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LINEAR GROUPS OVER MAXIMAL ORDERS

BY

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Counterexamples of O'Meara [6] show that the classical isomorphism theory of the linear groups breaks down when the coefficients are taken to be maximal orders in division algebras. The reason for this is simple in retrospect: Equivalences between categories of modules induce isomorphisms (of "equivalence type") between linear groups, and these can fall outside the scope of classical descriptions. Refer to [5]. The theory of [5] provides a new classification which takes this phenomenon into account. A special case of a theorem there asserts the following: If two free modules of ranks ≥ 3 over such maximal orders are given, then any isomorphism between their projective linear groups is either of equivalence type or the composite of an equivalence type with the transpose inverse isomorphism. The present article will extend this result in two directions: First to maximal orders in *central simple* algebras and from linear groups of free modules to those of *finitely generated projectives.*

More precisely, let R be a Dedekind domain with quotient field K, and let Λ be a maximal R-order in a central simple K-algebra A. Let $\mathfrak{M}_\Lambda$ be the category of right Λ-modules and let $M \in \mathfrak{M}_\Lambda$ be finitely generated projective. The *length* of M is by definition that of the right A-module $M \otimes_\Lambda A$, $GL(M)$ is the group of invertible Λ-homomorphisms on M, and $PGL(M)$ is the quotient of $GL(M)$ by its center. Denote by M^* the dual of M and by Λ° the R-order "opposite" Λ. Clearly $M^* \in \mathfrak{M}_{\Lambda^\circ}$. Let Λ_1 over R_1 be another such maximal order.

THEOREM. *Let M and M' be finitely generated projective Λ and Λ_1-modules, respectively, with lengths ≥ 3. If*

$$\Phi: PGL(M) \to PGL(M')$$

is an isomorphism of groups, then there is either

(i) *an equivalence $F: \mathfrak{M}_\Lambda \to \mathfrak{M}_{\Lambda_1}$ with $F(M) = M'$, such that F acting on homomorphisms induces Φ, i.e., $\Phi = F$,*

or

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(ii) *an equivalence* $E\colon \mathfrak{M}_{\Lambda^\circ} \to \mathfrak{M}_{\Lambda_1}$ *with* $E(M^*) = M'$, *such that* $\Phi = EC_M$, *where* $C_M\colon PGL(M) \to PGL(M^*)$ *is induced by transpose-inverse.*

This is a special case of Theorem (3.1) below. The paper concludes with Proposition (3.3) which has as consequence the fact that the conclusion of the theorem *does not hold* if Λ and Λ_1 are taken to be *hereditary* orders.

Terminology and basic facts will come from [7] for orders and modules, and [5] for the linear groups; [1], [2] and [4] are valuable general references.

1. Tensor products and R-equivalences

Let R be an integral domain with quotient field K and let A be a finite dimensional K-algebra. Let Λ be an R-algebra in A, i.e., Λ is an R-algebra contained in A, has its operations from A and spans A over K.

It is easy to see that

$$A \otimes_\Lambda A \cong A$$

as $(A\text{-}A)$-bimodules, with $a \otimes a_1 \to aa_1$. Note next that if ${}_\Lambda P$ is a left Λ-module, then there is a natural left A-module structure on $K \otimes_R P$. For $a \in A$, $ra \in \Lambda$ for a non-zero $r \in R$; set $a = r^{-1}\lambda$ with $\lambda \in \Lambda$ and put

$$a(k \otimes p) = r^{-1}k \otimes \lambda p.$$

One then has an isomorphism

$$A \otimes_\Lambda P \cong K \otimes_R P$$

of left A-modules with

$$a \otimes p \to r^{-1} \otimes \lambda p.$$

Completely analogous things can be done for a right Λ-module P_Λ.

Suppose now that B is another finite dimensional K-algebra and that Δ is an R-algebra in B. Let P be a $(\Lambda\text{-}\Delta)$-bimodule over R. Then there is an isomorphism of $(\Lambda\text{-}\Delta)$-bimodules

$$A \otimes_\Lambda P \cong P \otimes_\Delta B$$

given by $a \otimes p \to \lambda p \otimes r^{-1}$, where $a = \lambda r^{-1}$. The inverse is given analogously.

Recall the concept of R-functor from page 57 of [2]. An R-functor

$$F\colon \mathfrak{M}_\Lambda \to \mathfrak{M}_\Delta$$

is an *R-equivalence* if there is an R-functor $E\colon \mathfrak{M}_\Delta \to \mathfrak{M}_\Lambda$ such that EF and FE are naturally isomorphic to the identity functors on $\mathfrak{M}_\Lambda$ and $\mathfrak{M}_\Delta$ respectively. In this case Λ and Δ are *R-equivalent*. It is not difficult to check that if A and B are central simple K-algebras then they are K-equivalent if and only if they are in the same Brauer class.

