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HARMONIC AND SUPERHARMONIC FUNCTIONS ON COMPACT SETS

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In [4], T.W. Gamelin gives necessary and sufficient conditions which ensure that every continuous function on a compact subset K of $\mathbf{R}^2$, harmonic on the interior of K, can be approximated uniformly on K by functions harmonic in a neighborhood of K. Here we shall see that using [1] and [3] a stronger version of the same result can be proved even for arbitrary harmonic spaces.

In the following let K be a compact subset of a $\mathscr{P}$-harmonic space $(X, {}^*\mathscr{H})$ and let $\mathscr{C}(K)$ denote the space of all continuous real functions on K. For every finely open set V contained in K let $H(K, V)$ (resp. $S(K, V)$) be the set of all functions $g \in \mathscr{C}(K)$ such that $\varepsilon_x^{\complement G}(g) = g(x)$ (resp. $\varepsilon_x^{\complement G}(g) \leq g(x)$) for every $x \in V$ and every fine neighborhood G of x such that $\overline{G} \subset V$. The functions in $H(K, V)$ (resp. $S(K, V)$) are called finely harmonic (resp. finely superharmonic) on V. Evidently, $H(K, V) = S(K, V) \cap (-S(K, V))$.

This definition is useful in our context because of the following two facts. If V is open then $H(K, V)$ (resp. $S(K, V)$) is the set of all functions in $\mathscr{C}(K)$ which are harmonic (resp. superharmonic) on V [3, p. 264]. Furthermore, if V is the fine interior of K then $H(K, V)$ (resp. $S(K, V)$) is the uniform closure of the set $H(K)$ (resp. $S(K)$) of all functions in $\mathscr{C}(K)$ which are restrictions of harmonic (resp. superharmonic) functions on a neighborhood of K [1, p. 105], [3, p. 269].

A characterization of the Choquet boundary $Ch_{S(K,V)}K$ of K with respect to $S(K, V)$ involves the essential base of $\complement V$. Let us recall that for every subset A of X the base $b(A)$ of A is the set of all points $x \in X$ such that A is not thin at x whereas the essential base $\beta(A)$ of A (called quasi-base $\rho(A)$ in [5]) is the set of all points $x \in X$ such that A is not semi-polar at x, i.e., such that for every fine neighborhood V of x the set $A \cap V$ is not semi-polar. We note that $\beta(A)$ is the smallest finely closed subset F of X such that $A \setminus F$ is semi-polar. Moreover, if A is finely closed then $\beta(A)$ is the largest subset F of A such that $b(F) = F$.

If $V \subset K$ is finely open then

$$Ch_{S(K,V)}K = K \cap \beta(\complement V)$$

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and for every $x \in V$ the measure

$$\mu_x^V := \varepsilon_x^{\beta(\mathbf{C}V)}$$

is the minimal representing measure of x with respect to $S(K, V)$ (where we may replace $S(K, V)$ by $H(K, V)$ if $H(K, V)$ is linearly separating) [1, p. 101, 103]. If every semi-polar subset of X is polar (as for example in the classical case) then $\beta = b$ and $\mu_x^V = \varepsilon_x^{\mathbf{C}V}$; i.e., for open sets V the Choquet boundary $Ch_{S(K,V)}K$ is the set of all regular boundary points of V and the measures μ_x^V are the corresponding harmonic measures.

Let $\mathscr{P}$ denote the convex cone of all continuous real potentials on X. For every $p \in \mathscr{P}$, the fine support $\delta(p)$ is by definition the Choquet boundary of X with respect to $\mathscr{P} + \mathbf{R}p$. In fact, $\delta(p)$ is the smallest finely closed subset F of X such that p is finely harmonic on $\mathbf{C}F$, and the closure $C(p)$ of $\delta(p)$ is the smallest closed subset C of X such that p is harmonic on $\mathbf{C}C$.

LEMMA. *Let U and V be finely open subsets of X such that $V \setminus U$ is not semi-polar. Then there exists a potential $p \in \mathscr{P}$ such that $p \neq 0$ and $C(p) \subset V \setminus U$.*

Proof. Let $B(X)$ be the set of all Borel subsets of X. It is well known that there exist finely open sets $U', V' \in B(X)$ such that $U' \subset U$, $V' \subset V$ and the differences $U \setminus U'$ and $V \setminus V'$ are semi-polar. Moreover, there exists a semi-polar set $S \in B(X)$ such that $U \setminus U' \subset S$. Then

$$B := V' \setminus (U' \cup S) \in B(X),$$

$B \subset V \setminus U$ and B is not semi-polar since $V \setminus U \subset (V \setminus V') \cup B \cup S$.

By [5, p. 501], there exists a potential $p \in \mathscr{P}$ such that $p \neq 0$ and $C(p) \subset B$.

THEOREM. *Let U and V be finely open subsets of X which are contained in K. Then the following statements are equivalent*:

(1) $S(K, U) \subset S(K, V)$.

(2) $H(K, U) \subset H(K, V)$.

(3) $H(K, U) \subset S(K, V)$.

(4) *$V \setminus U$ is semi-polar.*

Proof. That (1) $\Rightarrow$ (2) $\Rightarrow$ (3) is obvious.

