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M-IDEALS OF L^∞/H^∞ AND SUPPORT SETS

BY

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1. Introduction

Let L^∞ be the usual space of bounded measurable functions on the unit circle T. Let H^∞ denote the subalgebra of L^∞ consisting of functions on T that are radial limits of bounded analytic functions of the open unit disk, and $H^\infty + C$ denotes the closed linear span of H^∞ and C, where C is the space of continuous functions on T. The norm of an L^∞-function f is denoted by $\|f\|$. If $H^\infty \subseteq A \subseteq L^\infty$, we let $M(A)$ denote the maximal ideal space of A. Elements of A may be identified with functions on $M(A)$. Such an algebra is commonly called a Douglas algebra.

If E is a generalized peak set for H^∞, we define

$$H_E^\infty = \{f \in L^\infty : f_{|E} \in H_{|E}^\infty\}.$$

The algebra $(H^\infty + C)_E$ is defined analogously. If E is a generalized peak set for $H^\infty + C$, then $(H^\infty + C)_E$ is closed. These algebras appeared in [16] and [11]. The reader is referred to [5], [3] and [9] for the theory of uniform algebras and to [6] and [13] for the general basic facts about H^∞.

If A is a closed subalgebra of $C(X)$, X is a compact space, then the essential set of A is the zero set of the largest closed ideal of $C(X)$ which lies in A. Equivalently, it is equal to $\overline{\bigcup \operatorname{supp} \mu}$, where $\mu \in A^\perp$.

The concept of M-ideals has been used by the authors of [10], [11], [16] and [17] in order to prove that L^∞/A is an M-ideal in L^∞/H^∞ for a certain Douglas algebra A. A subspace K of a Banach space Y is called an M-ideal of there exists an L-projection P from Y^* onto $K^\perp$, that is, P is a projection such that $\|y\| = \|Py\| + \|y - Py\|$ for all $y \in Y^*$. If K is an M-ideal of Y and if $x \in Y$ then there exists $m \in K$ such that $\operatorname{dist}(x, K) = \|x - m\|$ [1]. If $x \in Y \setminus K$ then

$$\operatorname{span}\{m : m \in K, \operatorname{dist}(x, K) = \|x - m\|\} = K \quad [7].$$

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2. The unit ball of L^∞/A and related results

In this section we prove the following theorem.

THEOREM 1. *If $A = H^\infty_E$ where E is a peak set for H^∞, or if A/H^∞ is an M-ideal of L^∞/H^∞ and $A \neq H^\infty_F$, where F is a generalized peak set for H^∞, then the unit ball of L^∞/A has not extreme points. Consequently, L^∞/A is not a dual space.*

The case $A = H^\infty_E$, where E is a peak set for H^∞ appeared in [11]. Another special case of Theorem 1 appeared in [17]. In [11], it is proved that if $A = H^\infty + L^\infty_F$, where F is an open subset of the unit circle, and

$$L^\infty_F = \{ f \in L^\infty : f \text{ is continuous on } F \},$$

then the unit ball of L^∞/A has no extreme points. In this case, it is not known whether A/H^∞ is a M-ideal or not.

All the above algebras share the property that they admit a best approximation. More importantly, they share also another property which says that if b is a Blaschke product such that $\bar{b} \notin A$ then the best approximation of $\bar{b}_{|E}$ to $A_{|E}$ is not unique, where E is the essential set of A.

Recently, K. Izuchi and the author [8] showed that there exists a generalized peak set F for H^∞ such that the unit ball of L^∞/H^∞_F has extreme points. In this case the algebra H^∞_F fails to have the second property that I mentioned above.

The proof of the theorem requires the following proposition which may be of some interest in its own right.

PROPOSITION 1. *Let A/H^∞ be an M-ideal of L^∞/H^∞, A is not of the form H^∞_E, where E is a peak set or generalized peak set for H^∞. If $f \in L^\infty$, $f \notin A$ then there exists $h \in A$ and $m \in M(A)\setminus M(L^\infty)$ such that* $\operatorname{dist}(f, A) = \|f - h\|$ *and h is not identically zero on* supp m, *the support of m.*

In [11], it was shown that if $A = H^\infty_E$, E is a generalized peak set for H^∞, then Proposition 1 is not valid. Moreover if E is a peak set for H^∞, then Proposition 1 is valid for $\bar{b} \notin H^\infty_E$, b is a Blaschke product [11, Proposition 1].

