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DECISION PROBLEMS FOR SOLUBLE GROUPS OF FINITE RANK

BY

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In Memoriam W.W. Boone

1. Introduction and results

In the present work we shall be concerned with three classical decision problems of group theory, the word problem, the generalized word problem and the conjugacy problem. It is now known that all three problems have negative solutions even in the class of finitely presented soluble groups. The first example of a finitely presented soluble group with insoluble word problem was given by Harlampovič [8]. Further examples have been found by Baumslag, Gildenhuys and Strebel [3].

In the light of these negative results it is of interest to discover finiteness conditions which are strong enough to imply solubility of one or more of the three decision problems. Indeed some such conditions are already known. Baumslag, Cannonito and Miller [1] showed that the word problem is soluble for nilpotent-by-polycyclic-by-finite groups which satisfy max-n, the maximal condition on normal subgroups; in particular this conclusion applies to finitely generated abelian-by-polycyclic-by-finite groups, by a well-known theorem of P. Hall [7]. At this point it is as well to note that Harlampovič's example has derived length 3 and is nilpotent of class 4-by-abelian; of course the group does not have max-n. Another cautionary remark; there are soluble groups of derived length 3 satisfying max-n which are not recursively presentable and thus have insoluble word problem; such examples are constructed in [16]. However it remains an open question whether a finitely presented (or even recursively presented) soluble group with max-n necessarily has soluble word problem.

Another positive result on the word problem was recently obtained by Cannonito and Robinson [4]; it was shown that a finitely generated soluble group with finite Prüfer rank has soluble word problem if and only if the

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group is recursively presented; furthermore the word problem is soluble for finitely generated soluble groups of finite Prüfer rank which are residually finite. On the other hand, cardinality arguments establish the existence of finitely generated soluble groups of finite Prüfer rank which have insoluble word problem.

Much less is known about the generalized word problem and the conjugacy problem. Both problems have positive solutions for polycyclic-by-finite groups since such groups have separable subgroups (Mal'cev [11]) and are conjugacy separable (Remeslennikov [13], Formanek [6]). Also it has been observed by Romanovskiĭ [17] that the generalized word problem is soluble for finitely generated abelian-by-nilpotent groups. Finally a recent article by Noskov [12] gives a positive solution to the conjugacy problem for finitely presented metabelian groups.

Soluble groups with finite rank. The aim of the present work is to determine to what extent the three decision problems can be solved for the various classes of soluble groups with finite rank. In this investigation we shall not restrict ourselves to finitely generated groups.

We begin with a review of the classes of soluble groups involved. If A is an abelian group, $r_p(A)$ and $r_0(A)$ denote respectively the *p-rank* and the *torsion-free rank* of A. The *total rank* of A is

$$r_0(A) + \sum_p r_p(A),$$

the sum being formed over all primes p.

A soluble group has *finite total rank* if for some series of finite length with abelian factors each factor has finite total rank. It is clear that a soluble group of finite total rank has finite *Prüfer rank* (i.e. there is a finite upper bound for the minimum number of generators of a finitely generated subgroup).

A *minimax group* is a group having a series of finite length whose factors satisfy either max (the maximal condition) or min (the minimal condition). It is straightforward to show that the soluble minimax groups are exactly the poly- (cyclic or quasicyclic) groups. If G is a finite extension of a soluble minimax group, the number of infinite factors in a series with cyclic or quasicyclic factors is an invariant

$$m(G)$$

called the *minimality* of G. Obviously every soluble minimax group has finite total rank. Finally it should be mentioned that for finitely generated soluble groups the properties "finite Prüfer rank", "finite total rank" and "minimax" coincide; for more on this see [14].

Results. Our conclusions relate to soluble groups which have finite total rank or are minimax and which have a recursive presentation. There are three main results.

THEOREM 2.3*. *Let G be a finite extension of a soluble group of finite total rank. Then the word problem can be solved for a presentation of G if and only if the presentation is recursive.*

THEOREM 3.1. *Let G be a finite extension of a soluble minimax group. Assume that G has a recursive presentation and that H is a subgroup of G which is recursively enumerable in terms of the presentation. Then there is an algorithm which decides membership of elements of G in H.*

THEOREM 4.1. *Let G be a finite extension of a soluble minimax group. Assume that G has a recursive presentation and that g is an element of G given in terms of the presentation. Then there is an algorithm which decides if an element of G is conjugate to g.*

Of these theorems the first is a generalization of the main result of [4]; it in turn is a special case of a more general but less quotable result (2.3) in §2. The third theorem represents a positive solution to a weak form of the conjugacy problem, the point being that the algorithm constructed is not uniform in g. Indeed Collins [5] has demonstrated that this form of the conjugacy problem is definitely weaker. We do not know if a uniform algorithm exists in the situation of Theorem 4.1. In addition it should be said that the algorithm of Theorem 3.1 is not uniform in H.

