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PROJECTIVE RESOLUTIONS OVER ARTIN ALGEBRAS WITH ZERO RELATIONS

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Introduction

The connection between categories of modules over a finite dimensional algebra over a field and categories of representations of a finite oriented graph satisfying a set of relations is well established. The finite oriented graph associated to such an algebra is called the quiver of the algebra.

More recently, coverings of the quiver of an algebra are playing an increasingly important role in the theory of representations of the algebra. On the other hand, quivers whose underlying graphs are trees, form an important and intensively studied class of such graphs.

We begin this paper by introducing a technique for computing the projective resolutions of simple representations of (possibly infinite) trees. This is done in Section 1. The section ends with an application to algebras whose quivers are trees. We show that the global dimension of such an algebra is bounded by one plus the number of relations.

In the second section we use the technique developed in Section 1 to study zero relations algebras. Although the quiver of such an algebra need not, in general, be a tree, there is a (possibly infinite) tree with relations that covers the quiver. Representations of this covering satisfying the appropriate relations are closely related to representations of the quiver of the algebra satisfying the generating relations of the algebra. In particular, we get again an algorithm for computing projective resolutions of the simple modules of such an algebra. We apply this to show that if Λ is a zero relation algebra with N generating relations, then either the global dimension of Λ is infinite, or it is bounded by $N^2 + 2$. The paper ends with a result showing that if Λ is a zero relations algebra of infinite global dimension, then the projective resolutions of the simple Λ-modules of infinite projective dimension satisfy a certain type of periodicity.

We assume the reader is familiar with the basic results connecting the category of modules over an algebra with the representations of a graph with relations [1], and also familiarity with [2] and [4] whose notation and terminology we use freely. Throughout this paper, k will denote a fixed field.

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1. The main method

Let T be an oriented (possibly infinite) tree and let ρ be a set of directed paths in T. If α is an arrow, we denote by $i(\alpha)$ its starting vertex and by $e(\alpha)$ its ending vertex. Similarly, if $r \in \rho$, $i(r)$ and $e(r)$ denote the starting vertex of the directed path r (its ending vertex respectively). We recall that a k-representation V of the tree T satisfying the relations ρ, assigns a k-vector space $V(v)$ to each vertex v of T, and a linear map $V(\alpha)$: $V(v) \to V(w)$ if there exists an arrow α: $v \to w$, and for every directed path $r = \alpha_1 \dots \alpha_n$ in ρ, then $V(\alpha_1) \dots V(\alpha_n) = 0$. Let us assume that T is locally finite, and given any vertex v, for every infinite directed path B with $i(B) = v$ there exists an $r \in \rho$ whose arrows and vertices are in B. (We recall that in a directed path all the arrows point in the *same* direction.)

We denote by rep(T, ρ) the category formed by the finite dimensional representations of T satisfying ρ, i.e., by those V such that $V(i) \neq 0$ for only a finite number of vertices and each $\dim_k V(i)$ is finite.

The morphisms of rep(T, ρ) are defined in the usual way and for further details we refer to [1].

To every vertex v of T we associate a *one dimensional representation* $S(v)$, having k assigned to v and 0 to every other vertex; for each arrow α, $V(\alpha)$ is the zero map. It is well known that rep(T, ρ) has enough projectives and that every finite dimensional representation has a unique projective cover. For a vertex v in T, we denote by $P(v)$ *the projective cover of* $S(v)$. Obviously $P(v)$ is an indecomposable representation. We note that $P(v)$ is isomorphic to the representation of T such that

$$V(w) = \begin{cases} k & \text{if there is a directed path from } v \text{ to } w \text{ containing no } r \in \rho \\ 0 & \text{otherwise} \end{cases}$$

and, if α is an arrow, then

$$V(\alpha) = \begin{cases} \text{identity} & \text{if } V(i(\alpha)) = V(e(\alpha)) = k \\ 0 & \text{otherwise} \end{cases}$$

DEFINITION. Let v, w be two vertices of T. Then $T(v, w)$ in the unique directed path (if it exists) with the property that $T(v, w)$ is a subtree of T, the start of $T(v, w)$ is v and its ending vertex is w.

