|} \log |\phi_a(re^{i\theta})|.
\]
Integrating on \(\theta\), we get \(\Delta(r) \leq \frac{4}{1 - |\alpha|} L(r, \alpha)\), which gives us (i).

To prove (ii) we use Fubini's theorem to see that
\[
\int L(r, \alpha) \, d\mu(\alpha) = \frac{1}{2\pi} \int_0^{2\pi} \tilde{\mu}(\phi(re^{i\theta})) \, d\theta.
\]
Next we write
\[
\frac{1}{2\pi} \int_0^{2\pi} \tilde{\mu}(\phi(re^{i\theta})) \, d\theta = \frac{1}{2\pi} \int_{E_r} \tilde{\mu}(\phi(re^{i\theta})) \, d\theta + \frac{1}{2\pi} \int_{E_r'} \tilde{\mu}(\phi(re^{i\theta})) \, d\phi,
\]
where
\[
E_r = \{\theta: 0 \leq \theta \leq 2\pi, |\phi(re^{i\theta})| \leq \rho\}
\]
and
\[
E_r' = \{\theta: 0 \leq \theta \leq 2\pi, |\phi(re^{i\theta})| > \rho\}.
\]
From the definition of \(V_\mu\) we see that
\[
\frac{1}{2\pi} \int_{E_r} \tilde{\mu}(\phi(re^{i\theta})) \, d\theta \leq V_\mu \frac{1}{2\pi} \int_{E_r} \, d\theta.
\]
To deal with the integral over \(E_r'\), we note that \(\tilde{\mu}\) is harmonic in
\[
\{z: \rho < |z| < 1/\rho\}
\]
and \(\tilde{\mu}(z) = 0\) if \(|z| = 1\). It follows that the function
\[
\tilde{\mu}(z) - V_\mu \frac{\log |z|}{\log \rho}
\]
is harmonic in the annulus \(A = \{z: \rho < |z| < 1\}\) and has a non-positive upper limit at each point of the boundary of \(A\). We conclude from the maximum principle that
\[
\tilde{\mu}(z) \leq V_\mu \frac{\log |z|}{\log \rho} \text{ if } z \in A.
\]
In particular, if $\theta \in E_\rho'$ then

$$\mu(\phi(re^{i\theta})) \leq V_\mu \frac{\log |\phi(re^{i\theta})|}{\log \rho}.$$  

Using the inequalities $1 - x \leq -\log x \leq (1 - x)/x$, valid if $0 < x < 1$, we see that

$$\frac{\log |\phi(re^{i\theta})|}{\log \rho} = \frac{-\log |\phi(re^{i\theta})|}{-\log \rho} \leq \frac{1 - |\phi(re^{i\theta})|}{(1 - \rho)|\phi(re^{i\theta})|} \leq \frac{1 - |\phi(re^{i\theta})|^2}{(1 - \rho)^2}.$$  

So we see that if $\theta \in E_\rho'$, then

$$\mu(\phi(re^{i\theta})) \leq \frac{V_\mu}{\rho(1 - \rho)} (1 - |\phi(re^{i\theta})|^2).$$

We can conclude that

$$\int L(r, \omega) \, d\mu(\omega) = \frac{1}{2\pi} \int_0^{2\pi} \mu(\phi(re^{i\theta})) \, d\theta$$

$$\leq V_\mu \frac{1}{2\pi} \int_{E_\rho} d\theta + \frac{V_\mu}{\rho(1 - \rho)} \frac{1}{2\pi} \int_{E_\rho'} (1 - |\phi(re^{i\theta})|^2) \, d\theta$$

$$= \frac{V_\mu}{\rho(1 - \rho)} \left[ \frac{1}{2\pi} \int_{E_\rho} (1 - \rho^2) \, d\theta + \frac{1}{2\pi} \int_{E_\rho'} (1 - |\phi(re^{i\theta})|^2) \, d\theta \right]$$

$$\leq \frac{V_\mu}{\rho(1 - \rho)} \frac{1}{2\pi} \int_0^{2\pi} (1 - |\phi(re^{i\theta})|^2) \, d\theta$$

$$= \frac{V_\mu}{\rho(1 - \rho)} \Delta(r),$$

since $1 - \rho^2 \leq 1 - |\phi(re^{i\theta})|^2$ if $\theta \in E_\rho$. This completes the proof.