Examples are given which show that Theorems 3.1 and 4.1 are not valid for soluble groups of finite total rank. Thus the class of soluble-by-finite minimax groups seems to represent the natural setting for theorems of this type.

As a consequence of the main theorems one concludes that the following properties coincide for a group G which is a finite extension of a soluble minimax group:

(i) G has a recursive presentation,
(ii) G has soluble word problem,
(iii) G has soluble generalized word problem,
(iv) G has soluble (weak) conjugacy problem.

In particular all four properties are enjoyed by finitely presented soluble groups of finite Prüfer rank, and by finitely generated residually finite soluble groups of finite Prüfer rank.

It does not seem possible to obtain results like Theorems 3.1 and 4.1 by appealing to separability of subgroups of conjugacy separability, as in the case of polycyclic groups. Indeed Jeanes and Wilson [10] have proved that a finitely generated soluble group has separable subgroups if and only if it is polycyclic, while Wehrfritz [20] has exhibited a finitely presented soluble minimax group which is not conjugacy separable.

Notation. If A is a module over a group Q, the set of Q-fixed points is written A^Q. Also $[A, Q]$ is the submodule generated by all $[a, x] \equiv a(x-1)$, $a \in A$, $x \in Q$.

2. The word problem

DEFINITION . A normal subgroup W of a group G is said to be *weakly minimal normal* if it is not a torsion group but all its proper G-invariant quotient groups are torsion. The subgroup generated by all weakly minimal normal subgroups of G is called the *weak socle*,

$$\mathrm{wsoc}(G);$$

this is taken to be the identity subgroup should G turn out to have no weakly minimal normal subgroups.

Recall that a set of normal subgroups of a group is termed *independent* if the subgroup which they generate is their direct product. It is a consequence of Zorn's Lemma that any group G possesses a maximal independent set of weakly minimal normal subgroups. Let $\{W_\lambda | \lambda \in \Lambda\}$ be such a set of subgroups of G, presumed non-empty. Choose an element g_λ of infinite order from W_λ; then the set $\{g_\lambda | \lambda \in \Lambda\}$ will be called a *basis* of wsoc(G). It follows easily from the definitions that

$$\mathrm{wsoc}(G)/\langle g_\lambda^G | \lambda \in \Lambda\rangle$$

is a torsion group. Also of course $\langle g_\lambda^G \rangle$ is weakly minimal normal in G.

Finally, a *prime base* of a group G is a set of elements of prime order whose conjugates exhaust the set of all elements of prime order in G.

With the aid of these definitions a rather general criterion can be given for a group to have soluble word problem. The proof of this result is in the spirit of arguments used by Huber-Dyson [9].

2.1. *Let G be a group with a recursive presentation such that*

(i) wsoc(G) *has a recursively enumerable* (*r.e.*) *basis* (*in terms of the presentation*),

(ii) *either G or its maximal normal torsion subgroup has a r.e. prime base P,*

(iii) *if N is a non-trivial normal subgroup of G, then either $N \cap$* wsoc(G) $\neq 1$ *or else N contains a non-trivial normal torsion subgroup of G. Then the word problem for the given presentation can be solved.*

Proof. Let

$$R \rightarrowtail F \overset{\pi}{\twoheadrightarrow} G$$

be the given presentation, F being free. Suppose that $\{a_1, a_2, \dots\}$ is a r.e. basis of wsoc(G), and let u_i be a preimage of a_i under π. Also let $\{v_1, v_2, \dots\}$ be a r.e. set of preimages under π of the elements of P. Suppose that f is a given element of F; it must be shown how to decide if f is in R or not. Put $g = f^{\pi}$ and $N = \langle g^G \rangle$; of course it is a question of deciding whether N is trivial.

If $N \cap P$ is empty, then N contains no non-trivial normal torsion subgroups of G. If $N \neq 1$, then by hypothesis $N \cap \mathrm{wsoc}(G) \neq 1$. Writing $N_i = \langle a_i^G \rangle$, we have

$$L \equiv N \cap \operatorname*{Dr}_i N_i \neq 1, \quad i = 1, 2, \dots$$

so that the projection of L into some direct factor N_i is non-trivial, and of course normal in G. Since N_i is weakly minimal in G, some positive power of a_i must lie in NM_i where

$$M_i = \langle N_j | j \neq i \rangle.$$

Therefore $N \neq 1$ if and only if either $N \cap P$ is non-empty or else $a_i^m \in NM_i$ for some positive i and m.

The algorithm which decides whether f is a relator consists of three procedures:

(i) enumerate the set of all relators R;
(ii) enumerate all words of the form xv_i where $x \in \langle f^F \rangle$ and $i = 1, 2, \dots$;
(iii) enumerate all words of the form $u_i^{-m} x w_1, \dots w_{i-1} w_{i+1} \dots w_r$ where i, r, m are positive integers, $x \in \langle f^F \rangle$ and $w_j \in \langle u_j^F \rangle$.

