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A BMO ESTIMATE FOR MULTILINEAR SINGULAR INTEGRALS

BY

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Introduction

Let

$$(1)\quad C(a,b;f)(x) = \text{p.v.}\int_{\mathbf{R}^n} \frac{\Omega(x-y)P_{m_1}(a;x,y)P_{m_2}(b;x,y)f(y)\,dy}{|x-y|^{n+M-2}}$$

where

$$P_m(a,x,y) = a(x) - \sum_{|\alpha|<m} \frac{1}{\alpha!} a^{(\alpha)}(y)(x-y)^\alpha$$

and $M = m_1 + m_2$. In this paper we establish the inequality

$$(2)\quad \|C(a,b;f)\|_p \le C_p \|\nabla^{m_1-1}a\|_{BMO} \|\nabla^{m_2-1}b\|_{BMO} \|f\|_p, \quad 1<p<\infty,$$

where Ω satisfies certain conditions and $\|\nabla^m a\|_{BMO} = \sum_{|\alpha|=m} \|a^{(\alpha)}\|_{BMO}$. BMO denotes the space of functions of bounded mean oscillation on $\mathbf{R}^n$.

The first result in this direction was established by Coifman, Rochberg, and Weiss [7] where it was shown that the commutator of the Hilbert transform and multiplication by a function A is bounded on $L^p(\mathbf{R})$, $1<p<\infty$, providing A is in BMO. The result for a single remainder of order 2 was proved by the first author in [3]. The methods used here are extensions of those in [3]. The main differences are: (1) a generalization of a basic estimate of Mary Weiss to Taylor series remainders (our lemma); (2) the boundedness of operators similar to $C(a,b;f)$ when a and b have appropriate derivatives in $L^q(\mathbf{R}^n)$ (see [2]); (3) a more complicated partition of the operator due to the presence of products and the fact that the order of the remainders is arbitrary.

Finally we note that the result proved here holds for any finite number of remainders. For simplicity we give the proof here for the case of two remainders. The authors wish to point out that while going from one re-

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mainder to a product of remainders presented no major obstruction in the present case, this is not always so. For example, the reader should compare the proofs in [4] for the commutators $[A, DH]$ and $[B[A, D^2H]]$. It is also of interest to point out that the methods developed by Coifman and Meyer [5] apply only to commutators with one remainder. Finally, we note that the result from [2] used in this paper is a non-trivial extension of the single remainder case in [1].

Preliminaries

Throughout this paper we will work in Euclidean space $\mathbf{R}^n$. Let $\alpha = (\alpha_1, \dots, \alpha_n)$ denote a multiindex and let $|\alpha| = \alpha_1 + \cdots + \alpha_n$ denote the order of α. If b is a smooth function on $\mathbf{R}^n$, $b^{(\alpha)}$ or b^α will denote the partial derivative

$$\frac{\partial^{|\alpha|} b}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2}, \dots, \partial x_n^{\alpha_n}}.$$

Let $P_m(b; x, y)$ denote the m^{th} order Taylor series remainder of b at x expanded about y. More precisely

$$P_m(b; x, y) = b(x) - \sum_{|\alpha| < m} \frac{b^{(\alpha)}(y)}{\alpha!}(x - y)^\alpha \tag{3}$$

where

$$\alpha! = \alpha_1! \alpha_2! \cdots \alpha_n! \quad \text{and} \quad (x - y)^\alpha = (x_1 - y_1)^{\alpha_1} \cdot \cdots \cdot (x_n - y_n)^{\alpha_n}.$$

Let $|E|$ denote the Lebesgue measure of a measurable set $E \subset \mathbf{R}^n$. In this paper Q will denote a cube with edges parallel to the co-ordinate axes and if b is an integrable function on Q, $m_Q(b)$ will denote the average of b over Q i.e., $|Q|^{-1} \int_Q b(x)\, dx$. A locally integrable function b is said to be of bounded mean oscillation, $b \in BMO$, provided there exists a constant C such that

$$\frac{1}{|Q|} \int_Q |b(x) - m_Q(b)|\, dx \leq C$$

for every Q. More generally, for $1 \leq q < \infty$, we let

$$S^q b(x) = \sup_{x \in Q} \left(\frac{1}{|Q|} \int_Q |b(t) - m_Q(b)|^q\, dt \right)^{1/q}. \tag{4}$$

Then it is well known (for example, see [8]) that $b \in BMO$ implies $S^q b \in$

$L^\infty(\mathbf{R}^n)$. If b is a function with m^{th} order derivatives in BMO, we let $S_m^q = \Sigma S^q(b^\alpha)(x)$ where the sum is taken over all α with $|\alpha| = m$. We will also use the following L^p, $p > 1$, version of the Hardy-Littlewood maximal function

$$(5) \qquad \Lambda_p f(x) = \sup_{x \in Q} \left(\frac{1}{|Q|} \int_Q |f(t)|^p \, dt \right)^{1/p}.$$

Finally we will let C be a constant that may vary from line to line.