Consider the functors $T\colon \mathfrak{M}_\Lambda \to \mathfrak{M}_A$ and $T_1\colon \mathfrak{M}_\Delta \to \mathfrak{M}_B$ given respectively by tensoring with ${}_\Lambda A$ and ${}_\Delta B$.

(1.1) *Let $F\colon \mathfrak{M}_\Lambda \to \mathfrak{M}_\Delta$ be an R-equivalence. Then there is a K-equivalence $\bar{F}\colon \mathfrak{M}_A \to \mathfrak{M}_B$ such that*

$$\begin{tikzcd} \mathfrak{M}_A \arrow[r, "\bar{F}"] & \mathfrak{M}_B \\ \mathfrak{M}_\Lambda \arrow[u, "T"] \arrow[r, "F"] & \mathfrak{M}_\Delta \arrow[u, "T_1"'] \end{tikzcd}$$

commutes, i.e., such that $\bar{F}T \cong T_1 F$.

Proof. By Theorem 3.1 page 60 of [2], for example, there is a set of equivalence data

$$(\Lambda, \Delta, {}_\Lambda P_\Delta, {}_\Delta Q_\Lambda, \mu, \tau)$$

with P and Q bimodules over R, such that F is isomorphic to the R-functor determined by tensoring with ${}_\Lambda P_\Delta$.

Now consider the bimodules

$${}_A\bar{P}_B = (A \otimes_\Lambda P) \otimes_\Delta B \quad \text{and} \quad {}_B\bar{Q}_A = (B \otimes_\Delta Q) \otimes_\Lambda A.$$

Making extensive use of the isomorphisms developed above and the properties of sets of equivalence data, shows that there is a set of equivalence data $(A, B, \bar{P}, \bar{Q}, \bar{\mu}, \bar{\tau})$.

The isomorphism $\bar{\mu}\colon \bar{P} \otimes_B \bar{Q} \to A$ is given by

$$\bar{\mu}[((a \otimes p) \otimes b) \otimes ((b_1 \otimes q_1) \otimes a_1)] = ar^{-1}r_1^{-1}\mu(p\delta\delta_1 \otimes q_1)a_1,$$

where $b = r^{-1}\delta$ and $b_1 = r_1^{-1}\delta_1$ with r, r_1 in R, δ, δ_1 in Δ. Similarly, $\bar{\tau}\colon \bar{Q} \otimes_A \bar{P} \to B$ is given by

$$\bar{\tau}[((b_1 \otimes q_1) \otimes a_1) \otimes ((a_2 \otimes p_2) \otimes b_2)] = b_1 s_1^{-1} s_2^{-1}\tau(q_1\lambda_1\lambda_2 \otimes p_2)b_2,$$

where $a_1 = s_1^{-1}\lambda_1$ and $a_2 = s_2^{-1}\lambda_2$ with s_1, s_2 in R and λ_1, λ_2 in Λ.

To prove the associativity properties use the isomorphisms

$$\bar{P} \simeq A \otimes_\Lambda (A \otimes_\Lambda P) \simeq A \otimes_\Lambda P$$

and

$$\bar{Q} \to B \otimes_\Delta (B \otimes_\Delta Q) \simeq B \otimes_\Delta Q.$$

These isomorphisms also show that $\bar{P}$ and $\bar{Q}$ are bimodules over K.

Let $\bar{F}\colon \mathfrak{M}_A \to \mathfrak{M}_B$ be the K-equivalence defined by tensoring with ${}_A\bar{P}_B$. Using the above isomorphisms once more shows that $\bar{F}T \cong T_1F$. Q.E.D.

(1.2) COROLLARY. *If Λ and Δ are R-equivalent, A and B are K-equivalent.*

Under more controlled conditions there is a converse for (1.2).

(1.3) *Suppose R is a Dedekind domain, that A and B are central simple K-algebras, and that Λ and Δ are maximal R-orders in A and B respectively.*

Then Λ and Δ are R-equivalent if and only if A and B are in the same Brauer class.

Proof. If Λ and Δ are R-equivalent then A and B are in the same Brauer class by (1.2).