Either f will be obtained by procedure (i), in which event f is a relator, or else a relator will be produced by procedure (ii) or (iii), which means that f is not a relator.

Definition. A group is said to satisfy the *weak minimal condition for normal subgroups*, wmin-n, if in every descending chain of normal subgroups all but a finite number of the factors are torsion groups.[2] For example, every group with finite torsion-free rank satisfies wmin-n (a group has *finite torsion-free rank* if it has a series of finite length whose non-torsion factors are infinite cyclic).

[2] Zaĭcev [21] has used this terminology in a different sense.

The following is an application of 2.1.

2.2. *Let G be a group satisfying the weak minimal condition on normal subgroups and let a presentation of G be given.*

(i) *If the word problem is soluble for the presentation, then the presentation is recursive and G has a r.e. prime base.*

(ii) *If the presentation is recursive and if either G or its maximal normal torsion subgroup has a r.e. prime base, then the word problem is soluble for the presentation.*

Proof. (i) This is clear; for if the word problem is soluble, the set of all elements of prime order can be enumerated and taken to form a prime base for G.

(ii) Let N be a non-trivial normal subgroup which contains no non-trivial normal torsion subgroups of G. If N did not contain a weakly minimal normal subgroup of G, there would exist an infinite descending chain of normal subgroups of G with non-torsion factors, violating wmin-n. It follows that $N \cap \mathrm{wsoc}(G) \neq 1$. Observe that wmin-n implies the existence of a finite basis of $\mathrm{wsoc}(G)$ (which may be empty). The result now follows from 2.1.

2.3. *Let G be a group which satisfies the weak minimal condition on normal subgroups. Assume that the maximal normal torsion-subgroup T of G is a Černikov group. Then the word problem is soluble for a presentation of G if and only if the presentation is recursive.*

Proof. It is well known that the elements of prime order in a Černikov group fall into finitely many conjugacy classes. Hence there is a finite prime base of T; the result is now a consequence of 2.2.

In particular this result applies to a finite extension of a soluble group with finite total rank; thus Theorem 2.3* follows at once. On the other hand, there is no such result for soluble groups of finite Prüfer rank, as a very simple example shows.

2.4. *There is a recursively presented abelian torsion group with finite Prüfer rank such that the word problem is insoluble for every presentation.*

Proof. Let π be a r.e. but non-recursive set of primes. Define G to be the group with generators x_p where p is any prime, and relations

$$[x_p, x_q] = 1 = x_p^p \quad (\forall p, q),$$

together with

$$x_p = 1 \text{ if } p \in \pi.$$

Evidently $G \cong \mathrm{Dr}_{p \in \pi'} C_p$ and the above is a recursive presentation. Suppose that the word problem were soluble for some presentation of G. Then it would be possible to compute the orders of the generators in that presentation, and hence to list all primes p for which there is an element of order p in G. However this is absurd since π' is not r.e..

Next we give an algorithm which decides if an element is contained in the maximal normal torsion subgroup; this will prove useful in the sequel.

2.5. *Let G be a group with finite torsion-free rank whose torsion-factors are locally finite and whose maximal normal torsion subgroup T is a Černikov group. Assume also that G is recursively presented. Then there is an algorithm which decides membership in T. Thus G/T has soluble word problem.*

Proof. It will be argued first that T is r.e. in terms of the given presentation. By a slight extension of a result of Mal'cev (cf. [14, Lemma 9.34]) the group G/T has a series of normal subgroups of finite length whose infinite factors are torsion-free abelian groups of finite rank. By [14, Theorem 9.39.3] there is a torsion-free normal subgroup of finite index in G/T. Consequently there is a positive integer m such that G^mT/T is torsion-free and G/G^mT is finite.

Let F denote the set of all elements of finite order in G. Then F is r.e. since one can enumerate all pairs (g, n) with $g \in G$, $n = 1, 2, \ldots,$ and in each case decide whether g^n equals the identity, using the positive solution of the word problem furnished by 2.3. Since G^m is obviously r.e., so is $G^m \cap F$. Moreover $G^m \cap F = G^m \cap T$ since G^mT/T is torsion-free. But $T/G^m \cap T$ is certainly finite, so it follows that T is r.e..

Finally G/T has a recursive presentation and by 2.3 the word problem is soluble for this group.

3. The generalized word problem

The main result in this section is the following:

3.1. *Let G be a finite extension of a soluble minimax group. Assume that G has a recursive presentation and that H is a subgroup of G which is recursively enumerable in terms of the presentation. Then there is an algorithm which decides membership of elements of G in H.*

Proof. Let m be the minimality of G. If $m = 0$, then G is finite and the result is clear. Assume $m > 0$. The proof is by induction on m. The group G contains an infinite abelian normal subgroup—for example, the centre of the

Fitting subgroup of a soluble normal subgroup of finite index. Consequently there is a non-trivial normal abelian subgroup A_0 which is either torsion-free or a divisible p-group for some prime p. If A_0 is torsion-free, define A to be the normal closure in G of some non-trivial element of A_0; then A is r.e.