Now let $T(v, w)$ be a subtree of T and let $\{v = x_1, \dots, x_n = w\}$ be the vertices of $T(v, w)$. Let $X = P(x_1) \amalg \cdots \amalg P(x_n) \in \text{rep}(T, \rho)$. Let us consider the functor

$$\alpha_X = \operatorname{Hom}_T(X, \cdot): \text{rep}(T, \rho) \to \operatorname{mod} \operatorname{End}_T(X)^{\text{op}},$$

where the latter is the category of the finitely generated left modules over the finite dimensional k-algebra $\mathrm{End}_T(X)^{\mathrm{op}}$.

Clearly $\mathrm{mod}\,\mathrm{End}_T(X)^{\mathrm{op}}$ is equivalent to $\mathrm{rep}(T(v,w),\rho')$ where ρ' is the restriction of ρ to $T(v,w)$. Let us denote by $Q(x_i) = \alpha_X(P(x_i))$, the indecomposable projective $T(v,w)$-representations, and by $U(x_i) = \alpha_X(S(x_i))$ the nonisomorphic simple $T(v,w)$-representations.

The next proposition will enable us to reduce the problem of finding the projective resolution of the simple T-representation $S(v)$ to the easier problem of determining the projective resolution of a $T(v,w)$-simple representation. First, we need the following:

Remarks. If a and b are two vertices of T, then $\mathrm{Hom}_T(P(a),P(b)) \neq 0$ implies the existence of a directed path from b to a. Moreover since there is at most one path from b to any other vertex we see that if x is a vertex and $S(x)$ is the simple T-representation then $S(x)$ is a composition factor of $P(b)$ at most once. More generally, consider

$$\cdots \to A_n \xrightarrow{f_n} A_{n-1} \xrightarrow{f_{n-1}} \cdots \to A_1 \xrightarrow{f_1} A_0 \to S(v) \to 0$$

be a minimal projective resolution of the simple T-resolution $S(v)$. Then for a given vertex x, $S(x)$ is a composition factor of at most one A_i and of multiplicity one for that A_i. The reason is as follows: As above, the result is true for A_0. If $P(y)$ is a summand of A_i then there is a directed path from b to y since $A_0 = P(b)$, $f_1, f_2, \dots, f_i$ is a sequence of nonzero homomorphisms, and, by minimality, $f_i(P(y)) \neq 0$. Now if $S(x)$ is a composition factor of A_i then $S(x)$ is a composition factor of some $P(y)$ which is a summand of A_i. Thus there is a directed path from b to x. Since this path is unique the result follows. The fact that for trees,

$$\dim_k \mathrm{Ext}^i_T(S(v),S(w)) \leq 1$$

will be used in the next section.

PROPOSITION 1.1. *Let v, w be vertices in T.*

(a) *Assume that there exists a j and a vertex x in $T(v,w)$ such that*

$$\mathrm{Ext}^j_{T(v,w)}(U(v),U(x)) \neq 0.$$

Then

$$\mathrm{Ext}^j_T(S(v),S(x)) \neq 0.$$

(b) *Let v, w be such that for some $i \geq 1$,*

$$\mathrm{Ext}^i_T(S(v),S(w)) \neq 0.$$

Then

$$\mathrm{Ext}^i_{T(v,w)}(U(v), U(w)) \neq 0.$$

Proof. (a) Let $X = P(x_1) \amalg P(x_2) \amalg \cdots \amalg P(x_n)$ where $\{x_1, x_2, \dots x_n\}$ are the vertices of $T(v, w)$ with $v = x_1$ and $w = x_n$. Let

$$\cdots \to A_m \xrightarrow{f_m} A_{m-1} \to \cdots \to A_1 \xrightarrow{f_1} A_0 \xrightarrow{f_0} S(v) \to 0$$