Conversely, assume that A and B are in the same Brauer class. So there is a central division algebra D over K, and finite dimensional right D-spaces V and W such that

$$A \cong \mathrm{End}\,(V_D) \quad \text{and} \quad B \cong \mathrm{End}\,(W_D)$$

as K-algebras. By (21.6) of [7] there is a maximal R-order Δ_0 in D and full right Δ_0-lattices N in V_D and N' in W_D such that

$$\Lambda \cong \mathrm{End}\,(N_{\Delta_0}) \quad \text{and} \quad \Delta \cong \mathrm{End}\,(N'_{\Delta_0})$$

as R-algebras. Since ${}_{\Lambda}N_{\Delta_0}$ and ${}_{\Delta}N'_{\Delta_0}$ are both progenerators in $\mathfrak{M}_{\Delta_0}$ and bimodules over R, tensoring with N and N' induces R-equivalences

$$\mathfrak{M}_{\Lambda} \to \mathfrak{M}_{\Delta_0} \quad \text{and} \quad \mathfrak{M}_{\Delta} \to \mathfrak{M}_{\Delta_0}. \qquad \text{Q.E.D.}$$

Let $P_{\Lambda} \in \mathfrak{M}_{\Lambda}$. There is a homomorphism

$$A \otimes_{\Lambda} P^* \to (P \otimes_{\Lambda} A)^*$$

of left A-modules, satisfying

$$a \otimes f \to [a \otimes f]$$

with $[a \otimes f](p \otimes a') = af(p)a'$, for a, a' in A, f in P^* and $p \in P$. If, in addition, both P_{Λ} and A_{Λ} are finitely generated over Λ this is an isomorphism. Define the inverse as follows: Let $g \in (P \otimes_{\Lambda} A)^*$, denote by $P \otimes \Lambda$ the obvious image of P in $P \otimes_{\Lambda} A$, and observe that $g(P \otimes \Lambda)$ is a finitely generated submodule of A_{Λ}. It follows that there is a non-zero $r \in R$ such that $rg(P \otimes \Lambda) \subseteq \Lambda$, so that the composite

$$P \longrightarrow P \otimes \Lambda \xrightarrow{rg} \Lambda$$

is in P^*. Denote this composite by rg and define

$$(P \otimes_{\Lambda} A)^* \to A \otimes_{\Lambda} P^*$$

by $g \to r^{-1} \otimes rg$. This definition does not depend on the choice of r. Use of the natural left K-vector space structure on $A \otimes_{\Lambda} P^*$ shows that this map is a homomorphism of left A-modules. It is easy to see that it is the inverse of the earlier map.

Suppose that A is central simple over K, and that $M \in \mathfrak{M}_\Lambda$ is finitely generated. Recall that the *length* $l(M_\Lambda)$ of M_Λ is defined to be equal to the length of the finitely generated right A-module $M \otimes_\Lambda A$. Do a similar thing for left Λ-modules. If A_Λ is finitely generated, the isomorphism

$$A \otimes_\Lambda M^* \cong (M \otimes_\Lambda A)^*$$

shows that $l({}_\Lambda M^*) = l(M_\Lambda)$, and hence that $l(M^*_{\Lambda^\circ}) = l(M_\Lambda)$, where Λ° is the opposite R-algebra of Λ in the central simple K-algebra A°. Finally, if B is also central simple over K, and if $F\colon \mathfrak{M}_\Lambda \to \mathfrak{M}_\Delta$ is an R-equivalence, then $l(M_\Lambda) = l(F(M)_\Delta)$. This follows from (1.1) above and (21.8) (right handed version) of [1].

2. Elementary properties of linear groups

Let R be a Dedekind domain with quotient field K, let D be a finite dimensional central division algebra over K, and let Λ be a maximal R-order in D.

Let W be a finite dimensional right vector space over D and let $M \subseteq W$ be a right Λ-module. Let $X = \{x_1, \ldots, x_n\}$ be a basis for W over D and let $\{f_1, \ldots, f_n\}$ be the dual basis. For $n \geq 2$, $i \neq j$ and $\alpha \in D$, let $\tau_{x_i\alpha, f_j}$ in $GL(W)$ be defined by

$$\tau_{x_i\alpha, f_j}(x) = x + x_i a f_j(x),$$

and observe that

$$\tau_{x_i\alpha, f_j}(M) \subseteq M \quad \text{if and only if} \quad \tau_{x_i\alpha, f_j}(M) = M.$$

Let $E_X(M)$ be the subgroup of $GL(W)$ generated by all the $\tau_{x_i\alpha, f_j}$ which stabilize M, and denote its derived group by $DE_X(M)$.

(2.1) *Suppose* $n \geq 2$ *and that* $M = x_1\mathfrak{a}_1 + \cdots + x_n\mathfrak{a}_n$ *with* $\mathfrak{a}_i$ *a right* Λ-*ideal in* D. *Then* $\tau_{x_i\alpha, f_j}(M) = M$ *if and only if* $\alpha\mathfrak{a}_j \subseteq \mathfrak{a}_i$.

Proof. Trivial.

