ILLINOIS JOURNAL OF MATHEMATICS
Volume 30, Number 2, Summer 1986

FREE ABELIAN X-GROUPS

BY

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Dedicated to the memory of W.W. Boone

1. Introduction

1.1 Let X be a ring with 1. Then a group G equipped with an action

$$g \mapsto g^{\alpha} \quad (g \in G, \alpha \in X, g^{\alpha} \in G)$$

of X on G and satisfying the axioms

$$g^1 = g, g^{\alpha+\beta} = g^{\alpha} g^{\beta}, (g^{\alpha})^{\beta} = g^{\alpha\beta},$$
$$g^{-1} h^{\alpha} g = (g^{-1} h g)^{\alpha} \quad (g, h \in G, \alpha, \beta \in X)$$

is termed an X-group.

X-groups were introduced by R.C. Lyndon in [6]. Lyndon's concern was with the so-called free X-groups when

$$X = \mathbf{Z}[x_1, \ldots, x_q] \tag{1}$$

is the polynomial ring in the finitely many variables $x_1, \ldots, x_q$ over $\mathbf{Z}$. Indeed Lyndon proved that the word problem for such free X-groups is solvable by concocting normal forms for its elements.

Shortly after Lyndon's paper appeared I noticed that the methods of my thesis [1] applied also to X-groups. The upshot is that the class of free X-groups studied by Lyndon can be constructed from ordinary free and free abelian groups by using generalised free products. The solution of the word problem is a byproduct of this constructive approach. This solution is neither as efficient nor as elegant as Lyndon's. However the method works not only when X is polynomial ring but also for a very wide class of not necessarily commutative rings.

Received March 22, 1985.

[1]Supported in part by the National Science Foundation.

This, however, is not the object of this paper. Instead I shall carry over techniques analogous to those in [1] to solve the word problem for free abelian X-groups, where for simplicity X is given by (1). These free abelian X-groups are the counterpart in this theory to the abelianisations of ordinary free groups, i.e. free abelian groups. Unlike the situation for free abelian groups, the free abelian X-groups turn out to be surprisingly complicated.

1.2 The rest of this paper will be concerned solely with abelian groups. All groups will henceforth be abelian and we will, for the most part, simply refer to abelian X-groups as X-groups.

X-groups can be viewed as groups with multiple operators, in the sense of P.J. Higgins [3]. Much of the ordinary theory of groups can be translated into a theory of groups with multiple operators. Indeed this is true also if one restricts oneself to X-groups, where the theory takes on a particularly sharp form (cf. [3]). In particular the theory of generalised soluble and generalised nilpotent groups has a counterpart in the theory of X-groups. Thus one has the notion of a nilpotent X-group, where the class 1 nilpotent X-groups are those X-groups satisfying the axioms

$$g^1 = g,\ g^{\alpha+\beta} = g^\alpha g^\beta,\ g^{\alpha\beta} = (g^\alpha)^\beta,\ (gh)^\alpha = g^\alpha h^\alpha \quad (g, h \in G, \alpha, \beta \in X).$$

In other words a nilpotent X-group of class 1 is simply an X-module.

The class of X-groups can also be viewed as a variety of universal algebras, in the sense of G. Birkhoff [2]. We shall freely avail ourselves of Birkhoff's theory.

Our main objective here is to prove the following theorem.

THEOREM A. *Let F be a free X-group on $a_1, \dots, a_n$ $(n < \infty)$. Then there is an algorithm which decides whether or not any X-word in $a_1, \dots, a_n$ takes on the value* 1 *in F.*

The proof of Theorem A is carried out by constructing F from a free abelian group on $a_1, \dots, a_n$ and then repeatedly forming direct products and direct limits.

1.3 The class of X-groups is a rich one. Thus, for example, we have the following counterpart of the following well-known theorem for groups due to B.H. Neumann [7].

THEOREM B. *There exist continuously many non-isomorphic* 2*-generator X-groups.*

Generation here is, of course, in the sense of X-group.

Theorem B suggests a host of problems about X-groups, which are analogues of known results about ordinary groups. For example, is there an analogue for X-groups of the Higman embedding theorem (G. Higman [4])?