Suppose, on the other hand, that A_0 is a divisible p-group. Let T be the unique maximum normal torsion subgroup of G. Then T is r.e. by 2.5. Also T is a Černikov group, so some T^n is a r.e. divisible abelian torsion group containing A_0. Since the word problem is soluble in G, the p-component of T^n is r.e.; define A to be this subgroup.

Using the r.e. subgroup A constructed in the last two paragraphs, one observes that $m(G/A) < m(G)$, while G/A has a recursive presentation. Since HA/A is r.e., induction shows that there is an algorithm to decide membership in HA/A.

Let g be a given element of G. One first decides whether gA belongs to HA/A. If it does not, then certainly $g \notin H$. Suppose that $g \in HA$. Enumerate all elements of the form $h^{-1}g$ with h in H, and in each case decide if $h^{-1}g$ belongs to A. This is possible because the word problem is soluble in G/A. By this means an element $a = h^{-1}g$ of A with h in H can be found. Now $g \in H$ if and only if $a \in H \cap A$. Thus it suffices to solve the word problem for $A/H \cap A$. But $H \cap A$ is r.e. since both H and A are r.e. Hence $A/H \cap A$ has a recursive presentation and the result follows from (2.3).

In the proof of (3.1) it is essential that we are dealing with a quotient-closed class of groups. Soluble groups of finite total rank do not form such a class, and in fact the theorem fails for this class.

3.2. *There is a recursively presented torsion-free abelian group of rank* 1 *with a recursively enumerable subgroup H such that no algorithm can decide membership in H.*

Proof. Let G be the additive group of rational numbers with square-free denominators. Let π be a r.e. non-recursive set of primes and define H to be the subgroup of G generated by the reciprocals of the primes in π. Then G has a recursive presentation and H is r.e. in terms of this presentation. However G/H is isomorphic with the group of 2.4, and this group has insoluble word problem for every presentation. Therefore it is impossible to decide membership in H algorithmically.

4. The conjugacy problem

The object of this section is to prove the following result.

4.1. *Let G be a finite extension of a soluble minimax group. Assume that G has a recursive presentation and that g is an element of G given in terms of the* *presentation. Then there is an algorithm which decides if an element of G is conjugate to g.*

The major difficulty of the proof resides in establishing the following result, in reality a special case.

4.2. *Let Q and A be minimax groups, A being abelian and Q soluble-by-finite. Assume that Q and A have recursive presentations and that A is a Q-module by means of an explicit action of Q on A expressed in terms of the presentation. Let a be a fixed element of A. Then there is an algorithm which decides of a given element b of A whether there is a q in Q such that $b = aq$.*

Deduction of 4.1 from 4.2.

Let us assume for the present that 4.2 has been proved. The proof of 4.1 is by induction on $m(G)$. If $m(G) = 0$, then G is finite and the result is obvious. Assume therefore that $m(G) > 0$.

(1) *There is an infinite r.e. normal abelian subgroup A_0 of G.*

In the first place there is an infinite normal abelian subgroup A of G, for example, the centre of the Fitting subgroup of soluble normal subgroup of finite index in G. If A_1 is reduced, some A_1^m with $m > 0$ is torsion-free. Define A_0 to be the normal closure in G of any non-trivial element of A_1^m. Then A_0 is certainly r.e.

If, on the other hand, A_1 contains a p^∞-subgroup for some prime p, the maximum normal torsion subgroup T of G will also contain this p^∞ subgroup. Also T is a Černikov group, so some T^m with $m > 0$ is divisible abelian. Let A_0 be the p-component of T^m. By 2.5 the subgroup T is r.e., whence so is T^m. Finally A_0 is r.e. because the word problem is soluble in G.

(2) *It may be assumed that $[g, G] \leq A_0$. Moreover it suffices to decide conjugacy to g of elements x such that $x \equiv g \bmod A_0$.*

Since A_0 is r.e., G/A_0 is recursively presented, and because $m(G/A_0) < m(G)$, there is an algorithm to decide conjugacy to gA_0 in G/A_0. If an element x of G is not conjugate to g modulo A_0, it is certainly not conjugate to g. Thus one can assume that xA_0 is conjugate to gA_0. Replacing x by a suitable conjugate, one can further assume that $x \equiv g \bmod A_0$. Put $K = C_G(gA_0)$; then $\langle x, g, A_0\rangle \leq K$ and x is conjugate to g in G if and only if it is conjugate to g in K. Also K is r.e. since the word problem for G/A_0 is soluble (by 2.3). Hence K has a recursive presentation. One can therefore replace G by K and assume that $[g, G] \leq A_0$.