Proof of Theorem 1. Consider $f + A$ with $\|f + A\| = 1$. Since A/H^∞ is an M-ideal, we may assume that $\|f\|_\infty = 1$. Following Axler [2] we write $f = \bar{b}g$ where $g \in H^\infty + C$ and b is a Blaschke product with $\bar{b} \notin A$. Choose $\tilde{h} \in A$ such that $\|\bar{b} - \tilde{h}\|_\infty = 1$ and $\tilde{h}$ is not identically zero on the support S of some representing measure m for A which is not a point mass. This follows from Proposition 1 if $A \neq H^\infty_E$, E is a peak set for H^∞; otherwise we refer to Proposition 1 of [11]. Let $h = -\tilde{h}g$; then there exists a point $x \in S$ such that

$|f(x) + \frac{1}{2}h(x)| < 1$. Suppose f is not zero at a point x where $\tilde{h}(x) \neq 0$. Then $|g(x)| = |f(x)| \neq 0$, so $h(x) \neq 0$. Thus

$$f(x) + \tfrac{1}{2}h(x) = \tfrac{1}{2}f(x) + \tfrac{1}{2}(f(x) + h(x)),$$

which implies that $|f(x) + \frac{1}{2}h(x)| < 1$. On the other hand if f is zero at any point $x \in S$, then

$$|f(x) + \tfrac{1}{2}h(x)| = \tfrac{1}{2}|f(x) + h(x)| < 1.$$

Let $g_1 = 1 - |f + \frac{1}{2}h|$. Then $g_1 \geq 0$ and moreover g_1 is not identically zero. Let W be a clopen neighborhood of x in $M(L^\infty)$ such that $S \setminus W \neq \emptyset$ and $1 - |f + \frac{1}{2}h| > 0$ on W. Let $c = \min\{g_1(x) : x \in W\}$; then $c > 0$. Hence $g_1 \geq c\chi_W$. Note that $\chi_W \notin A$, otherwise we get

$$\int \chi_W \, dm = \int \chi_W^2 \, dm = \left(\int \chi_W \, dm\right)^2.$$

This contradicts the inequality $0 < \int \chi_W \, dm < 1$. Thus

$$f \pm c\chi_W + A \neq f + A.$$

Furthermore,

$$\begin{aligned} \|f \neq c\chi_W + A\| &\leq \|f \neq c\chi_W + \tfrac{1}{2}h\|_\infty \\ &\leq \sup_{x \in M(L^\infty)} \left\{|f(x) + \tfrac{1}{2}h(x)| + g_1(x)\right\} \\ &= 1. \end{aligned}$$

Since $f + A = \frac{1}{2}\{f + c\chi_W + A\} + \frac{1}{2}\{f - c\chi_W + A\}$, we conclude that $f + A$ cannot be an extreme point of the unit ball of L^∞/A. This ends the proof of Theorem 1.

To prove Proposition 1, we start with the following proposition which is a refinement of a general result in function algebra.

PROPOSITION 2. *If B is a Douglas algebra, then*

$$\cup\{\operatorname{supp} m : m \in M(B) \setminus M(L^\infty)\}$$

is dense in the essential set of B.

Proof of Proposition 1. Since A/H^∞ is an M-ideal of L^∞/H^∞ then

(1) $$\operatorname{span}\{h + H^\infty : h \in A, \operatorname{dist}(f, A) = \|f - h\|\} = A/H^\infty.$$

Let E be the essential set of A. If for every h in equation (1) we have $h(E) = 0$, then we get $A_{|E} = H^\infty_{|E}$. Consequently, $H^\infty_{|E}$ is closed in $L^\infty_{|E}$ and since H^∞ is logmodular, we get E is a generalized peak set for H^∞ [5, page 65]. Since $A = \{f \in L^\infty\colon f_{|E} \in A_{|E}\}$, we get $A = H^\infty_E$. This contradiction shows that there exists $h \in A$ and $h(E) \neq 0$. By Proposition 2, there exists $m \in M(A) \setminus M(L^\infty)$ such that $h(\operatorname{supp} m) \not\equiv 0$. This ends the proof of Proposition 1.