be a minimal projective resolution $\mathscr{P}$ of $S(v)$ over $\mathrm{rep}(T, \rho)$. If $x \notin \{x_1, \dots, x_n\}$ and if $P(x)$ is a summand of A_j for some j, then there is a directed path from v to x and no directed path from x to any of the x_i for $i = 1, \dots n$. Therefore by our preceding remark, $\mathrm{Hom}_T(X, P(x)) = 0$. Therefore $\mathrm{Hom}_T(X, \mathscr{P})$ is a projective resolution of $U(v)$ over $T(v, w)$. To show its minimality, it is enough to show that each term of $\mathrm{Hom}_T(X, \mathscr{P})$ is indecomposable. This is clear again by the remarks preceding the proposition.

(b) Let $\mathscr{P}$ be a minimal projective resolution of $S(v)$,

$$\cdots \to A_m \xrightarrow{f_m} A_{m-1} \to \cdots \to A_0 \xrightarrow{f_0} S(v) \to 0$$

with $P(w)$ a direct summand of A_i. Then, for each $j = 0, \dots, i$ let x_j be such that $P(x_j)$ is a summand of A_j and there is a nonzero homomorphism $P(x_j) \to P(x_{j-1})$ with $P(w) = P(x_i)$. Then, let $T(v, w)$ be the complete subtree of T having $\{x_1, \dots x_i\}$ as subset of vertices. Using the remarks made earlier the proposition follows.

DEFINITION. Let T be a directed tree and v, w two vertices. Then, we define $v < w$ if there exists a directed path from v to w. Let (T, ρ) be a tree with relations. We assume that the set of relations is "minimal"; that is, $r \in \rho$ implies that r is not a proper subpath of any $s \in \rho$. We also assume that if $r \in \rho$, $l(r) \geq 2$ and $l(r) < \infty$ where $l(r)$ denotes the length of r, i.e., the number of arrows in r. We recall that if $r \in \rho$, by $i(r)$ we denote the origin of the directed path r and by $e(r)$ its end.

For the remainder of this section we assume that (B, ρ) is a *directed path* with minimal set of relations ρ. Again we assume that if v is any vertex of B, then there exists $r \in \rho$ such that $v \leq i(r)$ if the subpath originating from v is infinite. Before we go on with our next result we establish some notations that will greatly simplify our proofs.

First, given an indecomposable representation $M \in \mathrm{rep}(B, \rho)$ we write it as a closed interval $M = [a, b]$ where $S(a) = M/\mathbf{r}M$ where $\mathbf{r}M$ is the radical of the representation and $S(b) = \mathrm{Soc}\, M$. Given the structure of (B, ρ) if c is a vertex such that $\mathrm{Hom}_B(P(c), M) \neq 0$ then $a \leq c \leq b$.

Also, if M and N are indecomposable representations, we have a nonzero homomorphism f: $M \to N$ where $M = [a, b]$ and $N = [c, d]$ iff $c \leq a \leq d$ and $b \geq d$. In this case $\operatorname{Ker} f = 0$ if $b = d$ and $\operatorname{Ker} f = [d+1, b]$ if $b > d$. Since every $r \in \rho$ can also be expressed as an interval (x, y) it follows immediately that an indecomposable representation $M = [a, b]$ is projective if and only if either b is a sink (that is, there are no arrows starting at b) or there is a relation $r \in \rho$ such that $a \leq i(r)$, $e(r) = b + 1$ if b is not a sink. Next, given the structure of the indecomposable projective B-representations, it is clear that if $f \neq 0$ is a morphism between two indecomposable projective B-representations, then f is unique up to multiplication by an element of the ground field. Since the description of such nonzero homomorphisms is obvious, it is enough to give the terms of the minimal projective resolution of a B-representation in order to completely describe it.

Before starting our next result, we observe that since the set of relations ρ is minimal, given a vertex v and $r, r' \in \rho$ such that $i(r) = i(r') = v$, then $r = r'$.