2. Preliminaries

2.1 We emphasise again that all groups considered here are abelian and that X is given by (1). An X-group then is a group G equipped with an action of X on G satisfying the axioms

$$g^1 = g,\ g^{\alpha+\beta} = g^\alpha g^\beta,\ (g^\alpha)^\beta = g^{\alpha\beta} \quad (g \in G,\ \alpha, \beta \in X).$$

It is easy to deduce the following:

LEMMA 2.1. *Let G be an X-group. Then $g^0 = 1$ and $1^\alpha = 1$ $(g \in G, \alpha \in X)$.*

A subgroup H of an X-group G is termed an X-subgroup if $h^\alpha \in H$ whenever $h \in H$, $\alpha \in X$. If, in addition, $fg^{-1} \in H$ implies $f^\alpha g^{-\alpha} \in H$ for every $\alpha \in X$, H is termed an X-ideal of G and we can turn the factor group G/H into an X-group by setting $(gH)^\alpha = g^\alpha H$. A homomorphism ϕ from an X-group G into an X-group H is termed an X-homomorphism if

$$(g^\alpha)\phi = (g\phi)^\alpha \quad (g \in G,\ \alpha \in X).$$

LEMMA 2.2. *Let F be a free X-group freely generated by S. Then $G = gp(S)$, the group generated by S, is a free abelian group on S.*

Proof. Let H be the free abelian group on a set $S\hat{}$ in a one-to-one correspondence $s \mapsto s\hat{}$ with S. We turn H into an X-group by setting $a^x = a$ whenever $a \in H$, $x \in \{x_1, \dots, x_q\}$. Then the map $s \mapsto s\hat{}$ extends to an X-homomorphism of G onto H and the desired conclusion follows easily.

Lemma 2.2 suggests that the free X-group on S can be constructed from the free abelian group on S, explaining in part the approach we will take in Section 3.

3. The construction of free X-groups

3.1 Our objective here is to describe a procedure for constructing certain X-groups. This depends on the notion of a partial X-group explained below.

DEFINITION 3.1. A group G is termed a partial X-group if to each $g \in G$ there is an associated subset def g of X satisfying the following conditions:

(i) def g is an additive subgroup of X containing 1;
(ii) $\alpha \in$ def g if and only if $g^\alpha \in G$ is defined;
(iii) $g^1 = g$, def $1 = X$;

(iv) if $\alpha, \beta \in \operatorname{def} g$, then $g^{\alpha+\beta} = g^\alpha g^\beta$;
(v) if $\alpha\beta \in \operatorname{def} g$, $\alpha \neq 0 \neq \beta$, then $\alpha, \beta \in \operatorname{def} g$ and $g^{\alpha\beta} = (g^\alpha)^\beta$;
(vi) if $\beta \in \operatorname{def} g^\alpha$ then $\alpha\beta \in \operatorname{def} g$.

So a partial X-group is a group G on which the action of X is only partially defined; and when such an action is defined, the axioms of an X-group are satisfied.

We shall need certain additive subgroups X_n of $X = \mathbf{Z}[x_1, \ldots, x_q]$. X_n is the additive subgroup of X generated by all monomials in $x_1, \ldots, x_q$ of degree at most n including the integer 1. We put $X_0 = \mathbf{Z}$, $X_\infty = X$.

DEFINITION 3.2. A partial X-group G is termed a $\mathscr{P}$-group if:
(i) G is free abelian;
(ii) G is X-torsion-free, i.e., if $g \in G$, $\alpha \in \operatorname{def} g$, $\alpha \neq 0$, then $g^\alpha = 1$ only if $g = 1$;
(iii) if $g \in G$, $g \neq 1$, there exists an element $r \in G$ with the following properties:
(a) $g = r^\rho$, $\rho \in \operatorname{def} r$;
(b) if $r^\sigma = s^\tau$ then there exists $\mu \in \operatorname{def} r$ such that $s = r^\mu$ and $\sigma = \mu\tau$ ($s \in G$, $\sigma \in \operatorname{def} r$, $\tau \in \operatorname{def} s$, $\sigma \neq 0 \neq \tau$);
(c) $\operatorname{def} r = X_n$ for some $n \geq 0$.

We term such an element r in Definition 3.2 a *primitive root* of g. Notice that it is not hard to see that if r_1 is a second primitive root of g, then $r_1 = r^{\pm 1}$. So g has exactly two primitive roots.

The proof of the following lemma is straightforward and will be omitted.