(3) *Conclusion.*

Consider the r.e. subgroup $[A_0, g]$, which is normal in G because $[g, G] \leq A_0$. If this subgroup is infinite, then $m(G/[A_0, g]) < m(G)$ and there is an algorithm to decide conjugacy to $g \bmod [A_0, g]$. Therefore it is enough to consider elements x such that $x \equiv g \bmod [A_0, g]$. But then $x^{-1}g = [a_0, g]$ for some $a_0 \in A_0$, and $x = g^{a_0}$; thus such elements are always conjugate to g.

Now assume that $[A_0, g]$ has finite order m. Then $[A_0^m, g] = 1$. Clearly one can replace A_0 by A_0^m and then make the reductions of (2). Hence one can assume that $A = \langle A_0, g \rangle$ is a normal abelian subgroup of G. Of course A is recursively presented, as is $Q = G/A$; also A is a Q-module via conjugation and the Q-action is explicitly known. Hence 4.2 can be applied to decide conjugacy to g of any element x satisfying $x \equiv g \bmod A_0$.

Proof of 4.2. We begin with some terminology. Let Q be a group and let A be a Q-module. If a, b are elements of A, then b is said to be *Q-conjugate* to a if $b = aq$ for some q in Q. Thus our goal is to find an algorithm that decides Q-conjugacy to a.

Next assume that A is $\mathbf{Z}$-torsion-free. Then, of course, A is Q-rationally irreducible if $A \otimes_{\mathbf{Z}} \mathbf{Q}$ is a simple $\mathbf{Q}Q$-module. If A is K-rationally irreducible for every subgroup K with finite index in Q, then A will be called a *Q-plinth*. This is an extension of the terminology of [18].

The proof of 4.2 is by induction on $m(A) + m(Q)$. Notice that $m(A)$ may be assumed positive; otherwise A and $Q/C_Q(A)$ are finite and the result is clear.

(1) *There is a non-zero Q-submodule A_0 of A which is either $\mathbf{Z}$-torsion-free or a divisible p-group for some prime p.*

If A has a p^∞-subgroup for some prime p, let A_0 denote the maximal divisible p-subgroup of A. If however A is reduced, its torsion-subgroup is finite and there is a positive integer m such that A^m is torsion-free; in this case let $A_0 = A^m$.

(2) *The case A_0 a divisible p-group.*

We begin with an important reduction.

(2a) *It may be assumed that $A = \langle a, A^* \rangle$ where A^* is a non-zero r.e. divisible p-group and a Q-submodule, A/A^* is infinite, $Q/C_Q(A^*)$ is abelian-by-finite and $[a, Q] \leq A^*$. Furthermore it suffices to decide Q-conjugacy to a of elements b such that $b \equiv a$* mod A^*.

In the first place observe from its construction that A_0 is r.e. since the word problem is soluble in A. Next Q acts on A_0 as a linear group over the ring of p-adic integers. By the Lie-Kolchin-Mal'cev Theorem (see for example [14]) there is a $m > 0$ such that $N \equiv (Q^m)'$ acts unipotently on A_0. Hence there is a $i \geq 0$ for which $[A_0, {}_iN] > [A_{0,\,i+1}N] = 0$. Put $A^* = [A_0, {}_iN]$, observing that this Q-submodule is divisible and r.e.. Also $Q/C_Q(A^*)$ is abelian-by-finite since $C_Q(A^*) \geq N$ and Q/N is abelian-by-finite.

Since A/A^* is recursively presented and $m(A/A^*) < m(A)$, there is an algorithm to decide Q-conjugacy of $b + A^*$ to $a + A^*$. It is therefore clear that it suffices to discuss elements b satisfying $b \equiv a \bmod A^*$. Now

$$C_Q(a + A^*/A^*)$$

is r.e. because the word problem is soluble in A/A^*. Hence one may replace Q by $C_Q(a + A^*/A^*)$ and assume that $[a, Q] \leq A^*$. If A/A^* is finite, then A satisfies min; in this situation one need only enumerate the finite set of elements of A with the same order as a and check each element for Q-conjugacy to a. Thus one can suppose that A/A^* is infinite. Finally, it is clearly in order to replace A by $\langle a, A^* \rangle$ since the latter is a Q-submodule containing b.

(2b) *The case $Q/C_Q(A)$ abelian-by-finite.*

There is a positive integer m such that $[A, M'] = 0$ where $M = Q^m$. Clearly M is recursively presented and Q/M is finite. In fact it suffices to be able to decide M-conjugacy to a. For let $\{u_1, \dots, u_k\}$ be a transversal to M in Q. Let $q \in Q$ and write $q = xu_i$ with $x \in M$ and $1 \leq i \leq k$. Then $b = aq$ holds if and only if $bu_i^{-1} = ax$; thus it is enough to check each of the k elements bu_i^{-1} for M-conjugacy to a.