Definition. Let v_0 be a vertex of a directed path B with minimal set of relations ρ. We define inductively the *associated sequence of relations* $\mathscr{S}$ of v_0 (along B): If there is no $r \in \rho$ such that $i(r) = v_0$ then $\mathscr{S} = \emptyset$. Assume there is $r_1 \in \rho$ such that $i(r_1) = v_0$. Then, let r_2 be the relation $r \in \rho$ (if it exists) having the property that $i(r_1) < i(r_2) < e(r_1)$ and that $i(r_2)$ is minimal satisfying this double inequality.

Assume that we have constructed $r_1, r_2, \ldots, r_i$. Let

$$R_{i+1} = \{ r \in \rho \mid e(r_{i-1}) \leq i(r) < e(r_i) \}.$$

If $R_{i+1} \neq \emptyset$, let r_{i+1} be such that $i(r_{i+1})$ is minimal with $r_{i+1} \in R_{i+1}$. Then $\mathscr{S}$ is the sequence $r_1, r_2, \ldots, r_N$ where N is either ∞ if $R_i \neq \emptyset$ for all i, or N is such that $R_{N+1} = \emptyset$ but $R_N \neq \emptyset$.

Remark. From our definition we see that given a vertex of B, its associated sequence of relations may be empty, finite or infinite.

Theorem 1.2. *Let (B, ρ) be a directed path with minimal set of relations ρ. Let v_0 be a vertex of B and let v_1 be such that there exists an arrow $v_0 \to v_1$. Let $\mathscr{S} = \{r_i\}_{i=1}^N$ be the associated sequence of relations of v_0. Let $x_i = e(r_i)$.*

(a) *If $\mathscr{S} = \emptyset$, then $0 \to P(v_1) \to P(v_0) \to S(v_0) \to 0$ is a minimal projective resolution of the simple B-representation $S(v_0)$.*

(b) *Suppose $\mathscr{S} \neq \emptyset$. If $N < \infty$, then*

$$0 \to P(x_N) \to P(x_{N-1}) \to \cdots \to P(x_1) \to P(v_1) \to P(v_0) \to S(v_0) \to 0$$

is a minimal projective resolution of $S(v_0)$, and if $N = \infty$ then $pd_B S(v_0) = \infty$

and its minimal projective resolution is

$$\cdots \to P(x_i) \to P(x_{i-1}) \to \cdots \to P(x_1) \to P(v_1) \to P(v_0) \to S(v_0) \to 0.$$

Proof. (a) Let $P(v_0) = [v_0, a]$. Since there is no $r \in \rho$ such that $i(r) = v_0$ we have the following possibilities: a is a sink (i.e., no arrow leaves a and in this case $P(v_1) = [v_1, a]$), or there is an $r \in \rho$ such that $i(r) > v_0$. Let r' be such that for every $r \in \rho$ with $i(r) > v_0$, $v_0 < i(r') \leq i(r)$. Then $a = e(r') - 1$. Clearly $v_1 \leq i(r')$ so it follows again that $P(v_1) = [v_1, a]$.

Therefore $0 \to P(v_1) \to P(v_0) \to S(v_0) \to 0$ is a minimal projective resolution of $S(v_0)$.

(b) Induction on N. If $N = 1$ let $x_1 = e(r_1)$. We have

$$P(v_0) = [v_0, x_1 - 1]$$

and so

$$\Omega^1(v_0) = [v_1, x_1 - 1].$$

By the minimality of ρ, $P(v_1) = [v_1, a]$ with $a > x_1$, therefore $\Omega^2(v_0) = [x_1, a]$ and therefore $P(x_1)$ maps onto $\Omega^2(v_0)$. Assume we have proved our statement up to the i-th step, that is $r_1, r_2, \ldots r_{i-1}$ exist. We have to distinguish between two cases: when r_i exists and when it doesn't. We have $\Omega^i(v_0) \subset P(x_{i-2})$ and

$$0 \to \Omega^{i+1}(v_0) \to P(x_{i-1}) \to \Omega^i(v_0) \to 0$$

by the induction hypothesis. Therefore $\Omega^i(v_0) = [x_{i-1}, a]$ where $P(x_{i-2}) = [x_{i-2}, a]$.