**Corollary.**

(i) \[ \lim_{r \to 1} \frac{L(r, \omega)}{\Delta(r)} > 0 \] for all $\omega \in U$.  

(ii) \[ \lim_{r \to 1} \frac{L(r, \omega)}{\Delta(r)} < \infty \]

with the exception of a set of capacity 0.
Proof. Part (i) is clear. Part (ii) follows from a well known argument. Since the union of a countable number of sets of capacity 0 has capacity 0 it is enough to show that for each \( \rho, 0 < \rho < 1 \),

\[
E = \left\{ \alpha : |\alpha| \leq \rho \quad \text{and} \quad \lim_{r \to 1} \frac{L(r, \alpha)}{\Delta(r)} = \infty \right\}
\]

has capacity 0. If \( \mu \) is a distribution of the unit mass with support in \( E \), then by (ii) of the theorem we have

\[
\int \frac{L(r, \alpha)}{\Delta(r)} \, d\mu(\alpha) \leq \frac{V_{\mu}}{\rho(1 - \rho)}.
\]

We conclude from Fatou's lemma that

\[
\infty = \int \lim_{r \to 1} \frac{L(r, \alpha)}{\Delta(r)} \, d\mu(\alpha) \leq \frac{V_{\mu}}{\rho(1 - \rho)}
\]

and hence that \( V_{\mu} = \infty \). This implies that \( \gamma(E) = 0 \).

If we let

\[
E(\phi) = \{ \alpha \in U : \delta(\alpha) \neq 0 \}
\]

and

\[
E_1(\phi) = \left\{ \alpha \in U : \lim_{r \to 1} \frac{L(r, \alpha)}{\Delta(r)} = \infty \right\},
\]

then of course \( E(\phi) \subseteq E_1(\phi) \) and so Theorem 1 (ii) may be regarded as a generalization of Frostman's Theorem [4]. To show that it is a true generalization we give an example to show that \( E(\phi) \) and \( E_1(\phi) \) are not always the same.

Theorem 2. Suppose that \( B \) is a Blaschke product whose zeros lie on \((0, 1)\). Then

(i) \( E(B) = \emptyset \), and

(ii) \( \Delta(r) = O(\sqrt{1 - r}) \).

Proof. We assume \( B \) has infinitely many zeros. If \( \alpha \in U, \alpha \neq 0 \), and \( B_\alpha \) were not a Blaschke product then \( B_\alpha \) would have a radial limit equal to 0 somewhere. That is to say that \( B \) would have radial limit equal to \( \alpha \) at \( e^{i\theta} \) for some \( \theta, 0 \leq \theta \leq 2\pi \). If \( e^{i\theta} \neq 1 \), then \( B \) has a radial limit of modulus 1 at \( e^{i\theta} \). If \( B \) has a radial limit at 1, that limit must be 0 since \( B \) has infinitely many zeros on \((0, 1)\). This proves part (i). Part (ii) is proved by Carleson in [3], page 48, see also [1], Theorem 7, with \( \beta = 1 \).
Now, to get an example of a Blaschke product $B$ with $E(B) \neq E_1(B)$, let $B$ have the zeros $a_k = 1 - k^{-\alpha}$, $1 < \alpha < 2$. By Theorem 2, (i), $E(B) = \phi$. It is easy to calculate that

$$L(r, 0) = \int_0^1 \frac{n(t, 0)}{t} dt > \varepsilon(1 - r)^{(\alpha - 1)/\alpha},$$

for some $\varepsilon > 0$, and hence that

$$\frac{L(r, 0)}{\Delta(r)} \geq \delta(1 - r)^{(\alpha - 2)/2\alpha},$$

for some $\delta > 0$. It follows that $0 \in E_1(B)$.

3. Next we want to exploit the inequality in Theorem 1 (ii) to get some information about

$$L(r, \alpha) = \int_0^1 \frac{dt}{t\lambda(t)}$$

The method we use is analogous to one used in the theory of meromorphic functions by J. E. Littlewood [5], and refined by L. Ahlfors [2].

