ILLINOIS JOURNAL OF MATHEMATICS
Volume 29, Number 2, Summer 1985

APPROXIMATION OF COMPACT OPERATORS BY SUMS OF TRANSLATIONS

BY

CHING CHOU, ANTHONY TO-MING LAU[1] AND JOSEPH ROSENBLATT[2]

0. Introduction

Let G be a locally compact group. Let $VN(G)$ (resp. $C_\delta^*(G)$) denote the closure of the linear span of the left translation operators on $L^2(G)$ with respect to the weak operator topology (resp. norm topology). It is not hard to see that (Proposition 3.2) if $VN(G)$ contains a non-zero compact operator, then G is compact. In this case, the rank-one operator E on $L^2(G)$ defined by $E(f) = (\int f(x)\,dx)1_G$ is in $VN(G)$.

In this paper, we are concerned with the following problem: Does there exist an infinite compact group such that $E \in C_\delta^*(G)$?

Dunkl and Ramirez prove in [1] that if G is an infinite compact group which is amenable as discrete, then $E \notin G_\delta^*(G)$.

In this paper, we prove (Corollary 1.2), among other things, that if G is a compact group containing a dense subgroup with Kazhdan's property (T) (e.g., $SO(n, R)$, $n \geq 5$), then $E \in C_\delta^*(G)$. We also prove (Theorem 2.4) that if $C_\delta^*(G)$ contains a non-zero compact operator in $VN(G)$, then $C_\delta^*(G)$ must contain *all* compact operators in $VN(G)$.

1. Compact groups with property (A)

Let G be an infinite compact group with normalized Haar measure μ and let λ be the left regular representation of G on $L^2(G)$: for each x in G, $\lambda(x)$ is the isometry on $L^2(G)$ defined by $(\lambda(x)g)(y) = g(x^{-1}y)$, $g \in L^2(G)$, $y \in G$. Let $C_\delta^*(G)$ be the C^*-algebra generated by $\lambda(x)$, $x \in G$ and let E be the rank one operator $g \to (\int g(x)\,dx) \cdot 1_G$ on $L^2(G)$ where 1_G denotes the constant one function on G. We say G has *property* (A) if $E \in C_\delta^*(G)$. In this section we are concerned with the following natural question: does there exist a G with property (A)? Quite surprisingly, this question has a positive answer. However, it is easy to see that if G is abelian, then G does not have property (A).

Received March 1, 1983.
[1]Supported by an NSERC grant.
[2]Supported by a grant from the National Science Foundation.

Indeed, let G be an infinite compact abelian group. Then its dual group $\hat{G}$ is infinite and discrete. By Plancherel's Theorem, the map $f \to \hat{f}$, where $\hat{f}(\gamma) = \int f(y)\gamma(y)\, d\mu(y)$, $\gamma \in \hat{G}$, $f \in L_2(G)$, defines a linear isometry of $L^2(G)$ onto $l^2(\hat{G})$. Hence the operator $\Sigma_{i=1}^n c_i \lambda(x_i)$, $c_i \in \mathbf{C}$, $x_i \in G$, on $L^2(G)$, corresponds to multiplication on $l^2(\hat{G})$ by the trigonometrical function $\gamma \to \Sigma_{i=1}^n c_i \langle \gamma, x_i^{-1} \rangle$. Since $l^\infty(\hat{G})$ can be identified with multiplication operators on $l^2(\hat{G})$ by functions in $l^\infty(\hat{G})$, it follows that $C_\delta^*(G)$ can be identified with $AP(\hat{G})$, the algebra of almost periodic functions in $\hat{G}$, by density of trigonometrical polynomials in $AP(\hat{G})$. On the other hand the operator E is identified with the function δ_0 on $\hat{G}$ where $\delta_0(\gamma) = 1$ if $\gamma = 0$ and $\delta_0(\gamma) = 0$ if $\gamma \neq 0$. Since $\delta_0 \notin AP(\hat{G})$, $E \notin C_\delta^*(\hat{G})$.

A stronger result is known: in [1] Dunkl and Ramirez proved that if G_d, G considered as a discrete group, is amenable, then G does not have property (A).

Let π be a unitary representation of a discrete group N on a Hilbert space H. We say that π contains the trivial representation of N *weakly* if there exists a net $\{\nu_\alpha\}$ in H, $\|\nu_\alpha\| = 1$, such that

$$\lim_\alpha \|\pi(x)\nu_\alpha - \nu_\alpha\| = 0 \quad \text{for each } x \in N.$$

The following is our main result of this section.