Suppose now that $V \setminus U$ is not semi-polar. Then by the preceding lemma there exists a potential $p \in \mathscr{P}$ such that $p \neq 0$ and $C(p) \subset V \setminus U$. Let $x \in \delta(p)$ and let L be a compact fine neighborhood of x in V. Then $\varepsilon_x^{\mathbf{C}L} \neq \varepsilon_x$ and hence $\varepsilon_x^{\mathbf{C}L}(p) < p(x)$. Therefore the restriction of $-p$ to K is not finely superharmonic on V, but it is evidently finely harmonic on U. This shows that (3) implies (4).

Suppose finally that $V \setminus U$ is semi-polar. Then $\beta(\complement U) \subset \beta(\complement V)$. Let $s \in S(K, U)$, $x \in V$ and let G be a fine neighborhood of x such that $\overline{G} \subset V$. Since $\beta(\complement U) \subset \beta(\complement V) \subset \complement V \subset \complement G$ we conclude by [3, p. 264] that $\varepsilon_x^{\complement G}(s) \leq s(x)$. Thus $s \in S(K, V)$.

We note the following consequence which has already been proved in [1] and [3] in a slightly different way.

COROLLARY 1. *The following statements are equivalent*:

(1) $\overline{S(K)} = \mathscr{C}(K)$.

(2) $\overline{H(K)} = \mathscr{C}(K)$.

(3) *K has no finely interior points.*

Proof. Evidently, $S(K, \emptyset) = H(K, \emptyset) = \mathscr{C}(K)$. Let V be the fine interior of K. V is the empty set if and only if V is semi-polar. Thus the equivalences follow from the theorem since $\overline{S(K)} = S(K, V)$ and $\overline{H(K)} = H(K, V)$.

Let $\mathring{K}$ denote the interior of K and $\partial K = K \setminus \mathring{K}$ the boundary of K. Furthermore, let V be the fine interior of K. We recall that the points of

$$K \cap b(\complement K) = \partial K \cap b(\complement K)$$

are called stable boundary points of K. Hence

$$V = K \setminus b(\complement K) = \mathring{K} \cup (\partial K \setminus b(\complement K))$$

is the union of $\mathring{K}$ and the set $\partial K \setminus b(\complement K)$ of all unstable boundary points of K. Moreover, we note that

$$\beta(\complement V) = \beta(b(\complement K)) = b(\complement K)$$

and therefore the Choquet boundary $Ch_{S(K)}K$ is the set $K \cap b(\complement K)$ of all stable boundary points of K.

COROLLARY 2. *The following statements are equivalent*:

(1) $\overline{S(K)} = S(K, \mathring{K})$.

(2) $\overline{H(K)} = H(K, \mathring{K})$.

(3) $\overline{S(K)} \supset H(K, \mathring{K})$.

(4) *The set of all unstable boundary points of K is semi-polar.*

(5)(a) *For every $x \in \mathring{K}$, the measure $\mu_x^{\mathring{K}}$ is supported by the set of all stable boundary points of K.*

(b) *∂K has no finely interior points.*

(6) $\beta(\partial K) \subset b(\mathbf{C}K)$.

Proof. (1) ⇔ (2) ⇔ (3) ⇔ (4). Immediate consequence of the theorem.

(4) ⇒ (5). The fine interior of ∂K is empty since it is contained in the semi-polar set $\partial K \setminus b(\mathbf{C}K)$. Moreover, $\beta(\mathbf{C}\mathring{K}) = \beta(\mathbf{C}\mathring{V}) = b(\mathbf{C}K)$. Hence the measures $\varepsilon_x^{\beta(\mathbf{C}\mathring{K})}$, $x \in \mathring{K}$, are supported by the set $K \cap b(\mathbf{C}K)$ of all stable boundary points of K.

(5) ⇒ (6). Let $p \in \mathscr{P}$. By (5)(a) and [2, p. 75],

$$R_p^{\beta(\mathbf{C}\mathring{K})} = R_p^{b(\mathbf{C}K)} \quad \text{on } \mathring{K}$$

and hence

$$R_p^{\beta(\mathbf{C}\mathring{K})} = R_p^{b(\mathbf{C}K)} \quad \text{on } b(\mathring{K}).$$

Furthermore, trivially

$$R_p^{\beta(\mathbf{C}\mathring{K})} \geq R_p^{b(\mathbf{C}K)} = p \quad \text{on } b(\mathbf{C}K).$$

By (5)(b), the open set $\mathring{K} \cup \mathbf{C}K$ is finely dense in X; i.e., $b(\mathring{K}) \cup b(\mathbf{C}K) = X$. Therefore

$$R_p^{\beta(\mathbf{C}\mathring{K})} = R_p^{b(\mathbf{C}K)}.$$

This shows that $\beta(\mathbf{C}\mathring{K}) = b(\mathbf{C}K)$ and hence $\beta(\partial K) \subset b(\mathbf{C}K)$.

(6) ⇒ (4). The set $A := \partial K \setminus b(\mathbf{C}K)$ of all unstable boundary points of K satisfies

$$\beta(A) \subset \beta(\partial K) \subset b(\mathbf{C}K) \subset \mathbf{C}A.$$

Thus $A = A \setminus \beta(A)$ is semi-polar.

Remark. In view of Corollary 1 the result of [4] is the equivalence of (2) and (5) in Corollary 2 for the special case of classical potential theory on $\mathbf{R}^2$.

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