(2.2) *Suppose* $n \geq 3$ *and that* $M = x_1\mathfrak{b} + \cdots + x_{n-1}\mathfrak{b} + x_n\mathfrak{a}$, *with* $\mathfrak{a}$ *and* $\mathfrak{b}$ *right* Λ-*ideals in* D. *Then* $DE_X(M) = E_X(M)$.

Proof. Observe that for i, j and k distinct,

$$\tau_{x_i\alpha\beta, f_j} = [\tau_{x_i\alpha, f_k}, \tau_{x_k\beta, f_j}].$$

Now let $\tau_{x_i\alpha, f_j}$ be arbitrary with $\tau_{x_i\alpha, f_j}(M) = M$.

If $j = n$, $\alpha\mathfrak{a} \subseteq \mathfrak{b}$. Pick $k < n$, $k \neq i$. Then

$$\tau_{x_i\alpha, f_n} = [\tau_{x_i, f_k}, \tau_{x_k\alpha, f_n}],$$

and $\tau_{x_i\alpha, f_n} \in DE_X(M)$ by (2.1).

If $i = n$, proceed similarly. Assume therefore, that $i, j < n$. So $\alpha\mathfrak{b} \subseteq \mathfrak{b}$. Hence $\alpha \in \mathfrak{b}\mathfrak{b}^{-1} = (\mathfrak{b}\mathfrak{a}^{-1})(\mathfrak{a}\mathfrak{b}^{-1})$. So $\alpha = \sum_{\text{fin}} \gamma\delta$, with $\gamma \in \mathfrak{b}\mathfrak{a}^{-1}$ and $\delta \in \mathfrak{a}\mathfrak{b}^{-1}$. Since $\tau_{x_i\gamma, f_n}$ and $\tau_{x_n\delta, f_j}$ are both in $E_X(M)$,

$$\tau_{x_i\gamma\delta, f_j} \in DE_X(M).$$

Therefore $\tau_{x_i\alpha, f_j} = \prod_{\text{fin}} \tau_{x_i\gamma\delta, f_j}$ is in $DE_X(M)$. Q.E.D.

Assume now that M is a full right Λ-lattice in W, ie., M is a right Λ-lattice which spans V over K (or equivalently over D). By the right handed version of (27.8) of [7], there is a basis

$$X = \{x_1, \ldots, x_n\}$$

for W over D such that

$$M = x_1\Lambda + \cdots + x_{n-1}\Lambda + x_n\mathfrak{a},$$

where $\mathfrak{a}$ is a right Λ-ideal in D. This M is therefore a special case of that considered above. By (10.7) of [7], M is a finitely generated projective Λ-module. Identify

$$GL(M) = \{\sigma \in GL(W) \mid \sigma M = M\}.$$

Note that $E_X(M) \subseteq GL(M) \subseteq GL(V)$. Let $RL(M)$ be the subgroup of $GL(M)$ consisting of the invertible R-scalar transformations on M.

(2.3) *Suppose $n \geq 3$. Then the centralizer of $E_X(M)$ in $GL(M)$ is $RL(M)$.*

Proof. Proceed as in the proof of (2.14) of [5]. Use (2.1) repeatedly.

(2.4) *Let Λ and Λ_1 be maximal R-orders in D and let M and M_1 be full right Λ and Λ_1-lattices, respectively, in W. Put*

$$M = x_1\Lambda + \cdots + x_{n-1}\Lambda + x_n\mathfrak{a},$$

where $X = \{x_1, \ldots, x_n\}$ is a basis for W over D and $\mathfrak{a}$ is a right Λ-ideal in D.
Suppose $n \geq 2$ and $E_X(M) \subseteq GL(M_1)$. Then there is a normal ideal ${}_{\Lambda}\mathfrak{b}_{\Lambda_1}$ in D and a right Λ_1-ideal $\mathfrak{b}_n$ in D with $\mathfrak{a}\mathfrak{b} \subseteq \mathfrak{b}_n$, such that

$$M_1 = x_1\mathfrak{b} + \cdots + x_{n-1}\mathfrak{b} + x_n\mathfrak{b}_n.$$

Proof. Let $\mathfrak{b}_i = \{\alpha \in D \mid x_i\alpha \in M_1\}$. It is clear that $\mathfrak{b}_i$, $1 \leq i \leq n$, is a right Λ_1-module in D. We show $\mathfrak{b}_1 = \cdots = \mathfrak{b}_{n-1}$. We may assume $n \geq 3$. By (2.1) and the hypothesis, $\tau_{x_i\lambda, f_j}(M_1) \subseteq M_1$ for all $\lambda \in \Lambda$, and $i, j < n$. In particular,