THEOREM 1.1. *Let G be an infinite compact group. Then the following two conditions are equivalent*:

(a) *G has property* (A).

(b) *The left regular representation of G_d on*

$$L_0^2(G) = \left\{ g \in L^2 \colon \int g(x)\, dx = 0 \right\}$$

does not weakly contain the trivial representation.

Proof. (a) $\Rightarrow$ (b). Suppose G has property (A). Then there exist $x_1, \dots, x_n \in G$ and $c_1, \dots, c_n \in \mathbf{C}$ such that

(1) $\|E - \Sigma_{j=1}^n c_j \lambda(x_j)\| = \varepsilon < \frac{1}{2}$.

Let $M = \sup|c_i|$. Pick $\delta > 0$ such that $1 - \varepsilon - \delta \geq \frac{1}{2}$. We claim that

(2) if $g \in L_0^2(G)$, $\|g\|_2 = 1$ then $\|g - \lambda(x_j)g\|_2 > \delta/nM$ for some $1 \leq j \leq n$.

Note that (2) implies (b). To prove (2), note first that by (1) we have

$$\left\| E(1_G) - \sum_{j=1}^n c_j \lambda(x_j) 1_G \right\|_2 = \left| 1 - \sum_{j=1}^n c_j \right| \leq \varepsilon$$

and hence

(3) $|\Sigma_{j=1}^n c_j| \geq 1 - \varepsilon$.

If (2) were not true, then there would exist $g \in L_0^2(G)$, $\|g\|_2 = 1$ such that $\|g - \lambda(x_j)g\|_2 \leq \delta/nM$ for all j. Then by (1) we would have

$$\begin{aligned}
\tfrac{1}{2} > \varepsilon \geq \left\| E(g) - \sum_{j=1}^{n} c_j \lambda(x_j) g \right\|_2 &= \left\| \sum_{j=1}^{n} c_j \lambda(x_j) g \right\|_2 \\
&= \left\| \sum_{j=1}^{n} c_j \big(\lambda(x_j)g - g\big) + \left(\sum_{j=1}^{n} c_j \right) g \right\|_2 \\
&\geq \left| \sum_{j=1}^{n} c_j \right| \|g\|_2 - \left\| \sum_{j=1}^{n} c_j \big(\lambda(x_j)g - g \right\|_2 \\
&\geq 1 - \varepsilon - nM \sup \|\lambda(x_j)g - g\|_2, \quad \text{by (3)}, \\
&\geq 1 - \varepsilon - nM \cdot \frac{\delta}{nM} = 1 - \varepsilon - \delta \\
&\geq \tfrac{1}{2}.
\end{aligned}$$

Thus we have reached a contradiction.

(b) $\Rightarrow$ (a). If (b) holds then there exist $y_1, \dots, y_M \in G$ and $\varepsilon > 0$ such that for all $g \in L_0^2(G)$, $\|g\|_2 = 1$, for some i, $1 \leq i \leq M$,

$$\|\lambda(y_i)g - g\|_2 \geq \varepsilon.$$

Let $x_1 = e$, the identity of G, $x_2 = y_1, \dots, x_{M+1} = y_M$ and let $N = M + 1$. Let

$$A = \frac{1}{N} \sum_{k=1}^{N} \lambda(x_k).$$

We claim that $\|A\|_{L_0^2(G)} < 1$. That is, there exists δ, $0 < \delta < 1$, such that for all $g \in L_0^2(G)$, $\|A(g)\|_2 \leq (1 - \delta)\|g\|_2$. If not, then there exists a sequence $\{g_n\}$ in $L_0^2(G)$, $\|g_n\|_2 = 1$, such that $\|A(g_n)\|_2 \to 1$. Therefore,

$$\begin{aligned}
\|A(g_n)\|_2^2 &= \langle A(g_n), A(g_n) \rangle \\
&= \frac{1}{N^2} \sum_{i,j=1}^{N} \langle \lambda\big(x_j^{-1} x_i\big) g_n, g_n \rangle \to 1.
\end{aligned}$$

Since for each i, j, $|\langle \lambda(x_j^{-1} x_i) g_n, g_n \rangle| \leq 1$, we conclude that

$$\operatorname{Re}\langle \lambda\big(x_j^{-1} x_i\big) g_n, g_n \rangle \to 1.$$

But then

$$\|\lambda(x_i)g_n - \lambda(x_j)g_n\|_2^2 = 2 - 2\,\mathrm{Re}\langle\lambda(x_j^{-1}x_i)g_n, g_n\rangle \to 0$$

as $n \to \infty$. In particular, since $x_1 = e$ and $x_{k+1} = y_k$, $k = 1, \ldots, M$, we conclude that $\lim_n\|\lambda(y_k)g_n - g_n\|_2 = 0$ for each k, $1 \le k \le M$. This clearly contradicts the choice of $y_1, \ldots, y_M$. Thus, $\|A\|_{L_0^2(G)} < 1$, as claimed.