So $\Omega^i(v_0)$ is projective iff either a is a sink, or there is a relation $r \in \rho$ such that $i(r) \geq x_{i-1}$ and $e(r) = a + 1$. If r_i *does not exist*, then it follows immediately that $\Omega^i(v_0)$ is projective.

If r_i *exists*, then $a = x_i$ and obviously, from the minimality of ρ and the above comments, $\Omega^i(v_0)$ is not projective. Since

$$\Omega^i(v_0) = [x_{i-1}, x_i - 1] \quad \text{and} \quad P(x_{i-1}) = [x_{i-1}, b] \text{ with } b > x_i,$$

we see that $\Omega^{i+1}(v_0)$ is $[x_i, b]$ and its projective cover is $P(x_i)$.

Remark. As a first application we see that if Λ is an artin algebra whose quiver is given by a tree, then Proposition 1.1, together with the preceding theorem give us an algorithm for computing the projective resolutions of the simple Λ-modules.

COROLLARY 1.3. *Let Λ be an artin algebra whose quiver is given by a tree T together with a minimal set of relations ρ. Let n denote the cardinality of ρ. Then* $\operatorname{gl}\dim \Lambda \leq n + 1$.

Proof. Follows immediately from the previous theorem and from Proposition 1.1.

We must remark that there are numerous examples where we have equality in Corollary 1.3.

The next result will be useful for the next section. First we define $\mathrm{dist}(v, w)$ for $v < w$, where v, w are two vertices of T, to be the number of arrows in the unique directed path joining v and w.

PROPOSITION 1.4. *Let T be a directed tree with a set of minimal relations ρ. Let v_0 be a vertex of T. Assume that there is an l that bounds the lengths of all the relations in ρ. Let w be a vertex of T such that*

$$\mathrm{Ext}_T^n(S(v_0), S(w)) \neq 0.$$

Then $\mathrm{dist}(v_0, w) \leq (n-1)l$.

Proof. By Proposition 1.1 it is clear that it is enough to prove the proposition for the case T is a directed path containing v_0 and w. Let

$$\cdots \to P(w) \xrightarrow{f_{n-1}} P(x_{n-2}) \to \cdots \to P(v_1) \xrightarrow{f_1} P(v_0) \xrightarrow{f_0} S(v_0) \to 0$$

be a minimal projective resolution of $S(v_0)$ corresponding to the associated sequence of relations to v_0, as constructed in Theorem 1.2, with $x_i = e(r_i)$ and $w = x_{n-1} = e(r_{n-1})$. Since $d(x_i, x_{i+1}) \leq l$ for every i, we are done.

2. Zero relations algebras

Throughout this section we use the notations and terminology of [2] and [4]. We recall that a finite dimensional k-algebra with quiver Q, is called *a zero relations algebra* if the algebra is isomorphic to kQ/I where kQ denotes the path k-algebra associated to the oriented quiver Q and I is a two sided ideal of kQ generated by a set of paths of length ≥ 2 in Q.

Let Λ be a zero relation algebra and fix an identification of Λ with kQ/I where Q is the quiver of Λ and I is generated by paths $r_1, \ldots, r_N$. We assume that the set of relations $\rho = \{r_1, \ldots, r_N\}$ is minimal in the sense that if I is the ideal in kQ generated by $r_1, \ldots, r_N$, then the ideal generated by any proper subset of ρ is properly contained in I. We note that since Λ is a finite dimensional k-algebra we have the well known result:

LEMMA 2.1. *There exists a positive integer M such that each directed path in Q of length greater than M is in I.*