$$x_i\lambda\mathfrak{b}_j = x_i\lambda f_j(x_j\mathfrak{b}_j) \subseteq x_i\lambda f_j(M_1) \subseteq M_1,$$

so $\lambda \mathfrak{b}_j \subseteq \mathfrak{b}_i$ for all $\lambda \in \Lambda$. Taking $\lambda = 1$ and appealing to symmetry gives the result. Let $\mathfrak{b} = \mathfrak{b}_1 = \cdots = \mathfrak{b}_{n-1}$. Now let $j \neq n$, and $\alpha \in \mathfrak{a}$. By (2.1) and the hypothesis,

$$x_n \alpha \mathfrak{b} = x_n \alpha f_j(x_j \mathfrak{b}) \subseteq x_n \alpha f_j(M_1) \subseteq M_1.$$

So $\mathfrak{a}\mathfrak{b} \subseteq \mathfrak{b}_n$, and similarly $\mathfrak{a}^{-1}\mathfrak{b}_n \subseteq \mathfrak{b}$.

We show that $\mathfrak{b}$ is a $(\Lambda\text{-}\Lambda_1)$-bimodule and that

$$M_1 = x_1\mathfrak{b} + \cdots + x_{n-1}\mathfrak{b} + x_n\mathfrak{b}_n.$$

Let $x \in M_1$ be arbitrary, and let

$$x = x_1\alpha_1 + \cdots + x_n\alpha_n, \quad \alpha_i \in D.$$

For $\alpha \in \mathfrak{a}$, $\tau_{x_n\alpha, f_1}(M_1) \subseteq M_1$, so $\tau_{x_n\alpha, f_1}(x) = x + x_n\alpha\alpha_1$ is in M_1 and hence $\alpha\alpha_1 \in \mathfrak{b}_n$. So $\mathfrak{a}\alpha_1 \subseteq \mathfrak{b}_n$ and

$$\Lambda\alpha_1 = \mathfrak{a}^{-1}\mathfrak{a}\alpha_1 \subseteq \mathfrak{a}^{-1}\mathfrak{b}_n \subseteq \mathfrak{b}.$$

In particular $\alpha_1 \in \mathfrak{b}$. On the other hand letting $\alpha_1 \in \mathfrak{b}$ be arbitrary and taking $x = x_1\alpha_1$ gives $\Lambda\mathfrak{b} \subseteq \mathfrak{b}$ and hence that $\mathfrak{b}$ is a $(\Lambda\text{-}\Lambda_1)$-bimodule. Proceeding similarly shows that $\alpha_2, \ldots, \alpha_{n-1}$ are in $\mathfrak{b}$ hence that

$$x_1\alpha_1 + \cdots + x_{n-1}\alpha_{n-1} \in M_1.$$

Therefore, $x_n\alpha_n \in M_1$, $\alpha_n \in \mathfrak{b}_n$ and $M_1 = x_1\mathfrak{b} + \cdots + x_{n-1}\mathfrak{b} + x_n\mathfrak{b}_n$.

Finally, since R is Noetherian and M_1 finitely generated as R-module, $x_1\mathfrak{b}$ and $x_n\mathfrak{b}_n$ are finitely generated as R-modules. The same is of course true for $\mathfrak{b}$ and $\mathfrak{b}_n$. Since M_1 is a full right Λ_1-lattice in V, $K\mathfrak{b} = D = K\mathfrak{b}_n$, so ${}_{\Lambda}\mathfrak{b}_{\Lambda_1}$ is a normal ideal in D and $\mathfrak{b}_n$ is a right Λ_1-ideal in D. Q.E.D.

(2.5) *Let Λ and Λ_1 be two maximal R-orders in D and let M and M_1 be full right Λ and Λ_1-lattices, respectively, in W. Let*

$$M = x_1\Lambda + \cdots + x_{n-1}\Lambda + x_n\mathfrak{a},$$

where $X = \{x_1, \ldots, x_n\}$ is a basis for W over D and $\mathfrak{a}$ is a right Λ-ideal of D.

Suppose that $n \geq 2$, $E_X(M) \subseteq GL(M_1)$ and $E_X(M_1) \subseteq GL(M)$. Then there is a normal ideal ${}_{\Lambda}\mathfrak{b}_{\Lambda_1}$ in D such that $M_1 = M\mathfrak{b}$.

Proof. Use (2.4), then (2.1) to show that $\mathfrak{b}_n \subseteq \mathfrak{a}\mathfrak{b}$. Q.E.D.