Fix a positive integer m. For each $g \in L^2(G)$,

$$\begin{aligned}\|(A^m - E)g\|_2 &= \|A^m(g - E(g))\|_2 \\ &\le \|A^m\|_{L_0^2(G)}\|g - E(G)\|_2 \quad \bigl(\text{since } g - E(g) \in L_0^2(G)\bigr) \\ &\le 2\bigl(\|A\|_{L_0^2(G)}\bigr)^m\|g\|_2\end{aligned}$$

Therefore $\|A^m - E\| \le 2(\|A\|_{L_0^2(G)})^m \to 0$. That is $E \in C_\delta^*(G)$. □

A discrete group N is said to have Kazhdan's *property* (T) if any unitary representation of N which weakly contains the trivial representation has a 1-dimensional invariant subspace. From the definitions, it is clear, as pointed out by Margulis [6] and Sullivan [10] independently, that if G has a dense subgroup with property (T), then G satisfies (b) of Theorem 1.1. Therefore we have the following:

COROLLARY 1.2. *Let G be an infinite compact group containing a dense subgroup with property* (T). *Then G has property* (A).

According to Margulis [6, Proposition 4] (see also Sullivan [10]), if a connected simple compact Lie group G is not locally isomorphic to SO(3, R) or SO(4, R), then G contains a dense subgroup with property (T). Hence if $n \ge 5$, then SO(n, R) has property (A). Margulis has kindly communicated to one of us the interesting fact that SO(n), $n = 3, 4$ and SU(2) do not contain a dense subgroup with property (T). However, we are unable to decide whether or not these three groups have property (A). Note that it is important for us to know if these three groups have property (A). For if they do, then there is a unique invariant mean on $L^\infty(G)$ of these groups G (see Proposition 1.3), and this is very closely related to solving the Banach-Ruziewicz problem in S^2 and S^3 (see [7] and [10] for more details).

A linear functional m on $L^\infty(G)$ is called a *left invariant mean* on $L^\infty(G)$ if $\|m\| = 1$, $m \ge 0$ and $m({}_xf) = m(f)$ for $f \in L^\infty(G)$ and $x \in G$ where ${}_xf \in L^\infty(G)$ is defined by ${}_xf(y) = f(x^{-1}y)$. Note that the normalized Haar measure μ of G is a left invariant mean of $L^\infty(G)$. Granirer [3] and Rudin [8] have proved independently that if G_d is amenable then $L^\infty(G)$ has more than one left invariant mean (see also Lemma 3.3 of Stafney [9] for G second countable and abelian). For quite a while it was unknown whether there exists an infinite compact group G such that $L^{\infty}(G)$ has a unique left invariant mean. The following result is a direct consequence of results in Rosenblatt [7] and, in slightly different form, is also contained in [6] and [10].

PROPOSITION 1.3. *If G has property* (A), *then $L^{\infty}(G)$ has a unique left invariant mean.*

Proof. If $L^{\infty}(G)$ has more than one left invariant mean, then by Theorem 1.3 and Lemma 3.1 of [7], there exists a net $\{g_\alpha\}$ in $L_0^2(G)$, $\|g_\alpha\|_2 = 1$, such that $\lim_\alpha \|\lambda(x)g_\alpha - g_\alpha\|_2 = 0$ for each $x \in G$. That is, G does not satisfy condition (b) of Theorem 1.1. □

Remark 1.4. (a) The only known infinite compact groups G for which $L^{\infty}(G)$ has a unique left invariant mean are the ones containing a dense subgroup with property (T).

(b) If G has property (A), then G_d is not amenable (by Proposition 1.3 and Theorem 4.1 in [8]). But it is possible that G_d is not amenable, and G fails to have property (A). For example, take $G = H \oplus T$, H compact but arbitrary, T is the circle group. Then G has more than one left invariant mean on $L_{\infty}(G)$, but G_d is not amenable when H_d is not amenable.

