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DEFINABLE SUBGROUPS OF THE PRODUCT OF TWO GROUPS

BY

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In Memoriam W.W. Boone

In this article we investigate definable subgroups of the product of two groups. Our language contains only function symbols together with symbols of the theory of groups. Such a language can be realized in the product of two structures in an obvious way. For the sake of simplicity a structure for such a language will be called a group. In this article definable will always mean definable with parameters. If G and H are two groups pr_1 (resp. pr_2) will denote the obvious projection map $G \times H \to G$ (resp. $G \times H \to H$). We prove the following result:

THEOREM. *Let G and H be two groups. Let K be a definable subgroup of $G \times H$. Then there are $A \lhd pr(K)$, $B \lhd pr(K)$ and $g_1, \dots, g_n$ in G, $h_1, \dots, h_n$ in H such that*

(1) *A and B are definable subgroups of G and H respectively,*
(2) *$A \times B \subseteq K$ and $[K\colon A \times B]$ is finite,*
(3) *$K = (A \times B) \cup (g_1 A \times h_1 B) \cup \cdots \cup (g_n A \times h_n B)$,*
(4) *$g_i A \cap g_j A = h_i B \cap h_j B = \varnothing$ if $i \neq j$.*

Furthermore such A and B are unique.

If a group G has a minimal definable subgroup of finite index then it is unique. We call it the connected component of G and denote it G^0. If G is ω-stable or ω_0-categorical and stable then the connected component always exists (see [1]).

COROLLARY. *Suppose G^0, H^0 exist. Then $(G \times H)^0$ exists and $G^0 \times H^0 \subseteq (G \times H)^0$.*

Proof. Let C be a definable subgroup of $G \times H$ of finite index. By the theorem there are definable subgroups A and B of G and H respectively such that $A \times B \subseteq C$ and $[C\colon A \times B]$ is finite. Therefore $[G \times H\colon A \times B]$, and hence also $[G\colon A]$, $[H\colon B]$ are finite. This shows that $G^0 \subseteq A$, $H^0 \subseteq B$, so

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$G^0 \times H^0 \subseteq C$. We showed that $G^0 \times H^0$ is included in every definable subgroup of finite index of $G \times H$. Hence $(G \times H)^0$ exists and $G^0 \times H^0 \subseteq (G \times H)^0$. □

In the above corollary the equality does not always hold. Take

$$G = \Big(\bigoplus_I Z_p \Big) \oplus Z_q, \quad H = \Big(\bigoplus_J Z_q \Big) \oplus Z_p$$

where p and q are two distinct prime numbers, Z_p, Z_q are cyclic groups of order p and q respectively, and I and J are infinite sets. Then

$$G^0 = \bigoplus_I Z_p \neq G, \quad H^0 = \bigoplus_J Z_q \neq H.$$

but $(G \times H)^0 = G \times H$.

The following is immediate from the theorem.

COROLLARY. *If G is an infinite group then the diagonal subgroup of $G \times G$ is not definable.*

Before proving the theorem we would like to thank Simon Thomas for several very helpful suggestions.

Proof of the theorem. By Feferman-Vaught (see [2]), K is a finite union of sets of the form $A \times B$ where $A \subseteq G$, $B \subseteq H$ are definable subsets. By taking a (definable) subdivision of these subsets if necessary we may suppose that A's and B's are disjoint if they are distinct. Write

$$K = (A_1 \times B_1) \cup \cdots \cup (A_n \times B_n),$$

where $A_i \subseteq G$, $B_j \subseteq H$ are definable subsets and for all i, j either $A_i = A_j$ or $A_i \cap A_j = \varnothing$; similarly for B's, and $(A_i \times B_i) \cap (A_j \times B_j) = \varnothing$ if $i \neq j$.

The identity element $(1,1)$ of K is in one of the $A_i \times B_i$'s. Without loss of generality we may suppose that $(1,1) \in A_1 \times B_1$. Notice that $(A_i \times B_i) \cap K \neq \varnothing$ iff $A_i \times B_i \subseteq K$.