First consider the case where M acts unipotently on A^*, so that $[A^*, {}_iM] = 0$ for some least $i > 0$. Put $B = [A^*, {}_{i-1}M]$; then B is a r.e. Q-submodule and $[B, M] = 0$. Thus one can replace A^* by B—making once again the reductions of (2a)—and assume that $[A^*, M] = 0$.

In this case $[a, M]$ actually *consists* of elements of the form $a(x - 1)$, $x \in M$. Hence b is M-conjugate to a if and only if $b - a \in [a, M]$. Now $[a, M]$ is r.e., so the word problem is soluble in $A/[a, M]$. Therefore M-conjugacy to a is decidable.

Now consider the case where M does not act unipotently on A^*. Then there is an element x_0 of M which does not act unipotently on M. Consequently there is an integer j with the property $[A^*, {}_jx_0] = [A^*, {}_{j+1}x_0] \neq 0$. One now replaces A^* by $[A^*, {}_jx_0]$ and Q by M and makes the reductions of (2a). Thus one may assume that

$$A^* = [A^*, x_0];$$

this implies that $(A^*)^{\langle x_0 \rangle}$ is finite, say of order n. Hence $[A, x_0] \leq A^* = [A^*, x_0]$ and $A = A^* + A^{\langle x_0 \rangle}$. It follows that $A^{\langle x_0 \rangle}$ is not a torsion group. Next,

$$[A^{\langle x_0 \rangle}, M] \leq (A^*)^{\langle x_0 \rangle}$$

since $[A, M'] = 0$. Hence

$$\left[(A^{\langle x_0 \rangle})n, M\right] = [A^{\langle x_0 \rangle}, M]n \leq (A^*)^{\langle x_0 \rangle}n = 0.$$

Therefore $(A^{\langle x_0\rangle})n \leq A^M$ and so A^M cannot be a torsion group. It follows that $|A\colon A^M + A^*|$ is finite, equal to d say.

The next step is to show how to decide if ad and bd in $A^M + A^*$ are M-conjugate. Let $c = bd - ad \in A^*$ and write $ad = a_0 + a^*$ where $a_0 \in A^M$ and $a^* \in A^*$. Then $ad(x-1) = a^*(x-1)$ for all x in M. Clearly $c + a^*$ is M-conjugate to a^* if and only if bd is M-conjugate to ad. This is certainly decidable since A^* is r.e..

These considerations show that one can assume bd to be M-conjugate to ad, say $bd = (ad)x$ with x in M. Then $b - ax$ belongs to $L \equiv \{u \in A^* | ud = 0\}$. Thus one can assume that $b \equiv a \bmod L$.

Let $K = C_M(a + L/L)$ and $K_0 = C_K(\langle a, L\rangle)$. Then K/K_0 is finite since L is finite. It suffices to decide K-conjugacy, and hence K_0-conjugacy by a previous argument. But $b \in \langle a, L\rangle$, so b is K_0-conjugate to a if and only if $b = a$; the latter is certainly decidable.

(2c) *The general case.*

By construction, $Q/C_Q(A^*)$ is abelian-by-finite. Hence there is a positive integer n such that $N = (Q^n)'$ acts trivially on A^*. Of course N is r.e.. Also $[a, N] = [A, N] \leq A^*$. By (2b) one can decide Q-conjugacy of b to a modulo $[A, N]$ since Q/N is abelian-by-finite. Assume therefore that $b \equiv a \bmod [A, N]$. But then b is automatically N-conjugate to a since $b - a = a(x - 1)$ and $b = ax$ for some x in N. This completes the proof in the divisible case.

(3) *The case A_0 torsion-free.*

Here too we need to make a number of initial reductions.

(3a) *There is a r.e. subgroup K with finite index in Q and a r.e. K-submodule A^* of A_0 which is a K-finitely generated K-plinth.*

Write $K_0 = Q$. If A_0 is not a K_0-plinth, there is a subgroup K_1 of finite index in K_0 and a K_1-submodule A_1 of A_0 such that $0 < r_0(A_1) < r_0(A_0)$. The same argument may be applied to A_1 if it is not a K_1-plinth, and so on. It follows that there must exist a subgroup K_i with finite index in Q and a K_i-plinth A_i contained in A_0. Set $K = Q^t$ where $t = |Q\colon K_i|$ and define A^* to be the K-submodule of A_i generated by some non-zero element.

(3b) *It may be assumed that $A = \langle a, A^*\rangle$, A/A^* is torsion-free, A^* is a Q-finitely generated Q-plinth and $[a, Q] \leq A^*$. Furthermore it suffices to be able to decide Q-conjugacy to a of elements b satisfying $b \equiv a \bmod A^*$.*

In the first place it suffices to decide K-conjugacy because $|Q\colon K|$ is finite. Since $m(A/A^*) < m(A)$, one need only discuss elements b in $a + A^*$ and clearly it is $K_1 \equiv C_K(a + A^*)$-conjugacy which is at issue. If $|K : K_1|$ is infinite, then $m(K_1) < m(K) = m(Q)$ and the induction hypothesis applies. Assume therefore that $|K\colon K_1|$ is finite, so that A^* is a K_1-finitely generated

K_1-plinth. It is clear that one can replace Q, A_0 and A by K_1, A^* and $\langle a, A^* \rangle$.