2. Property (A) and compact operators

In this section, G will again denote an infinite compact group. We will prove, among other things, that if $C_\delta^*(G)$ contains a non-zero compact "operator", then G has property (A). But first of all we will need some notations and preliminary results in harmonic analysis on compact groups.

Let $VN(G)$ be the von Neumann subalgebra of $\mathscr{B}(L^2(G))$, the algebra of bounded operators on $L^2(G)$, generated by $\{\lambda(x); x \in G\}$, or equivalently, the von Neumann subalgebra generated by $\{\lambda(f); f \in L^1(G)\}$ where the operator $\lambda(f)$ on $L^2(G)$ is defined by $\lambda(f)(g) = f * g$. Note that $VN(G)$ also equals the commutant of the right convolution operators; i.e.,

$$VN(G) = \left\{T \in \mathscr{B}\left(L^2(G)\right): T(g * f) = T(g) * f, g \in L^2(G), f \in L^1(G)\right\}$$

(cf. Eymard [2]). Let $C_\lambda^*(G)$ be the C^*-algebra generated by $\lambda(f)$, $f \in L^1(G)$. If G is abelian, then $C_\lambda^*(G)$ can be identified with $C_0(\hat{G})$, the space of (continuous) functions on $\hat{G}$ which vanishes at infinity. Since $AP(\hat{G}) \cap C_0(\hat{G}) = \{0\}$, $C_\delta^*(G) \cap C_\lambda^*(G) = \{0\}$.

Let $\hat{G}$ be the dual of the compact group G; i.e., G is the equivalence classes of irreducible continuous unitary representations of G. For each $\alpha \in G$, choose $\Pi_\alpha \in \alpha$. Denote the coefficient functions of Π_α by $u_{i,j}^\alpha$, $1 \le i, j \le d_\alpha$ where d_α is the dimension of Π_α. Let V_α be the d_α^2-dimensional space generated by the $u^\alpha_{i,j}$'s. Then by the Peter-Weyl theorem, V_α is right and left translation invariant and

$$L^2(G) = \sum\{V_\alpha:\ \alpha \in \hat{G}\} \quad \text{(Hilbert sum)}.$$

Denote the linear span of $\bigcup\{V_\alpha:\ \alpha \in \hat{G}\}$ by $\operatorname{Trig} G$, the trigonometrical polynomials of G. Then $\operatorname{Trig} G$ is dense in $C(G)$ and $L^2(G)$. The following lemma is more or less known. For completeness, we present a proof here.

LEMMA 2.1. *Let $T \in VN(G)$. Then T is of finite rank if and only if $T = \lambda(u)$ for some $u \in \operatorname{Trig} G$.*

Proof. If $u \in \operatorname{Trig} G$, then there exists a *finite* set $F \subseteq \hat{G}$ such that

$$u \in \sum\{V_\alpha:\ \alpha \in F\}.$$

Then for $g \in L_2(G)$, $\lambda(u)(g) = u * g \in \Sigma\{V_\alpha:\ \alpha \in F\}$ (cf. Hewitt and Ross [4; p. 11]). Thus $\lambda(u)$ is of finite rank.

Conversely, assume that $T \in VN(G)$ is of finite rank. Since $T(V_\alpha) \subseteq V_\alpha$ for each α, there exists a finite set $F \subseteq G$ such that $T(V_\alpha) = \{0\}$ if $\alpha \notin F$. Denote the trace of $\alpha \in \hat{G}$ by χ_α. Let

$$\chi = \sum\{d_\alpha \chi_\alpha;\ \alpha \in F\}.$$

Then both χ and $T(\chi)$ belong to $\operatorname{Trig} G$. By [4, p. 14], we have

$$\chi * u^\alpha_{ij} = u^\alpha_{ij} \quad \text{if } \alpha \in F$$

and

$$\chi * u^\alpha_{ij} = 0 \quad \text{if } \alpha \notin F.$$

Therefore,

$$T(\chi) * u^\alpha_{ij} = T(\chi * u^\alpha_{ij}) = T(u^\alpha_{ij}) \quad \text{if } \alpha \in F$$

and

$$T(\chi) * u^\alpha_{ij} = T(0) = 0 \quad \text{if } \alpha \notin F.$$

But if $\alpha \notin F$, $T(u^\alpha_{ij}) = 0$. So $T(u^\alpha_{ij}) = T(\chi) * u^\alpha_{ij}$ for $\alpha \in \hat{G}$ and $1 \le i, j \le d_\alpha$. Thus $T = \lambda(T(\chi))$. □