**Theorem 3.** Suppose $\lambda$ is a positive decreasing function on $(0, 1)$ such that

$$\int_0^1 \frac{dt}{t\lambda(t)} < \infty.$$

Then for any inner function $\phi$, we have

$$\lim_{r \to 1} \frac{L(r, \alpha)}{\Delta(r)\lambda(\Delta(r))} = 0,$$

with the exception of a set of capacity 0.

**Proof.** Given $\rho$, $0 < \rho < 1$, it is enough to show that

$$\left\{ \alpha: |\alpha| \leq \rho, \lim_{r \to 1} \frac{L(r, \alpha)}{\Delta(r)\lambda(\Delta(r))} > 0 \right\}$$

has capacity 0. To show this it is enough to show that

$$\left\{ \alpha: |\alpha| \leq \rho, \lim_{r \to 1} \frac{L(r, \alpha)}{\Delta(r)\lambda(\Delta(r))} > 2 \right\}$$

has capacity 0, because once this is done we may replace $\lambda$ by $\varepsilon\lambda$, $\varepsilon > 0$. This being said, for all sufficiently large $n$ we may choose $r_n$ such that $\Delta(r_n) = 2^{-n}$ and let

$$E_n = \left\{ \alpha: |\alpha| \leq \rho, \frac{L(r_n, \alpha)}{\Delta(r_n)} \geq \lambda(2^{-n}) \right\}.$$
If $\mu$ is a distribution of the unit mass with support in $E_n$ we see that

$$\lambda(2^{-n}) = \int \lambda(2^{-n}) \, d\mu \leq \int \frac{L(r_n, \alpha)}{\Delta(r_n)} \, d\mu(\alpha) \leq \frac{V_\mu}{\rho(1 - \rho)}.$$ 

It follows that $V_n = V_{E_n} \geq \lambda(2^{-n})\rho(1 - \rho)$ and hence that

$$\frac{1}{V_{n+1}} \leq \frac{1}{\rho(1 - \rho)} \frac{1}{\lambda(2^{-(n+1)})} = \frac{1}{\rho(1 - \rho) \log 2 \int_{2^{-(n+1)}}^{2^{-n}} \frac{1}{t} \, dt} \leq \frac{1}{\rho(1 - \rho) \log 2 \int_{2^{-(n+1)}}^{2^{-n}} \frac{1}{t\lambda(t)} \, dt},$$

since $\lambda$ is decreasing. It now follows from the hypothesis on $\lambda$ that

$$\sum_{n=1}^{\infty} \frac{1}{V_n} < \infty.$$ 

And from this it follows that $E = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} E_k$ has capacity 0. Now if $\alpha \notin E$, $|\alpha| \leq \rho$, then $\alpha \notin \bigcup_{k=n}^{\infty} E_k$ for some $n$, and hence

$$\frac{L(r_k, \alpha)}{\Delta(r_k)} \leq \lambda(2^{-k})$$

for all $k \geq n$. Now fix $k \geq n$ and take $r, r_k \leq r \leq r_{k+1}$; then

$$\frac{L(r, \alpha)}{\Delta(r)} \leq \frac{L(r_k, \alpha)}{\Delta(r_{k+1})} = \frac{2L(r_k, \alpha)}{\Delta(r_k)} \leq 2\lambda(2^{-k}) = 2\lambda(\Delta(r_k)) \leq 2\lambda(\Delta(r)).$$

In other words, if $|\alpha| \leq \rho$, $\alpha \notin E$ then there is an $n$ such that

$$\frac{L(r, \alpha)}{\Delta(r)\lambda(\Delta(r))} \leq 2 \quad \text{for} \quad r \geq r_n.$$ 

In particular

$$\lim_{r \to 1} \frac{L(r, \alpha)}{\Delta(r)\lambda(\Delta(r))} \leq 2.$$
Remark. We note that the function

$$\lambda(t) = \left( \log \frac{1}{t} \right)^x$$

satisfies the hypotheses of Theorem 3, for $\alpha > 1$.