Suppose that A/A^* is finite and let T be the $\mathbf{Z}$-torsion subgroup of A. Then T is finite. Assume that Q-conjugacy to $a + T$ in A/T is decidable. Then one may restrict attention to elements b of $a + T$. Write $b = a + t$ with t in T. Now $b = aq$ with q in Q if and only if $t = a(q - 1)$. However, $a(q - 1) \in A^* \cap T = 0$ since $[a, Q] \leq A^*$. Hence b is Q-conjugate to a if and only if $b = a$, which is certainly decidable. Consequently one may assume that A is torsion-free.

(3c) *The case* $A = A^*$.

Since A is a torsion-free abelian group of finite rank and the word problem is soluble in A, the group $C_Q(A)$ is r.e. and therefore $Q/C_Q(A)$ has a recursive presentation. Hence one may suppose Q to act faithfully on A.

Now A is rationally irreducible; it is a well-known theorem of Mal'cev (see [14]) that Q is abelian-by-finite. Replacing Q by a suitable power one can assume that Q is abelian. Note that A remains a plinth for Q. By [14, Lemma 5.29.1], the group Q is finitely generated.

Write $S = \mathbf{Z}Q$, the integral group ring. Then S is a finitely generated commutative ring, so it is submodule computable [2]. Let R be the ring of endomorphisms of A generated by Q. Then $R \cong S/L$ where L is the annihilator of A in S. Since A is rationally irreducible as a Q-module, L is a prime ideal of S. Thus R is a finitely generated integral domain of characteristic 0. It follows by a result of Samuel ([19]) that the group of units R^* of R is a finitely generated abelian group. Clearly $Q \leq R^*$.

It must be shown how to decide if a given element b of A is Q-conjugate to a; one can, of course, assume that $a \neq 0$ and $b \neq 0$. Since A is rationally irreducible, A/aS is a torsion group, so there exist a positive integer m and a non-zero element s of S such that $bm = as$. Moreover, once b is given, such a pair (m, s) can be found by simply enumerating the elements of the form $bm - as$ and looking for one that is 0. Let s induce the endomorphism r of A.

Since A is torsion-free, $b = aq$ with q in Q if and only if $ar = aqm$. This is equivalent to $r = qm$ (in End A) since A is rationally irreducible. There is, therefore, only one possible candidate for q, namely the element rm^{-1} in the field of fractions of R. Thus b is Q-conjugate to a if and only if $rm^{-1} \in Q$. It remains to decide this point.

The first step is to decide whether rm^{-1} belongs to R, i.e. if $r \in Rm$. This amounts to deciding whether s belongs to $L + Sm$. But membership in the ideal $L + Sm$ can be decided by a uniform algorithm because S is submodule computable. If $r \notin Rm$, then certainly $rm^{-1} \notin Q$, so assume that $rm^{-1} \in R$. Next one decides whether $r^{-1}m \in R$, or equivalently whether m belongs to $L + Ss$; again this is possible by submodule computability. Assume that $r^{-1}m \in R$, so that $rm^{-1} \in R^*$. But R^* is a finitely generated abelian group with Q a subgroup, so there is an algorithm to decide whether rm^{-1} belongs to Q, and hence whether b is Q-conjugate to a.

(3d) *The case A/A^* infinite and $A = \langle a \rangle \oplus A^*$.*
Write $C = C_Q(A)$; two further case distinctions must be made.

(3e) *The case Q/C abelian-by-finite.*
There is an $m > 0$ such that $(Q^m)'$ acts trivially on A. Of course $N \equiv Q^m$ is r.e. and Q/N is finite. As usual it is enough to decide N-conjugacy to a. If $[A^*, N] = 0$, then $[a, N]$ consists of elements of the form $a(x-1)$, $x \in N$ and one proceeds as in (2b) (third paragraph). Assume therefore that

$$[A^*, N] \neq 0.$$

By rational irreducibility $(A^*)^N = 0$. Clearly $A/[A, N]$ is not torsion, so $A^N \neq 0$ (cf. [15, Lemma 5.12]). Since $A^N \cap A^* = 0$, it follows that $A/A^N + A^*$ is finite, of order d say. Since A is torsion-free, b is N-conjugate to a if and only if ad and bd are N-conjugate in $A^N + A^*$. The latter is true precisely when $(b-a)d = (ad)(x-1)$ for some x in N. Now $ad = a_0 + a^*$ with $a_0 \in A^N$ and $a^* \in A^*$, so $ad(x-1) = a^*(x-1)$. Put $b' = (b-a)d \in A^*$. It has to be decided if $b' + a^*$ is N-conjugate to a^* in A^*. By (3c) this can be done.