Let $RL(W)$ be the set of invertible K-scalar transformations on W. In reference to the situation of (2.5), consider the quotient maps

$$P_1\colon GL(W) \to GL(W)/RL(W) \quad \text{and} \quad P\colon GL(M) \to GL(M)/RL(M).$$

Observe that the kernel of the restriction of P_1 is $RL(M)$ and that P_1 induces an injection $PGL(M) \to P_1GL(W)$. So consider $PGL(M)$ as subgroup of $P_1GL(W)$ and do the same for $PGL(M_1)$. Denote by $SL(M)$ and $SL(M_1)$ the commutator subgroups of $GL(M)$ and $GL(M_1)$ respectively.

(2.6) *Suppose $n \geq 3$. Let Λ and Λ_1 be maximal R-orders in D, and let M and M_1 be full right Λ and Λ_1-lattices, respectively, in W. Suppose $PSL(M) \subseteq PGL(M_1)$ and $PSL(M_1) \subseteq PGL(M)$. Then there is a normal ideal ${}_\Lambda\mathfrak{b}_{\Lambda_1}$ in D such that $M1 = M\mathfrak{b}$.*

Proof. Put $M = x_1\Lambda + \cdots + x_{n-1}\Lambda + x_n\mathfrak{a}$ with $X = \{x_1, \ldots, x_n\}$ a basis for W and $\mathfrak{a}$ a right Λ-ideal of D. Now verify the hypothesis of (2.5).

Let $\tau_{x_i\alpha, f_j}$ be an arbitrary generator of $E_X(M)$ and show that it is in $GL(M_1)$.

Assume first that $j = n$. Pick $k < n$, $k \neq i$. By (2.1) and (2.2), $\alpha \in \mathfrak{a}^{-1}$ and $\tau_{x_i\alpha, f_n} = [\tau_{x_i, f_k}, \tau_{x_k\alpha, f_n}]$ with τ_{x_i, f_k} and $\tau_{x_k\alpha, f_n}$ in $SL(M)$. So $\alpha_1\tau_{x_i, f_k}$ and $\alpha_2\tau_{x_k\alpha, fn}$ are in $GL(M_1)$ for some non-zero α_1, α_2 in K. Since

$$\tau_{x_i\alpha, f_n} = [\alpha_1\tau_{x_i, f_k}, \alpha_2\tau_{x_k\alpha, f_n}],$$

$\tau_{x_i\alpha, f_n} \in GL(M_1)$. Proceed similarly if $i = n$. Now assume that $i, j < n$. So $\alpha \in \Lambda$ by (2.1). Since $\Lambda = \mathfrak{a}^{-1}\mathfrak{a}$, put $\alpha = \sum_{\text{fin}} \gamma\delta$, with $\gamma \in \mathfrak{a}^{-1}$ and $\delta \in \mathfrak{a}$. It suffices to show that $\tau_{x_i\gamma\delta, f_j}$ is in $GL(M_1)$, refer to the proof of (2.2). Since

$$\tau_{x_i\gamma\delta, f_j} = [\tau_{x_i\gamma, f_n}, \tau_{x_n\delta, f_j}],$$

this follows from (2.1) and the above. Therefore $E_X(M) \subseteq GL(M_1)$.

An application of (2.4) now gives $M_1 = x_1\mathfrak{b} + \cdots + x_{n-1}\mathfrak{b} + x_n\mathfrak{b}_n$, for some right Λ_1-ideals $\mathfrak{b}$ and $\mathfrak{b}_n$. Now refer to the proof of (2.2) and argue as above to show that $E_X(M_1) \subseteq GL(M)$. Q.E.D.

3. The main theorems

Let R and R_1 be Dedekind domains with quotient fields K and K_1 respectively. Let A be a central simple K-algebra and let Λ be an R-order in A. Analogously, let A_1 be a central simple K_1-algebra and let Λ_1 be an R_1-order in A_1.

(3.1) THEOREM. *Assume that Λ and Λ_1 are both maximal. Suppose that M and M' are finitely generated projective Λ and Λ_1-modules, respectively, with lengths ≥ 3.*

Let G be a subgroup of $PGL(M)$ containing $PSL(M)$ and let

$$\Phi\colon G \to PGL(M')$$

be a monomorphism such that $\Phi G \supseteq PSL(M')$. Then there exists either

(i) *a category equivalence $F\colon \mathfrak{M}_\Lambda \to \mathfrak{M}_{\Lambda_1}$ with $F(M) = M'$ such that $\Phi = F|_G$, or*

(ii) *a category equivalence $E\colon \mathfrak{M}_{\Lambda^\circ} \to \mathfrak{M}_{\Lambda_1}$ with $E(M^*) = M'$ such that $\Phi = EC_M|_G$.*

In particular, Λ or Λ° is Morita equivalent to Λ_1 and $l(M) = l(M')$.