PROPOSITION 2.2. *If G is compact, then*

$$C^*_\lambda(G) = \{T \in VN(G):\ T \text{ is a compact operator on } L^2(G)\}.$$

Proof. If T is a compact operator in $VN(G)$ and $T = U|T|$ is the polar decomposition of T, then U, $|T| \in VN(G)$ and $|T|$ is compact. Therefore by the spectral theorem, there exists a sequence of finite rank operators S_n in $VN(G)$ such that $S_n \to |T|$ in operator norm topology. Then $T_n = U \cdot S_n$ is a sequence of finite rank operators in $VN(G)$ such that $\|T_n - T\| \to 0$. By the above lemma, each $T_n = \lambda(u_n) \in C_\lambda^*(G)$, for some $u_n \in \operatorname{Trig} G$. So $T \in C_\lambda^*(G)$.

The converse is well known (see for example [11, p. 47]; we thank the referee for pointing this out to us). □

Remark 2.3. The fact that each $\lambda(f)$, $f \in L^1(G)$, is compact can be viewed as a generalization of the Riemann Lebesgue lemma for (compact) abelian groups. Indeed, if G is abelian, then $\lambda(f)$ corresponds to the multiplication by $\hat{f}$ (the Fourier transform of $\hat{f}$) operator on $l^2(\hat{G})$. But a bounded function h on $\hat{G}$ viewed as a multiplication operator on $l^2(\hat{G})$ is compact if and only if $\hat{h} \in C_0(\hat{G})$.

The predual of $VN(G)$ can be realized as an algebra of continuous functions on G, namely, $A(G)$, the Fourier algebra of G. Indeed, each $u \in A(G)$ can be written as $h * \tilde{k}$ where $h, k \in L^2(G)$, $\tilde{k}(x) = \overline{k(x^{-1})}$, $x \in G$. For $T \in VN(G)$, and $u = h * \tilde{k} \in A(G)$, $\langle T, \check{u} \rangle$ equals the inner product of $T(h)$ and k in $L^2(G)$ where $\check{u} \in A(G)$ is defined by $u(x) = u(x^{-1})$. $A(G)$ with pointwise multiplication and the norm

$$\|u\| = \inf\{\|h\|_2 \|k\|_2 \colon u = h * \tilde{k}, h, k \in L^2(G)\}$$

is a commutative Banach algebra. Further, $VN(G)$ is an $A(G)$-module where for $u \in A(G)$, $T \in VN(G)$, $u \cdot T$ is defined by $\langle u \cdot T, v \rangle = \langle T, uv \rangle$, $v \in A(G)$. For more details on the algebras $A(G)$ and $VN(G)$, see Eymard [2]. Note that if $f \in L^1(G)$, $x \in G$ and $u \in A(G)$, then $u \cdot \lambda(f) = \lambda(uf)$ and $u \cdot \lambda(x) = u(x)\lambda(x)$. Hence, as easily checked, $C_\lambda^*(G)$ and $C_\delta^*(G)$ are both $A(G)$-submodules of $VN(G)$.

We are now ready to give the main result of this section.

THEOREM 2.4. *Let G be a compact group. The following are equivalent*:

(a) *G has property* (A).
(b) *$C_\delta^*(G)$ contains a non-zero compact operator.*
(c) *$C_\delta^*(G)$ contains all compact operators in $VN(G)$.*
(d) *$C_\lambda^*(G) \subseteq C_\delta^*(G)$.*
(e) *$C_\lambda^*(G) \cap C_\delta^*(G) \neq \{0\}$.*

Proof. That (a) ⇒ (b) is clear.

(b) ⇒ (c). If $C_\delta^*(G)$ contains a non-zero compact operator, then it contains a nonzero positive compact operator. By the spectral theorem for compact normal operators, we see that $C_\delta^*(G)$ contains a nonzero operator T of finite rank. By Lemma 2.1, there exists $u_0 \in \operatorname{Trig} G \subseteq A(G)$, $u_0 \neq 0$, such that $T = \lambda(u_0)$.