4. Next we show that if we allow a slightly larger exceptional set we get a somewhat better result. We recall some notions from the theory of Hausdorff measures. A positive increasing function, $h$, defined on $(0, \infty)$ such that $\lim_{r \to 0} h(r) = 0$ is called a measure function. If $\alpha \in \mathbb{C}$ and $r \geq 0$, $\Delta(a, r)$ will denote $\{z: |z - a| < r\}$. We have the set function

$$M_h(E) = \inf \{\sum h(r_k): E \subseteq \bigcup \Delta(a_k, r_k)\}.$$

The set function $M_h$ is monotone and subadditive. For $\varepsilon > 0$ there is the set function

$$M_h^\varepsilon(E) = \inf \{\sum h(r_k): E \subseteq \bigcup \Delta(a_k, r_k), r_k \leq \varepsilon\}$$

and finally,

$$\Lambda_h(E) = \lim_{\varepsilon \to 0} M_h^\varepsilon(E).$$

The set function $\Lambda_h$ is actually a measure on the Borel sets, also $M_h$ and $\Lambda_h$ have the same null sets. In [4], Frostman has shown that if the measure function $h$ satisfies

$$\int_0^1 \frac{h(t)}{t} \, dt < \infty$$

then for any Borel set $E$ such that $\gamma(E) = 0$ we must have $\Lambda_h(E) = 0$. This cannot be a consequence of an inequality involving $\gamma$ and $\Lambda_h$ because $\gamma$ is finite on bounded sets but $\Lambda_h$ is in general infinite. We will show that under some additional mild assumptions on $h$ there is a general inequality between $M_h$ and $\gamma$. We will assume that the measure function $h$ is continuous and that $h(r)/r^2$ is decreasing, and that

$$\int_0^1 \frac{h(t)}{t} \, dt < \infty.$$

We define

$$\hat{h}(\varepsilon) = \int_0^\varepsilon \frac{h(t)}{t} \, dt.$$

Lemma. (i) There is a constant $C$ such that for every compact set $K \subseteq U$ we have $M_h(K) \leq C \hat{h}(\gamma(K))$. 

If there is a constant $C_0$ such that

$$h(t) \leq C_0 t^{1/2} e^{-h(t)/h(t)}$$

then $M_h(K) \leq C_1 h(y(K))$ for some constant $C_1$, independent of $K$.

Remarks. It follows from the assumption that $h(r)/r^2$ is decreasing that

$$h(cr) \leq c^2 h(r) \quad \text{for any } c \geq 1.$$

It then follows that (*) holds any time that $h(t) \leq c h(t)$ for some constant $c$. This is the case for example if $h(r) = r^\alpha$, $0 < \alpha < 2$. If we take

$$h(r) = \left( \log \frac{1}{r} \right)^{-\alpha} \quad \text{with } \alpha > 1$$

then (*) is still true but it is no longer true that $h(r) \leq c h(r)$ for some constant $c$. Condition (*) fails for

$$h(r) = \log \frac{1}{r} \left( \log \log \frac{1}{r} \right)^{-1} \quad \alpha > 1.$$

Proof of lemma. The proof is a modification of the proof of Frostman [4] and depends on his basic result that says that there is a constant $a > 0$, independent of $h$, such that if $K \subseteq U$ is compact, there is a positive Borel measure $\mu$ on $K$ such that $\mu(\Delta(z, r)) \leq h(r)$ for all $z \in \mathbb{C}$ and $r \geq 0$ and $\mu(K) \geq a M_h(K)$. We calculate

$$\hat{\mu}(z) = \int \log \left| \frac{1 - z}{\bar{z} - z} \right| d\mu(\xi) \leq \int_0^R \log \frac{1}{r} d\Omega(r) + \mu(K) \log 2,$$

where $R$ is chosen so that $\Delta(z, R) \supseteq K$, and $\Omega(r) = \mu(\Delta(z, r))$. After integrating by parts we find that

$$\hat{\mu}(z) \leq \Omega(R) \log \frac{1}{R} + \int_0^R \frac{\Omega(r)}{r} dr + \mu(K) \log 2$$