Proof. (1) Suppose first that A and A_1 are division algebras. Denote them by D and D_1 respectively. Set $\Lambda_1 = \Delta$ and note that Λ and Δ are integral domains with division rings of quotients D and D_1 in the sense of Section 6 of [6].

To prove the theorem in case (1) it suffices to show that the conclusions of Theorem (3.3) of [5] hold in present context (replace "rank" by "length") and then to set $F = F^g F_P$ or $E = F^h F_Q$. To verify the conclusions of this theorem proceed as in the finite rank case of its proof, making use of the following observations.

Put $V = M \otimes_\Lambda D$ and identify M with its canonical image in V under $m \to m \otimes 1$. Let $\dim V_D = n$ and note that $n \geq 3$. Since M is a fintely generated projective Λ-module, M is a full Λ-lattice in V. By (27.8) of [7],

$$M = x_1\Lambda + \cdots + x_{n-1}\Lambda + x_n\mathfrak{a},$$

where $\{x_1, \ldots, x_n\}$ is a D-basis for V and $\mathfrak{a}$ is a right Λ-ideal in D. Since $r\mathfrak{a} \subseteq \Lambda$ for a non-zero $r \in R$,

$$M \subseteq x_1\Lambda + \cdots + x_{n-1}\Lambda + (x_n r^{-1})\Lambda.$$

In particular, M is a bounded Λ-module on V_D in the sense of Section 6 of [6]. A completely analogous thing is true for M'_Δ. Since G is full of projective transvections in $P_1GL(V)$, and similarly for ΦG, one can apply the theorems of O'Meara–Sosnovskii. Then make use of (2.6).

(2) In the general case, choose division algebras D and D_1 over K and K_1 respectively, and finite dimensional vector spaces V over D and W over D_1 such that $A \cong \mathrm{End}\,(V_D)$ and $A_1 \cong \mathrm{End}\,(W_{D_1})$ as K and K_1-algebras respectively.

Now let Δ be any maximal R-order in D. By (1.3) there is an R-equivalence $F_1\colon \mathfrak{M}_\Lambda \to \mathfrak{M}_\Delta$, with associated group isomorphism

$$F_1\colon PGL(M) \to PGL(F_1(M)).$$

Refer to Section 2 of [5]. Similarly, letting Δ_1 be a maximal R_1-order in D_1, there is an R_1-equivalence $F_2\colon \mathfrak{M}_{\Lambda_1} \to \mathfrak{M}_{\Delta_1}$ with associated isomorphism

$$F_2\colon PGL(M') \to PGL(F_2(M')).$$

Applying (1.2) of [5] to an inverse of F_2, gives an equivalence

$$E_2\colon \mathfrak{M}_{\Delta_1} \to \mathfrak{M}_{\Lambda_1} \quad \text{with } E_2(F_2(M')) = M'.$$

Let E_2: $PGL(F_2(M')) \to PGL(M')$ be the associated isomorphism. Define the isomorphism Φ_1 by requiring the diagram

$$\begin{tikzcd} G \arrow[r, "\Phi"] \arrow[d, "F_1"'] & \Phi G \\ F_1 G \arrow[r, "\Phi_1"] & E_2^{-1}(\Phi G) \arrow[u, "E_2"'] \end{tikzcd}$$

to commute. Use of the properties of length shows that case (1) applies to Φ_1. So either there is

(i) an equivalence

$$F_3\colon \mathfrak{M}_\Delta \to \mathfrak{M}_{\Delta_1}$$

with $F_3(F_1(M)) = F_2(M')$ and $\Phi_1 = F_3\,|_{G'}$, or

(ii) an equivalence

$$E_3\colon \mathfrak{M}_{\Delta^\circ} \to \mathfrak{M}_{\Delta_1}$$

with $E_3(F_1(M)^*) = F_2(M')$ and $\Phi_1 = E_3\, C_{F_1(M)^*}\,|_G$.

In case (i) let $F = E_2 F_3 F_1$ to finish the proof. In case (ii) apply (2.12) and (1.2) both of [5] to show that there is an equivalence E_1: $\mathfrak{M}_{\Lambda^\circ} \to \mathfrak{M}_{\Delta^\circ}$ with $E_1(M^*) = F_1(M)^*$, such that the diagram

$$\begin{tikzcd} GL(M) \arrow[r, "C_M"] \arrow[d, "F_1"'] & GL(M^*) \arrow[d, "E_1"] \\ GL(F_1(M)) \arrow[r, "C_{F_1M^*}"'] & GL(F_1(M)^*) \end{tikzcd}$$

commutes. Using the projective version of this diagram and letting $E = E_2 E_3 E_1$ completes the proof in this case.