Proof of Proposition 2. If $B = L^\infty$, then the essential set of B is empty. So assume that $B \neq L^\infty$. If $m \in M(B) \setminus M(L^\infty)$ then $\operatorname{supp} m$ lies in E [9], where E is the essential set of B. Consequently if S if the closure of

$$\cup \{\operatorname{supp} m\colon m \in M(B) \setminus M(L^\infty)\}$$

in $M(L^\infty)$ then $S \subset E$. Note that by the Chang-Marshall Theorem [4], [12], the set S is non-empty. Let $f \in L^\infty$ be such that $f(S) = 0$. Clearly $f_{|\operatorname{supp} m} \in B_{|\operatorname{supp} m}$ for every $m \in M(B)$. By [14], we get $f \in B$. This shows that $S = E$. This proves Proposition 2.

In order to answer question 4 in [11], we need the following proposition.

PROPOSITION 3. *If S is a peak set for $H^\infty + C$ then S is the essential set of* $(H^\infty + C)_S$.

Proof of Proposition 3. *Step* 1. If F is a peak set for H^∞, $F \subset X_\alpha$ for some fiber X_α, then F is the essential set of H^∞_F.

First, let us show that the essential set of H^∞_F is non-empty. Let g be a peaking function for F. By [6, p. 171] there exists a clopen set W in $M(L^\infty)$ such that $g = 1$ on $W \cap X_\alpha$. Consequently, $W \cap X_\alpha \subset F$. Since X_α is the essential set of $H^\infty_{X_\alpha}$ [17], there exists $\mu \perp H^\infty_{X_\alpha}$ such that $|\mu|(W \cap X_\alpha) > 0$. Since $\chi_F\mu \perp H^\infty_F$ and $\chi_F\mu = \mu$ on $W \cap X_\alpha$, we get $\chi_F\mu \neq 0$. Consequently $H^\infty_F \neq L^\infty$. Hence the essential set E of H^∞_F is non-empty.

If possible, let $x \in F$ but $x \notin E$. There exists a clopen set W in $M(L^\infty)$ such that $x \in W$ and $W \cap E = \emptyset$. Let $h = \chi_W g$. Then $h(x) = 1$ and $|h| < 1$ on $(X_\alpha \setminus F) \cup E$. There exists a clopen set V in $M(L^\infty)$ such that $h = 1$ on $V \cap X_\alpha$. Note that $V \cap X_\alpha \subset F \setminus E$. Since X_α is the essential set of $H^\infty_{X_\alpha}$, there exists $\mu \perp H^\infty_{X_\alpha}$ such that $|\mu|(V \cap X_\alpha) > 0$. Consequently

$$|\chi_F\mu|(V \cap X_\alpha) > 0.$$

This means that $\operatorname{supp} \chi_F\mu$ intersects V. Since $\operatorname{supp} \chi_F\mu \subset E$, we get a contradiction. This prove that F is the essential set of H^∞_F.

Step 2. Let $x \in S$; then $x \in X_\alpha$ for some fiber X_α. If $X_\alpha \subset S$ then the essential set of $H^\infty_{X_\alpha}$ which is X_α lies in the essential set of $(H^\infty + C)_S$. Thus

$x \in E$, E is the essential set of $(H^\infty + C)_S$. If $X_\alpha \not\subset S$, let $F = X_\alpha \cap S$. Then F is a peak set for H^∞ and so F is the essential set of H_F^∞. Since $(H^\infty + C)_S \subset H_F^\infty$, then $F \subset E$. Thus $x \in E$. Hence S is the essential set of $(H + C)_S$.