$$\leq \mu(K) \log \frac{1}{R} + \int_0^\varepsilon \frac{h(r)}{r} dr + \mu(K) \log [\log R - \log \varepsilon] + \mu(K) \log 2$$

$$= \tilde{h}(\varepsilon) + \mu(K) \log \frac{1}{\varepsilon} + \mu(K) \log 2,$$

for any $\varepsilon > 0$. Now the measure $\nu = \mu/\mu(K)$ is a distribution of the unit mass on $K$ and

$$\hat{\nu}(z) \leq \frac{\tilde{h}(\varepsilon)}{\mu(K)} + \log \frac{1}{\varepsilon} + \log 2.$$
To prove (i) we just choose $\varepsilon = \frac{1}{\log (\mu(K))}$ and get

$$V(z) \leq C + \log \frac{1}{h^{-1}(\mu(K))};$$

this means that

$$V_K \leq C + \log \frac{1}{h^{-1}(\mu(K))},$$

so

$$\mu(K) \leq h(\varepsilon \gamma(K)) \leq e^{2\varepsilon} h(\gamma(K)).$$

Since $\mu(K) \geq aM_h(K)$, the proof of (i) is complete.

To prove (ii) we return to the inequality

$$V(z) \leq \frac{h(\varepsilon)}{\mu(K)} + \log \frac{1}{\varepsilon} + \log 2.$$ 

This time we let $\varepsilon = h^{-1}(\mu(K))$. Now we check that (*) yields

$$\frac{\log \frac{1}{h^{-1}(\mu(K))} + \log \frac{1}{h^{-1}(\mu(K))} + \log 2 \leq \log \frac{1}{h^{-1}(\mu(K)/c)}}.$$ 

We conclude that

$$V_K \leq -\log h^{-1} \left( \frac{\mu(K)}{c} \right),$$

and hence

$$M_h(K) \leq \frac{1}{a} \mu(K) \leq \frac{c}{a} h(\gamma(K)).$$

**COROLLARY.** Let $\emptyset \subseteq U$ be open. Then

(i) $M_h(\emptyset) \leq c\tilde{h}(\gamma(\emptyset))$,

and

(ii) if (*) holds then $M_h(\emptyset) \leq c\tilde{h}(\gamma(\emptyset))$.

**Proof.** From (i) of the lemma we conclude that

$$\sup \{ M_h(K) \mid K \subseteq \emptyset, K \text{ is compact} \} \leq c\tilde{h}(\gamma(\emptyset)).$$

But Carleson has shown [3] that

$$M_h(\emptyset) \leq 24 \sup \{ M_h(K) \mid K \subseteq \emptyset, K \text{ compact} \}.$$ 

This proves (i) of the corollary; (ii) is proved in the same way.
THEOREM 4. Suppose \( h \) is a measure function and \( \lambda \) is a positive decreasing function on \( (0, 1) \) such that

\[
\int_0^1 \frac{h(e^{-\lambda(t)})}{\lambda(t)} \, dt < \infty.
\]

Then there is a set \( E \subseteq U \), such that \( M_h(E) = 0 \) and

\[
\lim_{r \to 1} \frac{L(r, \alpha)}{\Delta(r)} < \infty \quad \text{for all } \alpha \notin E.
\]

Proof. Since \( M_h \) is subadditive it is enough to show that for each \( \rho, \)

\[
0 < \rho < 1,
\]

\[
\left\{ \alpha : |\alpha| \leq \rho, \lim_{r \to 1} \frac{L(r, \alpha)}{\Delta(r)\lambda(\Delta(r))} = \infty \right\}
\]

is a null set for \( M_h \). Fix such a \( \rho \) and choose \( r_n \) such that \( \Delta(r_n) = 2^{-n} \) and let

\[
\mathcal{O}_n = \left\{ \alpha : |\alpha| < \rho, \frac{L(r_n, \alpha)}{\Delta(r_n)} > \frac{\lambda(\Delta(r_n))}{\rho(1 - \rho)} \right\}.
\]