Since the lengths of $F_1(M)$ and $F_2(M')$ are the same, so are those of M and M' by Section 1. Q.E.D.

(3.2) Corollary. *Let M and M' be finitely generated projective modules over Λ and Λ_1, respectively, with lengths ≥ 3. Then the following are equivalent:*

(i) *There is a category equivalence F: $\mathfrak{M}_\Lambda \to \mathfrak{M}_{\Lambda_1}$ with $F(M) = M'$, or E: $\mathfrak{M}_{\Lambda^\circ} \to \mathfrak{M}_{\Lambda_1}$ with $E(M^*) = M'$.*

(ii) *$GL(M) \cong GL(M')$.*

(iii) *$PGL(M) \cong PGL(M')$.*

(iv) *$SL(M) \cong SL(M')$.*

(v) *$PSL(M) \cong PSL(M')$.*

Proof. Proceed as in the proof of (3.1a) of [5]. Use (2.2) and (2.3) above.

(3.3) *Suppose $R = R_1$, also $K = K_1$, and that A and A_1 are in the same Brauer class. Suppose that Λ and Λ_1 are both hereditary, and let n be a positive integer.*

Then there are finitely generated projective modules $M \in \mathfrak{M}_\Lambda$ and $M' \in \mathfrak{M}_{\Lambda_1}$ both of length n, such that $GL(M) \cong GL(M')$ and $PGL(M) \cong PGL(M')$.

Proof. Begin by letting Δ and Δ_1 be maximal R-orders in A and A_1 containing Λ and Λ_1 respectively. By (1.3) there is an R-equivalence F: $\mathfrak{M}_\Delta \to \mathfrak{M}_{\Delta_1}$.

Now let M_Δ be any finitely generated projective Δ-module. Put $F(M) = M'_{\Delta_1}$. By (21.6) and (21.8) of [1] (right handed versions) $M' \in \mathfrak{M}_{\Delta_1}$ is finitely generated projective. By Section 2 of [5], $GL(M_\Delta) \cong GL(M'_{\Delta_1})$ and $PGL(M_\Delta) \cong PGL(M'_{\Delta_1})$. By Section 1, $l(M_\Delta) = l(M'_{\Delta_1})$.

Since M is a right Δ-lattice, it is a right Λ-lattice. So M is a finitely generated projective Λ-module by (10.7) of [7]. By Example 1, page 378 of [7], $GL(M_\Lambda) = GL(M_\Delta)$, and since $\text{Cen}\, \Lambda = \text{Cen}\, \Delta = R$, $PGL(M_\Lambda) = PGL(M_\Delta)$. Since

$$M \otimes_\Lambda A \cong M \otimes_R K \cong M \otimes_\Delta A,$$

as right A-modules (refer to Section 1), $l(M_\Lambda) = l(M_\Delta)$. The facts developed above for M are analogously true for M'. Therefore

$$GL(M_\Lambda) \cong GL(M'_{\Lambda_1}) \quad \text{and} \quad PGL(M_\Lambda) \cong PGL(M'_{\Lambda_1}),$$

and $l(M_\Lambda) = l(M'_{\Lambda_1})$.

It remains to show that for a positive integer n, there is a finitely generated projective M_Δ with $l(M_\Delta) = n$.

Choose a division algebra D over K and a finite dimensional right vector space V over D such that $A \cong \text{End}\,(V_D)$ as K-algebras. By (21.6) of [7], there is a maximal R-order Δ_0 in D and a full right Δ_0-lattice N in V, such that $\Delta \cong \text{End}\,(N_{\Delta_0})$ by restriction. By (27.8) of [7] there is a basis $\{x_1, \ldots, x_n\}$ for V over D, and a right Δ-ideal $\mathfrak{a}$ in D, such that

$$N = x_1\Delta_0 + \cdots + x_{n-1}\Delta_0 + x_n\mathfrak{a}.$$

Clearly, $N \otimes_{\Delta_0} D \cong V$. It now follows from Section 1 that $A \otimes_\Delta N \cong V$ as left A-algebras. Therefore $l({}_\Delta N) = 1$. Hence $l(N^*_\Delta) = 1$ from Section 1. Now take an appropriate direct sum. Q.E.D.

Remarks. It is now easy to construct specific examples of hereditary orders for which the conclusions of Theorem (3.1) fail. Refer to Section 3D of [5] for example. The underlying reason for the failure seems to be that finitely generated projectives are progenerators over maximal orders and that this is no longer the case for hereditary ones. It is probable that (3.1) remains true for progenerators over hereditary orders. Indeed Bolla [3] proves the endomorphism ring analogue of (3.1) for progenerators over arbitrary rings.

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