Considering (Q, ρ) as a graph with relations, since ρ consists of directed paths, the *universal cover* of (Q, ρ), Φ: $(\tilde{Q}, \tilde{\rho}) \to (Q, \rho)$ consists of the topological universal cover Φ: $\tilde{Q} \to Q$ of Q together with $\tilde{\rho}$, the set of all paths in $\tilde{Q}$ which map via Φ to a path in ρ. Thus $\tilde{Q}$ is a tree, which, if Q contains a (possibly non oriented) cycle, is infinite. In any case $\tilde{Q}$ is locally finite. Let F: $\operatorname{rep}(\tilde{Q}, \tilde{\rho}) \to \operatorname{rep}(Q, \rho)$ be the functor induced from Φ, mapping the category of finite dimensional k-representations of $\tilde{Q}$ satisfying $\tilde{\rho}$ to the category of finite dimensional k-representations of Q satisfying ρ.

Recall that $\operatorname{rep}(Q, \rho)$ is equivalent to $\operatorname{mod} \Lambda$, and there is a grading of Λ given by the fundamental group of the underlying graph of Q, so that $\operatorname{rep}(\tilde{Q}, \tilde{\rho})$ is equivalent to $\operatorname{gr}(\Lambda)$, the category of finitely generated graded Λ-modules and degree 1 maps, where 1 denotes the identity element of the fundamental group. We have a commutative diagram:

$$\begin{array}{ccc} \operatorname{rep}(\tilde{Q}, \tilde{\rho}) & \approx & \operatorname{gr}(\Lambda) \\ \downarrow{\scriptstyle F} & & \downarrow \\ \operatorname{rep}(Q, \rho) & \approx & \operatorname{mod} \Lambda, \end{array} \tag{2.2}$$

where $\operatorname{gr}(\Lambda) \to \operatorname{mod} \Lambda$ is the forgetful functor. Using (2.2) we identify $\operatorname{mod} \Lambda$ with $\operatorname{rep}(Q, \rho)$ and $\operatorname{gr}(\Lambda)$ with $\operatorname{rep}(\tilde{Q}, \tilde{\rho})$. We say that an object in $\operatorname{rep}(Q, \rho)$ is *gradable* if it is in the image of F. By arguments similar to those of [2], we see immediately that the indecomposable projective and the simple Λ-modules are gradable.

Moreover, if

$$\cdots \to A_n \xrightarrow{f_n} A_{n-1} \to \cdots \to A_0 \xrightarrow{f_0} X \to 0$$

is a minimal projective resolution in $\operatorname{rep}(\tilde{Q}, \tilde{\rho})$, then

$$\cdots F(A_n) \xrightarrow{F(f_n)} F(A_{n-1}) \to \cdots \to F(A_0) \xrightarrow{F(f_0)} F(X) \to 0$$

is a minimal projective resolution of $F(X)$ in $\operatorname{rep}(Q, \rho)$. Thus, questions relating to projective resolutions of simple Λ-modules can be "lifted" to questions about projective resolutions of simple representations of $\tilde{Q}$ satisfying $\tilde{\rho}$. In regard to this, we have:

THEOREM 2.3. *Let Λ be a finite dimensional zero relations k-algebra. Then, the construction of the projective resolution of the simple Λ-modules is algorithmic. That is, the determination of the projective Λ-modules in a minimal projective resolution of a simple Λ-module reduces to the construction of associated sequences in tree $\tilde{Q}$ with relations $\tilde{\rho}$. In particular, keeping the notations introduced above and in* §1, *let S_1 and S_2 be simple Λ-modules. Suppose v_i is the* *vertex in Q associated to S_i, $i = 1, 2$. Let v_1^* be a vertex in $\tilde{Q}$ lying over v_1; i.e., $\Phi(v_1^*) = v_1$. Then, for $n \geq 2$,*

$$\dim_k \operatorname{Ext}^n_\Lambda(S_1, S_2) = \sum_{\substack{v_2^* \in \Phi^{-1}(v_2) \\ \operatorname{dist}(v_1^*, v_2^*) \leq (n-1)L}} \dim_k \operatorname{Ext}^n_{\tilde{Q}}\big(S(v_1^*), S(v_2^*)\big)$$

where L is the length of the longest path in ρ.