COROLLARY [11, question 4]. *Let S be a peak set for $H^\infty + C$ and b a Blaschke product such that $\bar{b} \in (H^\infty + C)_S$. Then there exists h in $(H^\infty + C)_S$ and $m \in M((H^\infty + C)_S) \setminus M(L^\infty)$ such that*

$$\operatorname{dist}(\bar{b}, (H^\infty + C)_S) = \|\bar{b} - h\| \quad \text{and} \quad h(\operatorname{supp} m) \neq 0.$$

Proof. If S is a peak set for H^∞, then the result is proved in [11]. So assume that S is not a peak set for H^∞. Since S is a G_δ-set and S is not a peak set for H^∞, then S cannot be a generalized peak set for H^∞ [5, Lemma 12.1]. An easy argument using Proposition 3 one can show that $(H^\infty + C)_S$ cannot be of the form H_E^∞, where E is a generalized peak set for H^∞. Now use the fact that $(H^\infty + C)_S / H^\infty$ is a M-ideal together with Proposition 1 to end the proof of the corollary.

Remark. The above corollary has been obtained independently by Pamela Gorkin. Combining the above corollary with Theorem 1, one can conclude that the closed unit ball of $L^\infty/(H^\infty + C)_S$ has no extreme points.

3. Countably generated algebras

First let us remark that if E is a peak set for $H^\infty + C$, then

$$\cup \{\operatorname{supp} m \colon m \in M(H^\infty + C) \setminus M(L^\infty), \operatorname{supp} m \subset E\}$$

is dense in E. This follows by applying Propositions 3 and 2 to the algebra $(H^\infty + C)_E$. Another remark is that if E is a non-singleton closed antisymmetric set for $H^\infty + C$, then

$$\cup \{\operatorname{supp} m \colon m \in M(H^\infty + C) \setminus M(L^\infty)\}$$

is dense in E. This follows from the observation that E is the essential set of $(H^\infty + C)_E$ and Proposition 2.

If A is a Douglas algebra and $f_1, f_2, \ldots \in L^\infty$, the algebra

$$A[f_1, f_2, \ldots]$$

will denote the smallest closed subalgebra of L^∞ containing A and the functions $f_1, f_2, \ldots$.

THEOREM 2. *If A and B are Douglas algebras and B is countably generated over A, then $\cup\{\operatorname{supp} m\colon m \in M(B) \setminus M(L^\infty)\}$ is dense in the essential set of A.*

Proof. Let $B = A[f_1, f_2, \dots]$. There exists a Blaschke product b such that $B \subset A[\bar{b}]$ [15]. Also, there exists a Blaschke product b_0 such that $b_0\bar{b}^n \in H^\infty + C$ for all n (see [2], [6]). Thus $b_0 B \subset A$. By [15], L^∞ is not countably generated over A. Hence the essential set E of B is a non-empty closed subset of $M(L^\infty)$.

Let $\mathscr{T}_E = \{f \in L^\infty: f(E) = 0\}$. Then $\mathscr{T}_E$ lies in B. Since $b_0\mathscr{T}_E = \mathscr{T}_E$, we conclude that $\mathscr{T}_E \subset A$. Now, using the fact that E is closed and lies in the essential set of A, together with $\mathscr{T}_E \subset A$ to conclude that E is the essential set of A. By using Proposition 2, we conclude that

$$\cup\{\operatorname{supp} m: m \in M(B)\setminus M(L^\infty)\}$$

is dense in the essential set of A.

The following corollary is immediate.

COROLLARY. *If A and B are Douglas algebras and b is a Blaschke product such that $b.B \subset A$, then $\cup\{\operatorname{supp} m: m \in M(B)\setminus M(L^\infty)\}$ is dense in the essential set of A.*

Remark. Theorem 2 shows that if A and B are Douglas algebras and B is countably generated over A, then A and B have the same essential set. In case $A = H^\infty + C$, Theorem 2 says that $\cup\{\operatorname{supp} m: m \in M(B)\setminus M(L^\infty)\}$ is dense in $M(L^\infty)$. To see that, it suffices to shown that the essential set of $H^\infty + C$ is $M(L^\infty)$. But if the essential set of $H^\infty + C$ is a proper subset of $M(L^\infty)$, then $\{f \in L^\infty: f(E) = 0\} \subset H^\infty + C$. Consequently, $H^\infty + C$ contains characteristic functions. This is impossible because the maximal ideal space of $H^\infty + C$ is connected

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