Let $I = \{u \in A(G): \lambda(u) \in C_\delta^*(G)\}$. We claim that I is a closed ideal of $A(G)$. Indeed, if $v \in A(G)$ and $u \in I$, then $v \cdot \lambda(u) \in C_\delta^*(G)$; but $v \cdot \lambda(u) = \lambda(vu)$ and hence $vu \in I$. Thus I is an ideal. If $u_n \in I$, $u \in A(G)$ and $\|u_n - u\|_{A(G)} \to 0$, then, since

$$\|\lambda(u_n) - \lambda(u)\| \leq \|u_n - u\|_1 \leq \|u_n - u\|_\infty \leq \|u_n - u\|_{A(G)},$$

we conclude that $\lambda(u) \in C_\delta^*(G)$ and hence $u \in I$.

Let $x \in G$. Since $u_0 \neq 0$, there exists $x_0 \in G$ such that $u_0(x_0) \neq 0$. Now if $y = x_0 x^{-1}$, then

$$\lambda({}_y u_0) = \lambda(x_0 x^{-1})\lambda(u_0) \in C_\delta^*(G);$$

i.e., ${}_y u_0 \in I$. But ${}_y u_0(x) = u_0(x_0) \neq 0$. Thus we conclude that for each $x \in G$ there exist $u \in I$ such that $u(x) \neq 0$. By the generalized Wiener's Tauberian theorem (cf. Eymard [2, p. 223]), we see that $I = A(G)$, or, $\lambda(A(G)) \subseteq C_\delta^*(G)$. Since $\lambda(A(G))$ is dense in $\lambda(L^1(G))$, we conclude that $C_\lambda^*(G)$ is contained in $C_\delta^*(G)$.

(c) $\Rightarrow$ (d) and (b) $\Leftrightarrow$ (e) follow from Proposition 2.2, and (d) $\Rightarrow$ (a) is clear. □

Remark 2.5. That (a) $\Rightarrow$ (d) in Theorem 2.4 can be proved directly. Indeed, if $u \in A(G)$, then $u \cdot E = u \cdot \lambda(1) = \lambda(u)$; i.e., $\lambda(u) \in C_\delta^*(G)$. Since $A(G)$ is L^1-dense in $L^1(G)$, if $f \in L^1(G)$ and $\varepsilon > 0$ are given, then there exists $u \in A(G)$ such that $\|u - f\|_1 < \varepsilon$. So

$$\|\lambda(f) - \lambda(u)\| \leq \|f - u\|_1 < \varepsilon.$$

Therefore $\lambda(f) \in C_\delta^*(G)$. This implies that $C_\lambda^*(G) \subseteq C_\delta^*(G)$.

What we are interested in here in studying property (A) is getting at the operators $\sum_{i=1}^n a_i \lambda(x_i) = T$, where $a_1, \ldots, a_n \in \mathbf{C}$ and $x_1, \ldots, x_n \in G$, as they act on $L^2(G)$ or on the appropriate parts of the dual $\hat{G}$. For example, we have:

PROPOSITION 2.6. *G has property* (A) *if and only if for each $\varepsilon > 0$ and $\alpha \in \hat{G}$, there exists $T = \sum_{i=1}^n c_i \lambda(a_i)$, $c_1, \ldots, c_n \in \mathbf{C}$, $a_1, \ldots, a_n \in G$, such that $\|T\|_{V_\alpha} > 1 - \varepsilon$ and $\|T\|_{V_\tau} \leq \varepsilon$ for all $\tau \in \hat{G}$, $\tau \neq \alpha$.*

Proof. If G has property (A), let $\varepsilon > 0$, $\alpha \in G$ be fixed. Let P_α be the orthogonal projection of $L_2(G)$ onto V_α. Then P_α is compact. Also since V_α and its orthogonal complement $\Sigma\{V_\tau: \tau \neq \alpha, \tau \in \hat{G}\}$ are both translation invariant, P_α commutes with right translations on $L^2(G)$. Hence $P_\alpha \in VN(G)$. In particular, $P_\alpha \in C_\delta^*(G)$ by Theorem 2.4. Choose $T = \sum_{i=1}^n c_i \lambda(a_i)$,

$c_1, \dots, c_n \in \mathbf{C}$ and $a_1, \dots, a_n \in G$, such that $\|P_\alpha - T\| < \varepsilon$. Then for each $g \in V_\alpha$, $\|g\|_2 = 1$,

$$\|P_\alpha(g) - T(g)\|_2 < \varepsilon \quad \text{or} \quad \|g - T(g)\|_2 < \varepsilon.$$

Hence $\|T(g)\|_2 > 1 - \varepsilon$; i.e., $\|T\|_\alpha > 1 - \varepsilon$. Also, if $g \in V_\tau$, $\alpha \neq \tau$, $\|g\|_2 = 1$,

$$\|T(g)\|_2 = \|P_\alpha(g) - T(g)\|_2 < \varepsilon.$$

So $\|T\|_{V_\tau} \leq \varepsilon$.