Proof. Let S_1 and S_2 be simple Λ-modules and v_1 and v_2 the vertices of Q associated to S_1 and S_2 respectively. Let v_1^* be a vertex in $\tilde{Q}$ such that $\Phi(v_1^*) = v_1$. Then $F(S(v_1^*)) = S(v_1) \approx S_1$ and $S(v_2) \approx S_2$. By the relationship given by the covering Φ: $(\tilde{Q}, \tilde{\rho}) \to (Q, \rho)$ we see that if $\tilde{v}$ is a vertex in $\tilde{Q}$, then $F(S(\tilde{v})) = S(v_2)$ iff $\Phi(\tilde{v}) = v_2$. Thus for $n \geq 0$ we have

$$\operatorname{idim}_k \operatorname{Ext}^n_\Lambda(S_1, S_2) = \sum_{v^* \in \Phi^{-1}(v_2)} \dim_k \operatorname{Ext}^n_{\tilde{Q}}\big(S(v_1^*), S(v^*)\big)$$

As noted earlier, $\tilde{\rho}$ consists of all paths in $\tilde{Q}$ which map by Φ to a path in ρ. Thus, if L is the length of the longest path in ρ, it follows that L also bounds the lengths of the paths in $\tilde{\rho}$. Therefore, applying Proposition 1.4 and 2.1, for $n \geq 2$ we have

$$\dim_k \operatorname{Ext}^n_\Lambda(S_1, S_2) = \sum_{\substack{v^* \in \Phi^{-1}(v_2) \\ \operatorname{dist}(v_1^*, v^*) \leq (n-1)L}} \dim_k \operatorname{Ext}^n_{\tilde{Q}}\big(S(v_1^*), S(v^*)\big).$$

By the results of Section 1, the construction of the projective resolutions of the simple representations is algorithmic. Furthermore, the computation of the first n terms of the Λ-projective resolution of S_1 for $n \geq 2$, is given by computing the $(\tilde{Q}, \tilde{\rho})$-projective resolution of $S(v_1^*)$ which is determined by the finite full subgraph $\tilde{Q}_0$ of $\tilde{Q}$ with vertex set

$$\big\{ v^* \colon \operatorname{dist}\big(v_1^*, v^*\big) \leq (n-1)L \big\}$$

and relation set consisting of those relations of $\tilde{\rho}$ which are paths in $\tilde{Q}_0$. Therefore the computations are finite.

The next result is, in a sense, an application of the above result.

THEOREM 2.4. *Let Λ be a finite dimensional zero relation k-algebra. Keeping the notation of this section, let N be the number of relations in the relation set ρ. Then, if there exists a simple Λ-module S, such that $pd_\Lambda S \geq N^2 + 3$, then* $\operatorname{gl\,dim} \Lambda = \infty$.

Proof. As above let Φ: $(\tilde{Q}, \tilde{\rho}) \to (Q, \rho)$ be the universal cover of (Q, ρ). Suppose that S is a simple Λ-module with $pd_\Lambda S \geq N^2 + 3$. Let v_0 be the vertex of Q associated to S and let v_0^* be a vertex in $\tilde{Q}$ so that $\phi(v_0^*) = v_0$. By (2.3) there exists a simple representation $S(v^*)$ of rep$(\tilde{Q}, \tilde{\rho})$ such that $\mathrm{Ext}_{\tilde{Q}}^{N^2+3}(S(v_0^*), S(v^*)) \neq 0$. By (1.1) and (1.2) there must be a path p from v_0^* to v^* and an associated sequence of relations on that path $r_1, r_2, \dots, r_{N^2+2}$. Let us consider the pairs $(\Phi(r_i), \Phi(r_{i+1}))$ of relations in $\rho \times \rho$, for $i = 1, \dots, N^2 + 1$. Since the cardinality of ρ is N, we conclude that there must be integers a, b such that $1 \leq a < b \leq N^2$ such that