Claim 1. If two of $A_1 \times B_j$, $A_i \times B_1$, $A_i \times B_j$ are in K then the third one is also in K.

Proof. First, suppose $A_1 \times B_j$ and $A_i \times B_1$ are in K. Let $a \in A_i$ and $b \in B_j$. Then $(a,1)$, $(1,b)$ are in K, so their product (a,b) is also in K. Hence $K \cap (A_i \times B_j)$ is not empty, i.e., $A_i \times B_j \subseteq K$. Suppose now $A_i \times B_1$ and $A_i \times B_j$ are in K. Let a, b be as above. If $(1,b)$ is not in K, then $(a,b) = (a,1)(1,b) \in K^c \cap (A_i \times B_j)$, i.e., $A_i \times B_j$ is in K^c, a contradiction. So $(1,b) \in K$, i.e., $A_i \times B_j \subseteq K$. The third case is similar. □

Let $I = \{i / A_i \times B_1 \subseteq K\}$ and $J = \{j / A_1 \times B_j \subseteq K\}$. Set $A = \bigcup_I A_i$, $B = \bigcup_J B_j$. Then by Claim 1, $A_i \times B_j \subseteq K$ for all i in I, j in J. Furthermore (again by Claim 1) if for some $i \in I$, $A_i \times B_j$ is in K then j is in J and vice versa. Since

$$A = pr_1(K \cap (G \times \{1\})),$$

A is a subgroup of G. Similarly B is a subgroup of H. From now on (changing notation if necessary) we suppose that $A \times B$ is part of our decomposition.

We will now show that if $(x, y) \in A_i \times B_j \subseteq K$ then $A_i B_j \subseteq (x, y)A \times B$. Let x, y be such elements. Suppose for some x_1 in A_i, some y_1 in B_j we have $(x^{-1}x_1, y^{-1}y_1)$ not in $A \times B$. Then either $x^{-1}x_1$ is not in A or $y^{-1}y_1$ is not in B. Say $x^{-1}x_1 \notin A$. Then $(x^{-1}x_1, 1) \notin K$. But

$$(x^{-1}x_1, 1) = (x^{-1}, y^{-1})(x_1, y) \in (A_i \times B_j)^{-1} \cdot (A_i \times B_j) \subseteq K,$$

a contradiction. Hence $A_i \times B_j \subseteq (x, y)A \times B$.

Since we have only finitely many $A_i \times B_j$'s in K, this shows that $[K: A \times B]$ is finite. Also by changing notation if necessary we may replace A_i's and B_j's by cosets of A and B respectively. We will now prove that $A \triangleleft pr_1(K)$. Let $a \in A$, $x \in pr_1(K)$. If $y \in H$ is such that $(x, y) \in K$, then $(x, y)^{-1}(a, 1)(x, y) = (x^{-1}ax, 1) \in K$. Hence $x^{-1}ax \in A$. Similarly $B \triangleleft pr_2(K)$.

We now show the fourth point of the theorem.

Lemma. *If $A_i \times B_j$, $A_i \times B_k$ are in K then $B_j = B_k$.*

Proof. Let $a \in A_i$, $b \in B_j$, $b_1 \in B_k$. Then $(1, bb_1^{-1}) = (a, b)(a, b_1^{-1}) \in K$. Hence $bb_1^{-1} \in B$. This shows that $B_j B_k^{-1} \subseteq B$. Since B_j, B_k are cosets of B, and $B \triangleleft pr_2(K)$ the equality must hold. So $B_j B_k^{-1} = B = B_k B_k^{-1}$, i.e. $B_j = B_k B_k^{-1} B_k$, i.e. $bB = b_1 B B^{-1} b_1^{-1} b_1 B = b_1 B$. Therefore $B_j = B_k$.

The dual statement of the lemma also holds. This proves the fourth point of the theorem. Let us now prove the uniqueness of A and B.

Let C and D be as in the theorem. If $d \in D$, then $(1, d) \in K$. Hence $d \in B$. This shows that $D \subseteq B$. Similarly $C \subseteq A$. Changing the roles of A and C, B and D we get the inverse inclusion. The proof of the theorem is now complete.

References

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