To prove the converse, let $0 < \varepsilon < 1/2$ and α be the trivial representation of G. Then $V_\alpha = \{c1_G; c \in \mathbf{C}\}$. Choose

$$T = \sum_{i=1}^{n} c_i \lambda(x_i), \quad c_1, \dots, c_n \in \mathbf{C}, x_1, \dots, x_n \in G,$$

such that $\|T\|_{V_\alpha} > 1 - \varepsilon$ and $\|T\|_{V_\tau} \leq \varepsilon$ for all $\tau \in G, \tau \neq \alpha$. Since

$$\|T\|_{V_\alpha} = |T(c1_G)| \quad \text{for some } c \in \mathbf{C}, |c| = 1,$$

we have

$$\left|\sum_{i=1}^{n} c_i\right| = |c| \left|\sum_{i=1}^{n} c_i\right| = |T(c1_G)| = \|T\|_{V_\alpha} > 1 - \varepsilon.$$

Also, $V_\alpha^\perp = L_0^2(G)$. So $\|T\|_{L_0^2(G)} = \sup\{\|T\|_{V_\tau}; \tau \in G; \tau \neq \alpha\} \leq \varepsilon$. Hence if $g \in L_0^2(G)$, $\|g\|_2 = 1$, $\|\sum_{i=1}^n c_j \lambda(x_j) g\|_2 \leq \varepsilon < \frac{1}{2}$. An argument similar to that of Theorem 1.1 (a) $\Rightarrow$ (b) shows that condition (b) of Theorem 1.1 holds. Hence G has property (A) by Theorem 1.1. □

PROPOSITION 2.8. *Let K be a closed normal subgroup of G. If G has property* (A), *then G/K also have property* (A).

Proof. Let j: $L^2(G/K) \to L^2(G)$ be defined by $j(h') = h'\sigma$, where σ is the canonical homomorphism of G onto G/K, (G, H and G/K equipped with the normalized Haar measure). Then J is a linear isometry of $L^2(G/K)$ onto $L_K^2(G)$, the space of L^2-functions on G which are constant on cosets. Let $T \in VN(G)$, $h' \in L^2(G/K)$. Then $T(j(h')) \in L_K^2(G)$. Define $\tilde{j}(T) \in VN(G)$ by $\tilde{j}(T)(h') = (j^{-1} \circ T \circ j)(h')$ (see Eymard [2, p. 217]). Then $\tilde{j}$ is continuous, $\tilde{j}(\lambda(x)) = \lambda(x)$, $x \in G$, and $\tilde{j}(\lambda(f)) = \lambda(f')$, $f \in L^1(G)$, where $f' \in L^1(G/K)$ is defined by

$$f'(x) = \int f(xt)\, d\mu_K(t).$$

Thus

$$\tilde{j}(C_\lambda^*(G) \cap C_\delta^*(G)) \subset C_\lambda^*(G/K) \cap C_\delta^*(G/K).$$

In particular, if $E = \lambda(1_G) \in C_\delta^*(G)$, then $\tilde{j}(E) = \lambda(1_{G/K})$ is in $C_\delta^*(G/K)$; i.e., G/K has property (A) also. □

Remark 2.9. It follows from above that $SO(5) \times T$ does not have property (A). This also follows from the fact that $L^\infty(SO(5) \times T)$ has more than one left invariant mean (see Proposition 1.3).

3. Non-compact groups

Let G be a locally compact group with a fixed left Haar measure and with modular function Δ. For each $x \in G$, define the right translation operator $r(x)$ on $L^2(G)$ by $r(x)g(t) = \Delta(x)^{1/2}g(tx)$, $t \in G, g \in L^2(G)$. We need the following result due to Lau [5, Theorem 4.6].