$$(\Phi(r_a), \Phi(r_{a+1})) = (\Phi(r_b), \Phi(r_{b+1})).$$

Set $v_1 = \Phi(e(r_{a+1}))$. Now, consider the path p in $\tilde{Q}$ again. As one moves along p from v_0^* to v_1^*, one passes first through $e(r_{a+1})$, then through $e(r_{b+1})$. Let p_1 be the subpath of p from v_0^* to $e(r_{a+1})$ and p_2 be the subpath of p from $e(r_{a+1})$ to $e(r_{b+1})$. Then $\Phi(p_2)$ is an oriented cycle with origin and terminus $v_1 = \Phi(e(r_{a+1}))$. Finally consider the infinite path q in Q consisting of first $\Phi(p_1)$ and then, following the cycle $\Phi(p_2)$ a countable number of times. The path q lifts uniquely to an infinite path $\tilde{q}$ in $\tilde{Q}$ starting at v_0^*. Note that $\tilde{q}$ coincides with p until at least $e(r_{b+1})$.

Thus we get an associated sequence for v_0^* on $\tilde{q}$ beginning with $r_1, r_2, \dots, r_{a+1}, \dots r_{b+1}$.

From the definition of the associated sequence of relations along a path, we see that to determine the $(b + 2)$-nd relation we only need to know the vertex $e(r_b)$ and $i(r_{b+1})$ and the starting vertices of relations starting between $i(r_{b+1})$ and $e(r_b)$. But after p_1, $\tilde{q}$ follows over various liftings of the same cycle path from $\phi(e(r_{b+1})) = e(\phi(r_{a+1}))$ and r_{a+2} which lies on this cycle is determined by r_a and r_{a+1}. From this, we conclude that if $r_1^*, r_2^*, \dots, r_M^*$ is the associated sequence of relations for v_0^* along $\tilde{q}$, then $M = \infty$ and $r_i^* = r_i$ for $i = 1, \dots, b + 1$ and $\Phi(r_{lb+j}^*) = \Phi(r_{a+j})$ for $l \geq 1$ and $0 \leq j \leq b - a - 1$. From Theorem 1.2(b) we find that $pd_\Lambda S = \infty$.

The fact that we were able to construct a path $\tilde{q}$ in $\tilde{Q}$ where $\Phi(\tilde{q})$ is a path in Q which eventually winds up around a fixed cycle in Q, leads to a certain type of periodicity in the projective resolution of the simple Λ-module S. The next result makes this precise.

COROLLARY 2.5. *Let Λ be a finite dimensional zero relation k-algebra. Suppose S is a simple Λ-module with infinite projective dimension. Let*

$$\cdots A_n \xrightarrow{f_n} A_{n-1} \to \cdots \to A_1 \xrightarrow{f_1} A_0 \xrightarrow{f_0} S \to 0$$

be a minimal projective resolution of S. Then, there exists a sequence of indecomposable projective Λ-modules $P_1, P_2, \dots, P_t$ and a positive integer l such that P_i is a direct summand of A_{l+mt+i} for all $m \geq 0$ and $i = 1, \dots, t$.

Proof. The result is a direct consequence of the proof of (2.4). Namely, using the notation of the proof of (2.4), since $pd_\Lambda S \geq N^2 + 3$, we may construct the path $\tilde{q}$ in $\tilde{Q}$ with associated sequence of relations $r_1^*, \ldots, r_{a-1}^*, r_a^*, r_{a+1}^*, \ldots, r_b^*$. Let $\tilde{x}_j = i(r_{la+j})$ for $j = 1, \ldots, b-a$, for $l \geq 1$. Then, by setting $P_j = P(\Phi(\tilde{x}_j))$ our result follows with $t = b - a$, $l = a - 1$.

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