Then \( \mathcal{O}_n \) is an open set and if \( \mu \) is any distribution of the unit mass with support in \( \mathcal{O}_n \) we have from Theorem 1 (ii), \( \lambda(\Delta(r_n)) \leq V_\mu \) and hence

\[
\lambda(\Delta(r_n)) \leq V_{\mathcal{O}_n}.
\]

From the corollary we see that

\[
M_h(\mathcal{O}_n) \leq c\tilde{h}(\exp(-V_{\mathcal{O}_n})) \leq c\tilde{h}(\exp(-\lambda(\Delta(r_n)))).
\]

We conclude as before that

\[
M_h(\mathcal{O}_{n+1}) \leq \frac{c}{\log 2} \int_{2^{-n+1}}^{2^{-n}} \frac{h(e^{-\lambda(t)})}{\lambda(t)} \, dt
\]

and hence that \( \Sigma M_h(\mathcal{O}_n) < \infty \). Since \( M_h \) is monotone and subadditive we see that \( M_h(E) = 0 \), where \( E = \bigcap_k \bigcup_{n \geq k} \mathcal{O}_n \). As before we conclude that if \( \alpha \notin E, \)

\[
|\alpha| \leq \rho, \text{ then}
\]

\[
\lim_{r \to 1} \frac{L(r, \alpha)}{\Delta(r)\lambda(\Delta(r))} \leq \frac{2}{\rho(1 - \rho)}.
\]

Remark. Of course if condition (*) of the lemma holds, then the hypothesis of Theorem 4 may be weakened to read

\[
\int_0^1 \frac{h(e^{-\lambda(t)})}{\lambda(t)} \, dt < \infty.
\]
**Corollary.** Let $M_\beta$ be the set function associated to the measure function $h(r) = r^\beta$, $0 < \beta \leq 2$. Suppose $\lambda$ is a positive decreasing function on $(0, 1)$ such that

$$\int_0^1 \frac{e^{-c\lambda(t)}}{t} \, dt < \infty \quad \text{for some constant } C.$$

Then there is a set $E \subseteq U$ such that $M_\beta(E) = 0$ for all $\beta, 0 < \beta \leq 2$, and

$$\lim_{r \to 1} \frac{L(r, \alpha)}{\Delta(r)\lambda(\Delta(r))} < \infty \quad \text{if } \alpha \notin E.$$

(Note that $\lambda(t) = \log \log (1/t)$ will work.)

**Proof.** Fix $\beta, 0 < \beta \leq 2$, and let $\Lambda(t) = c\lambda(t)/\beta$. Then

$$\int_0^1 \frac{[e^{-\Lambda(t)}]^\beta}{t} \, dt < \infty$$

and by Theorem 4 there is a set $E_\beta$ with $M_\beta(E_\beta) = 0$ such that

$$\lim_{r \to 1} \frac{L(r, \alpha)}{\Delta(r)\lambda(\Delta(r))} < \infty \quad \text{for } \alpha \notin E_\beta.$$

Of course this means that

$$\lim_{r \to 1} \frac{L(r, \alpha)}{\Delta(r)\lambda(\Delta(r))} < \infty \quad \text{for } \alpha \notin E_\beta.$$

Now choose $\beta_0 > 0$ and define

$$E = \bigcap_{n=1}^\infty \bigcup_{k \geq n} E_{\beta_k}.$$

If $\alpha \notin E$ then $\alpha \notin \bigcup_{k \geq n} E_{\beta_k}$ for some $n$. In particular $\alpha \notin E_{\beta_n}$ and hence

$$\lim_{r \to 1} \frac{L(r, \alpha)}{\Delta(r)\lambda(\Delta(r))} < \infty.$$

Fix $\beta, 0 < \beta \leq 2$; then $\beta > \beta_0$ for some $n$. Now

$$E = \bigcup_{k \geq n} E_{\beta_k}, \quad \text{so} \quad M_\beta(E) \leq \sum_{k \geq n} M_\beta(E_{\beta_k}).$$

But clearly $M_\beta(E_{\beta_k}) \leq M_\beta(E_{\beta_k}) = 0$ because $\beta > \beta_k$; that is, $M_\beta(E) = 0$.

**References**


UNIVERSITY OF WISCONSIN

MADISON, WISCONSIN