LEMMA 3.1. *Let $G \in \mathscr{P}$, i.e., let G be a $\mathscr{P}$-group, $u, v \in G$, $\alpha \in \operatorname{def} u$, $\beta \in \operatorname{def} v$, $u \neq 1 \neq v$, $\alpha \neq 0 \neq \beta$ If $u^\alpha = v^\beta$ then*:
(i) u and u^α have the same primitive roots;
(ii) u and v have the same primitive roots;
(iii) if r is a primitive root of u, $u = r^\rho$, $v = r^\sigma$ and $\alpha\rho = \beta\sigma$;
(iv) if r is a primitive root of u then the primitive roots of r are r and r^{-1}; in particular if $r = s^\tau$, $s = r^{\pm 1}$ and $\tau = \pm 1$.

Next we record some useful terminology.

DEFINITION 3.3. Let G and H be partial X-groups. Then a group homomorphism θ: $G \to H$ is termed a partial X-homomorphism (X-monomorphism if θ is 1-1) if
(i) $\alpha \in \operatorname{def} g$ implies $\alpha \in \operatorname{def}(g\theta)$;
(ii) $(g^\alpha)\theta = (g\theta)^\alpha$.

We shall have occasion to consider the case when θ is actually an inclusion of G in H. Then θ is a partial X-homomorphism if and only if $\operatorname{def}_G g \subseteq \operatorname{def}_H g$ where $\operatorname{def}_G g = \operatorname{def} g$ and we are thinking of g as an element of G and similarly for H. Thus g^α has the same meaning irrespective of whether g is viewed as an element of G or an element of H provided g^α is already defined in G. We then say G is a partial X-subgroup of the partial X-group H. The main step in our embedding procedure is:

PROPOSITION 3.1. *Let $G \in \mathscr{P}$, $g \in G$, $\alpha \in X$. If $\alpha \notin \operatorname{def} g$ then we can construct a $\mathscr{P}$-group H and a partial X-monomorphism θ: $G \to H$ with the following properties*:

(i) $\operatorname{def}(g\theta) \ni \alpha$;
(ii) *if s is a primitive root in G, s is a primitive root in H*;
(iii) *if r is a primitive root of g in G and if* $\operatorname{def} r = X_n$, *then* $\operatorname{def}(r\theta) = X_m$ *where $m > n$ can be chosen finite or infinite as desired*;
(iv) *if $b \in G$, $b \notin \{r^\beta | \beta \in X_n\}$, then* $\operatorname{def} b = \operatorname{def}(b\theta)$.

Proof. Since $\alpha \notin \operatorname{def} g$, $n < \infty$ (see (iii)). Let

$$R = \left\{ r^\beta | \beta \in X_n \right\}.$$

Now $G \in \mathscr{P}$. It follows that R is a free abelian group. Indeed if M is the set of all monomials

$$x_{i_1} x_{i_2} \cdots x_{i_l} \quad \left(x_{i_j} \in \{x_1, \dots, x_q\},\ l \leq n \right)$$

of degree at most n together with 1, then the elements r^μ $(\mu \in M)$ form a basis for R.

Observe next that R is an isolated subgroup of G, i.e., if l is a positive integer and if $b \in G$, then $b^l \in R$ only if $b \in R$. For if $b^l = r^\beta$ $(\beta \in X)$ then $b = r^{\beta'}$ by Lemma 3.1. Now an isolated subgroup of finite rank of a free abelian group is a direct factor. Thus

$$G = R \times S.$$

Now $g = r^\gamma$, $\gamma \neq 0$ and $\gamma\alpha \in X_m$ for some $m > n$, where we allow $m = \infty$. Consider then the free X-group F of rank 1 on a. Notice that

$$F = \{ a^\gamma | \gamma \in X \}$$

is simply a multiplicative copy of a free X-module on a. Let

$$T = \{ a^\tau | \tau \in X_m \}.$$

T can be viewed as a partial X-group in the obvious way. Moreover $T \in \mathscr{P}$. Next let

$$H = T \times S$$

and define θ: $G \to H$ by

$$\theta\colon r^\rho s \to a^\rho s \quad (\rho \in X_n,\ s \in S)$$

Is is clear that θ is monic. In order to complete the proof of Proposition 3.1 we turn H into a partial X-group as follows:

(i) $\operatorname{def}_H(a^\tau) = \operatorname{def}_T(a^\tau)$ $(\tau \in X_m)$;

(ii) $\operatorname{def}_H(a^\tau s) = \mathbf{Z}$ if $\tau \in X_m$, $\tau \notin X_n$, $s \neq 1$;

(iii) $\operatorname{def}_H(a^\tau s) = \operatorname{def}_G(r^\tau s)$ if $\tau \in X_n$, $s \neq 1$; here, for $\sigma \in \operatorname{def}_G(r^\tau s)$, we define $(a^\tau s)^\sigma = ((r^\tau s)^\sigma)\theta$.