(3f) *The case Q/C not abelian-by-finite.*
Let $D = C_Q(A^*)$; then $C \leq D$ and Q/D is abelian-by-finite. Thus $C \neq D$ and $[a, D]$ is a non-zero Q-submodule of A^*. Since $C_Q(A)$ is obviously r.e., one can suppose without loss of generality that Q acts faithfully on A. Rational irreducibility shows that $A^*/[a, D]$ is a torsion group. Since A^* is finitely generated Q-module, $A^*/[a, D]$ is in fact finite.
The decomposition $A = \langle a \rangle \oplus A^*$ shows that each element q of Q may be identified with a matrix of the form

$$\begin{pmatrix} 1 & q' \\ 0 & q^{\varphi} \end{pmatrix}$$

where $q' \in A^*$ and $\varphi\colon Q \to \operatorname{Aut} A^*$ is a homomorphism with kernel D. Let $H = \operatorname{Im} \varphi$. Define E to be the group of all matrices of the form

$$\begin{pmatrix} 1 & a^* \\ 0 & h \end{pmatrix}$$

where $a^* \in A^*$ and $h \in H$. Then Q is a subgroup of E. The groups A^* and H can be identified with subgroups of E by associating a^* with

$$\begin{pmatrix} 1 & a^* \\ 0 & 1 \end{pmatrix}$$

and h with

$$\begin{pmatrix} 1 & 0 \\ 0 & h \end{pmatrix}.$$

Of course $A^* \triangleleft E$ and $E = QA^*$.

Let $u \in [a, D] \leq A^*$; then $u = a(q-1)$ with q in D, and so $aq = a + u$. Hence $u = q'$ and

$$\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & q' \\ 0 & q^{\varphi} \end{pmatrix} \in Q.$$

Therefore $[a, D] \leq Q$. Since $A^*/[a, D]$ is finite, it follows that $|E: Q|$ is finite.

Let $\{t_1, \ldots, t_k\}$ be a transversal to $H \cap Q$ in H. Consider an element b of A such that $b \equiv a \bmod A^*$. Let

$$e = \begin{pmatrix} 1 & a^* \\ 0 & h \end{pmatrix}$$

be an element of E. Then $b = ae$ holds if and only if $b - a = a^*$. It must therefore be decided whether there is an element h of H such that

$$e = \begin{pmatrix} 1 & a^* \\ 0 & h \end{pmatrix}$$

belongs to Q where $a^* = b - a$.

For any $h \in H$, write

$$\begin{pmatrix} 1 & 0 \\ 0 & h \end{pmatrix} = st_i$$

with $s \in H \cap Q$ and $1 \leq i \leq k$. Then

$$e = \begin{pmatrix} 1 & a^* \\ 0 & h \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & h \end{pmatrix}\begin{pmatrix} 1 & a^* \\ 0 & 1 \end{pmatrix} = st_i\begin{pmatrix} 1 & a^* \\ 0 & 1 \end{pmatrix}.$$

Thus e belongs to Q if and only if the element

$$t_i\begin{pmatrix} 1 & a^* \\ 0 & 1 \end{pmatrix}$$

belongs to Q. Hence to decide whether b is Q-conjugate to a one must decide whether one of the k elements

$$t_i\begin{pmatrix} 1 & a^* \\ 0 & 1 \end{pmatrix}$$

of E belongs to Q. But this is certainly possible. For E is a recursively presented soluble minimax group (actually a semidirect product $H \ltimes A^*$) and Q is a r.e. subgroup, so 3.1 may be applied. (Alternatively a simple direct argument may be applied since $|E: Q|$ is finite.) This completes the proof of 4.2.

An example. Finally it will be shown that 4.1 fails for soluble groups of finite total rank.

4.3. *There is a recursively presented torsion-free nilpotent group of class* 2 *and rank* 3 *which has an element g such that no algorithm can decide conjugacy to g.*

Proof. Let π be a r.e. non-recursive set of primes. Define S to be the additive group of square-free rationals and let S_0 be the subgroup of S generated by the reciprocals of the primes in π. Let

$$A = \mathbf{Z} \oplus S.$$

To each s in S there corresponds an automorphism ξ_s of A which is described by the matrix

$$\begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}.$$

Then $Q = \{\xi_s | s \in S_0\}$ is a subgroup of Aut A isomorphic with S_0. Now form the semidirect product

$$G = Q \ltimes A.$$

Evidently G is nilpotent of class 2 and rank 3; it is also evident that G has a recursive presentation. Let $g = (1, 0) \in A$. If $s \in S$, then $(1, s)$ in A is conjugate to g if and only if $s \in S_0$. However it has been shown in (3.2) that no algorithm can decide membership in S_0. Hence no algorithm can decide conjugacy to g in G.

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