LEMMA 3.1. *If G is non-compact and K is non-empty compact convex subset of $L^2(G)$ such that $r(x)(K) \subset K$ for each $x \in K$, then $K = \{0\}$.*

The spaces $VN(G)$, $C_\lambda^*(G)$, $C_\delta^*(G)$ and $A(G)$ can be defined for any locally compact group G. But we have the following:

PROPOSITION 3.2. *If G is locally compact group and if $VN(G)$ contains a non-zero compact operator T, then G is compact.*

Proof. Let $K = \{T(g)\colon g \in L^2(G)$ and $\|g\|_2 \le 1\}$. Then K is a relatively compact convex subset of $L^2(G)$ and $K \neq \{0\}$. Furthermore, if $x \in G$,

$$r(x)(T(g)) = T(r(x)(g)),$$

since T commutes with right translations on $L^2(G)$, $g \in K$. Therefore $r(x)(K) \subseteq K$. Hence G is compact by Lemma 3.1. □

While $VN(G)$ contains no non-zero compact operators if G is non-compact, it is still interesting to ask: for what kind of non-compact non-discrete group G is $C_\lambda^*(G) \cap C_\delta^*(G) = \{0\}$? (Compare with Theorem 2.4.) It is known that if G is a non-discrete locally compact group and if G_d is amenable, then $C_\lambda^*(G) \cap C_\delta^*(G) = \{0\}$. (See Dunkl and Ramirez [1].) However, when G_d is not amenable, $C_\lambda^*(G) \cap C_\delta^*(G)$ can be non-zero for non-discrete and non-compact G. In fact, let H be a non-discrete compact group with property (A). Let Z be the integer group and let $G = H \oplus Z$. Let $f = 1_{H \oplus \{0\}}$. Then $\lambda(f) * h = p$ is the function on G defined by

$$p(g,m) = \int_{H \oplus \{m\}} h\, du_G, \quad h \in L^2(G).$$

Let T_n be a sequence consisting of linear combinations of left translates on $L^2(H)$ such that $\|T_n - \lambda(1_H)\| \to 0$. If

$$T_n = \sum_{i=1}^{k} c_i \lambda(x_i), \quad c_1, \ldots, c_n \in \mathbf{C} \quad x_1, \ldots, x_n \in H_i,$$

let $\tilde{T}_n = \sum_{i=1}^{n} c_i \lambda(x_i, 0)$. Then, as is readily checked, $\|\tilde{T}_n - \lambda(f)\| \to 0$. In particular $C_\lambda^*(G) \cap C_\delta^*(G) \neq \{0\}$.

The following questions seem to be interesting:

(1) If a nondiscrete locally compact group G contains a dense subgroup with property (T) (as a discrete group), does it follow that $C_\lambda^*(G) \cap C_\delta^*(G) \neq \{0\}$?

(2) If $C_\lambda^*(G) \cap C_\delta^*(G) \neq \{0\}$, does it follow that $C_\lambda^*(G) \subseteq C_\delta^*(G)$?

References

1. C.F. Dunkl and D.E. Ramirez, *C^*-algebras generated by Fourier-Stieltjes transforms*, Trans. Amer. Math. Soc., vol. 164 (1972), pp. 435–441.
2. P. Eymard, *L'algèbra de Fourier d'un groupe localement compact*, Bull. Soc. Math. France, vol. 92 (1964), pp. 181–236.
3. E. Granirer, *Criteria for compactness and for discreteness of locally compact amenable group*, Proc. Amer. Math. Soc., vol. 40 (1973), pp. 615–624.
4. E. Hewitt and K.A. Ross, *Abstract harmonic analysis*, vol. 2, Springer Verlag, New York, 1970.
5. A.T. Lau, *Closed convex invariant subsets of $L_p(G)$*, Trans. Amer. Math. Soc., vol. 232 (1977), pp. 131–142.
6. G.A. Margulis, *Some remarks on invariant means*, Monatsheite für Mathematik, vol. 90 (1980), pp. 233–235.
7. J. Rosenblatt, *Uniqueness of invariant means for measure-preserving transformations*, Trans. Amer. Math. Soc., vol. 265 (1981), pp. 623–636.
8. W. Rudin, *Invariant means on L_∞*, Studia Math., vol. 44 (1972), pp. 219–227.
9. J.D. Stanfey, *Arens multiplication and convolution*, Pacific J. Math., vol. 14 (1964), pp. 1423–1447.
10. D. Sullivan, *For $n > 3$ there is only one finitely additive rotationally invariant measure on the n-sphere defined on all Lebesgue measurable subsets*, Bull. Amer. Math. Soc., vol. 4 (1981), pp. 121–123.
11. A. Weil, *L'integration dans les groupes topologiques et ses applications*, Actualités Sci. Indust., vols. 869, 1145, Herman, Paris, 1941 and 1951.

State University of New York at Buffalo
Amherst, New York

University of Alberta
Edmonton, Alberta, Canada

Ohio State University
Columbus, Ohio