Finally, we need to check that $H \in \mathscr{P}$; this follows by inspection of (i), (ii), (iii) above which define the way in which H is turned into a $\mathscr{P}$-group.

We denote this group H by the triple $\langle G, g, \alpha\rangle$ and identify G with its image $G\theta$ in $\langle G, g, \alpha\rangle$. Then $\langle G, g, \alpha\rangle$ satisfies the following universal mapping property.

LEMMA 3.2. *Let φ be a partial X-homomorphism of the $\mathscr{P}$-group G into the X-group J. Then φ can be continued to a partial X-homomorphism φ^+ of $\langle G, g, \alpha\rangle$ into J.*

Proof. By its very definition,

$$\langle G, g, \alpha\rangle = T \times S.$$

Define a map φ^+ of $\langle G, g, \alpha\rangle$ into J by

$$\varphi^+\colon a^\tau s \to (a\varphi)^\tau s\varphi \quad (\tau \in X_m,\ s \in S)$$

It is clear that φ^+ is well-defined. Moreover it is certainly a homomorphism of groups. We have only to check that it is a partial X-homomorphism. To this end we need to inspect the three conditions set down in the proof of Proposition 3.1, viz. (i), (ii), and (iii). The fact that φ^+ then is a partial X-homomorphism follows immediately.

LEMMA 3.3. *Let I be a well-ordered index set, $\{G_i \mid i \in I\}$ a family of $\mathscr{P}$-groups indexed by the elements of I. Suppose that if $i \leq j$ then G_i is a direct factor of G_j. Suppose furthermore that the inclusion $G_i \hookrightarrow G_j$ $(i \leq j)$ is a partial X-homomorphism. Finally suppose that if $r \in G_i$ is primitive, then r is a primitive element in G_j whenever $i \leq j$. Then the ascending union*

$$G = \bigcup_{i \in I} G_i$$

viewed as a partial X-group in the obvious way, is a $\mathscr{P}$-group.

Proof. We check the conditions for G to be a $\mathscr{P}$-group in turn.

First of all to see that G is free abelian pick a basis for G_1, supplement it so as to provide a basis for G_2, supplement it to provide a basis for G_3 and so on. This yields ultimately a basis for G.

Next observe that G is X-torsion-free because each G_i is X-torsion-free.

Finally in order to complete the verification that $G \in \mathscr{P}$ we need only to examine what happens to an element $r \in G_i$ which is primitive and is such that $\operatorname{def}_{G_j} r$ alters an infinite number of times as j ranges over the elements of I exceeding i. But in this situation we can find a subsequence $i < j_1 < j_2 < \dots$ such that

$$\operatorname{def}_{G_{j_k}}(r) = X_{n_k} \quad \text{and} \quad n_1 < n_2 < \dots .$$

So $\operatorname{def}_G r = X$ as needed.

We come now to our main embedding theorem:

THEOREM 3.1. *Let G be a $\mathscr{P}$-group. Then G can be embedded in a $\mathscr{P}$-group $\hat{G}$ such that*

(i) *$\hat{G}$ is an X-group,*

(ii) *for every X-group and every partial X-homomorphism θ from G into H there is a unique X-homomorphism $\hat{\theta}$ from $\hat{G}$ into H.*

Proof. Let $G_0 = G$, $\theta_0 = \theta$. For each ordinal η we define a $\mathscr{P}$-group G_η and a partial X-homomorphism θ_η from G_η into H by transfinite induction. Firstly if η has a predecessor, say $\eta - 1$, then $G_{\eta-1} \in \mathscr{P}$ and $\theta_{\eta-1}\colon G_{\eta-1} \to H$ has already been defined. If $G_{\eta-1}$ is an X-group, we define $G_\eta = G_{\eta-1}$ and $\theta_\eta = \theta_{\eta-1}$. If $G_{\eta-1}$ is not an X-group, there exists $g \in G_{\eta-1}$ and $\alpha \in X$ such that $\alpha \notin \operatorname{def}_{G_{\eta-1}}(g)$.

Let

$$G_\eta = \langle G_{\eta-1}, g, \alpha \rangle.$$

By Lemma 3.3, $\theta_{\eta-1}$ can be extended to a partial X-homomorphism $\theta_\eta\colon G \to H$. Finally if η is a limit ordinal then we proceed as follows.

Suppose inductively that for

$$I = \{\mu | \mu < \eta\}$$

the family of $\mathscr{P}$-groups $\{G_\mu | \mu \in I\}$ satisfies the conditions of Lemma 3.3 and that partial X-homomorphisms $\theta_\mu\colon G_\mu \to H$ have been defined so that θ_μ agrees with θ_λ whenever $\mu \leq \lambda < \eta$. We let

$$G_\eta = \bigcup G_\mu, \quad \mu < \eta.$$

Then, by Lemma 3.3, $G_\eta \in \mathscr{P}$ and $\theta_\eta\colon G_\eta \to H$ is a partial X-homomorphism. Now let ρ be an ordinal chosen sufficiently large so as to insure that $G_\rho = G_{\rho+1}$. We put

$$\hat{G} = G_\rho, \; \hat{\theta} = \theta_\rho.$$

This completes the proof of Theorem 3.1.

COROLLARY 3.1. *Let G be a free abelian group freely generated by the set Y. Furthermore turn G into a partial X-group by defining*

$$\operatorname{def} g = \mathbf{Z} \text{ if } g \in G,\ g \neq 1, \qquad \operatorname{def} 1 = X$$

Then ($G \in \mathscr{P}$ *and*) $\hat{G}$ *is a free X-group freely generated by Y.*

4. Properties of free X-groups

4.1 There are a number of properties of free X-groups that can be deduced from Corollary 3.1. Two of them involve notions from P.J. Higgins theory of groups with multiple operators [3].

DEFINITION 4.1. Let G be an X-group, $g \in G$. Then the centraliser $C(g)$ of g is defined by

$$C(g) = \left\{ h \in G \mid (h^{\alpha}g)^{\beta} = h^{\alpha\beta}g^{\beta} \text{ for all } \alpha, \beta \in X \right\}.$$

DEFINITION 4.2. Let G be an X-group. Then the center $C(G)$ of G is defined by

$$C(G) = \bigcap_{g \in G} C(g).$$

Notice that in general $C(g)$ need not, on the face of it, be an X-subgroup. However it is not hard to check that $C(G)$ is an X-ideal.

The proof of Corollary 3.1 yields the following properties of a free X-group.

THEOREM 4.1. *Let F be a free X-group. Then the following hold.*

(a) *If* $f \in F$, $f \neq 1$, *then*

$$C(f) = \{ g^{\alpha} \mid \alpha \in X \}$$

for some $g \in G$, *i.e.,* $C(f)$ *is a cyclic X-group.*

(b) *If F is of rank at least two,* $C(F) = 1$.

(c) *F is a free abelian group and if* $f \in F$, $f \neq 1$ *then* $f^{\alpha} = 1$ ($\alpha \in X$) *only if* $\alpha = 0$, *i.e., F is X-torsion-free.*

4.2 Now we come to our main result. Thus suppose F is a free X-group on $x_1, \dots, x_q$. Then every element of F can be represented as an X-word $w(x_1, \dots, x_q)$ in $x_1, \dots, x_q$, i.e., a meaningful expression involving $x_1, \dots, x_q$, the group operations and the operations of X on F. The following theorem holds.

THEOREM 4.2. *The word problem for F is solvable, i.e., there is an algorithm whereby one can decide whether or not any X-word $w(x_1, \ldots, x_q) = 1$ in F.*

The proof of Theorem 4.2 depends heavily on Proposition 3.1, which is crucial in the proof of the following:

LEMMA 4.1. *Let G be a free abelian group with basis $y_1, \ldots, y_p (p < \infty)$ and let*

$$G_0 = G_1,\ G_1 = \langle G_0, g_1, \alpha_1 \rangle, \ldots, G_k = \langle G_{k-1}, g_k, \alpha_k \rangle$$

be a sequence of successive applications of the construction in Proposition 3.1. Then given any X-word $w = w(y_1, \ldots, y_p)$ there is an algorithm which decides whether or not $w \in G_k$. Moreover if $w \in G_k$ there is an algorithm for expressing w in terms of a free basis for G_k. If $w \notin G_k$ then there is an algorithm for constructing finitely many further adjunctions

$$G_{k+1} = \langle G_k, g_{k+1}, \alpha_{k+1} \rangle, \ldots, G_{k+l} = \langle G_{k+l-1}, g_{k+l}, \alpha_{k+l} \rangle \tag{2}$$

so that $w \in G_{k+l}$, and an algorithm for expressing w in terms of a free basis for G_{k+l}.

Proof. We may assume that each of $G_0, \ldots, G_k$ is actually a finitely generated abelian group. Notice that Proposition 3.1 provides a basis for each of the groups $G_0, \ldots, G_k$.

The proof of Lemma 4.1 will be by induction on the number m of non-constant elements of X that occur in w. If $m = 0$, $w \in G$ and the desired conclusion follows on re-expressing the elements $y_1, \ldots, y_p$ in terms of the new basis for G_k given by the proof of Proposition 3.1.

Suppose then that $m > 0$. Now there are two cases to consider.

Case 1. $w = w_1^\lambda$ $(\lambda \in X, \lambda \notin \mathbf{Z})$. In this case w_1 involves $m - 1$ elements of X and so inductively we can find a series (2) so that $w_1 \in G_{k+l}$. We can then constructively express w_1 in terms of a basis for G_{k+l}. Proposition 3.1 can then be applied to determine whether or not $\lambda \in \operatorname{def}_{G_{k+l}}(w_1)$-the point here is that the very construction of G_{k+l} carries with it the information as to what elements of X act on the elements of G_{k+l}.

If $\lambda \in \operatorname{def}_{G_{k+l}}(w_1)$ we compute w_1^λ in G_{k+l} and thence backtrack to decide whether or not $w_1^\lambda \in G_k$. If $\lambda \notin \operatorname{def}_{G_{k+l}}(w_1)$ we apply the construction of Proposition 3.1 once again to yield

$$G_{k+l+1} = \langle G_{k+l}, w_1, \lambda \rangle,$$

a free basis for G_{k+l+1} and an explicit representation of $w = w_1^\lambda$ in terms of this basis.

Case 2. $w = w_1 w_2$ where both w_1 and w_2 involve fewer than m elements of X. Here we follow the procedure in Case 1 for w_1 and w_2 in turn.

Case 3. $w = w_1^\lambda w_2$ where $\lambda \in X$, $\lambda \notin \mathbf{Z}$, w_1 involves less than m elements of X and w_1 is a group word in $y_1, \ldots, y_p$. Here we apply the procedure in Case 1 again for w_1. In the final step here after obtaining the group G_{k+l} for w_1 we check whether $w_1^\lambda w_2 \in G_k$ and also re-express $w_1^\lambda w_2$ in terms of a free basis for G_{k+l}.

This completes the proof of Lemma 4.1 and, on allying it with Corollary 3.1, the proof of Theorem 4.1.

5. Counting finitely generated X-groups

5.1 Our objective here is to prove the following:

THEOREM 5.1. *There are continuously many non-isomorphic 2-generator X-groups.*

We consider the free nilpotent X-group G of class two on a and b (see P. Higgins [3]) where we assume to begin with that $X = \{x\}$. It is easy to see that G, qua ordinary group, is free abelian with basis

$$a, a^x, \ldots, b, b^x, \ldots, c_{\alpha,\beta}, c_{\alpha,\beta}^x, \ldots \quad (\alpha, \beta \in X, \alpha \neq 0 \neq \beta)$$

where $c_{\alpha,\beta}$ is defined by the equation

$$(a^\alpha b^\beta)^x = a^{\alpha x} b^{\beta x} c_{\alpha,\beta}.$$

Thus the center of G is a free X-group of countably infinite rank. So it contains continuously many ideals which implies G itself contains continuously many ideals. Therefore G has continuously many non-isomorphic quotients. Notice now that G can also be viewed as an X-group, where $X = \{x_1, \ldots, x_q\}$, on setting $g^{x_i} = g^x$ $(i = 1, \ldots, q)$.

This completes the proof of Theorem 5.1.

In conclusion let me point out that it seems likely that the celebrated theorem of Higman, Neumann and Neumann [5] that every countable group is a subgroup of a 2-generator group has a counterpart for X-groups.

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