Statements of results

Our main result is the following:

THEOREM. *Let Ω be homogeneous of degree zero, satisfy $|\Omega(x) - \Omega(y)| \leq C|x - y|$ for $|x| = |y| = 1$, and have vanishing moments up to order $M - 2$ over the unit sphere in $\mathbf{R}^n$. Let $f \in L^p(\mathbf{R}^n)$, $1 < p < \infty$, and let a and b be functions with derivatives of order $m_1 - 1$ and $m_2 - 1$ respectively in BMO. Then if $C(a, b; f)$ is defined by* (1), *we have*

$$(6) \qquad \|C(a, b; f)\|_p \leq C_p \|\nabla^{m_1 - 1} a\|_{BMO} \|\nabla^{m_2 - 1} b\|_{BMO} \|f\|_p.$$

We introduce the maximal operator

$$(7) \qquad C_*(a, b; f)(x) = \sup_{\varepsilon > 0} \left| \int_{|x-y| > \varepsilon} \frac{P_{m_1}(a; x, y) P_{m_2}(b; x, y) \Omega(x - y) f(y) \, dy}{|x - y|^{n+M-2}} \right|.$$

The theorem above follows by standard arguments from the following "good λ" estimate on the maximal operator.

MAIN ESTIMATE. *For $\gamma > 0$ sufficiently small,*

$$(8) \qquad \left| \left\{ x \in \mathbf{R}^n \colon C_*(a, b; f)(x) > 3\lambda, \, S_{m_1 - 1}^q a(x) S_{m_2 - 1}^q b(x) \Lambda_p f(x) \leq \gamma\lambda \right\} \right| \\ \leq C\gamma^r \left| \left\{ x \in \mathbf{R}^n \colon C_*(a, b; f)(x) > \lambda \right\} \right|.$$

where $1/p + 1/q = 1/r$ and $q > \max(n, p')$ where $1/p + 1/p' = 1$.

The pointwise estimate

In proving the main estimate one analyzes the operator by writing

$$C(a, b, f) = C(a, b; f_1) + C(a, b; f_2)$$

where f_1 is supported on a cube $\bar{Q}$ and f_2 is supported on the complement of $\bar{Q}$. The estimate for $C(a, b; f_1)$ (see (17) and (26)) is obtained from the theorem of Cohen and Gosselin on multilinear singular integrals [2]. The estimate for $C(a, b; f_2)$ is obtained from a pointwise estimate for the Taylor series remainder $P_m(a; x, y)$ in terms of L^p averages of the m^{th} order derivatives of a over cubes containing x and y and having volume comparable to $|x - y|^n$. This estimate generalizes a lemma of Mary Weiss which in one form states

$$(9) \qquad |a(x) - a(y)| \leq C_n \sum_{|\alpha|=1} \left\{ \frac{1}{|Q(x, y)|} \int_{Q(x, y)} |\nabla a(y)|^q \, dy \right\}^{1/q}$$

where $Q(x, y)$ is the cube centered at x with edges parallel to the axes and having diameter $2\sqrt{n}|x - y|$. (For one version of this lemma see C.P. Calderon [6, page 145]). For our purposes we prove the following:

LEMMA. *Let $b(x)$ be a function on $\mathbf{R}^n$ with m^{th} order derivatives in $L^q(\mathbf{R}^n)$ where $q > n$. Then*

$$|P_m(b; x, y)| \leq C_{m,n}|x - y|^m \sum_{|\alpha|=m} \left(\frac{1}{|Q(x, y)|} \int_{Q(x, y)} |b^\alpha(t)|^q \, dt \right)^{1/q}.$$

where $Q(x, y)$ is the cube centered at x with edges parallel to the axes and having diameter $5\sqrt{n}|x - y|$.

The estimate for $C(a, b; f_2)$ is somewhat technical. The primary term is the first integral in (28). The cases of $\varepsilon \approx \operatorname{diam} \tilde{Q}_j$ and $\varepsilon \gg \operatorname{diam} \tilde{Q}_j$ are handled separately. In each case, $k(x, y) - k(x_0, y)$ is written as a sum of several terms. The pointwise estimate is used repeatedly.

Proof of the lemma We use induction on m. For $m = 1$, the result is that of Mary Weiss cited above. We now assume the result is valid for $1 \leq j \leq m - 1$. Let z be on the perpendicular bisector of the line segment from x to y and such that the angle betwen the line segment from z to x and the line segment from x to y is $\leq \pi/4$. Then

$$(10) \qquad P_m(b; x, y) = P_m(b; x, z) + P_m(b; z, y) + \sum_{0<|\alpha|<m} \frac{(x - y)^\alpha}{\alpha!} P_{m-|\alpha|}(b^\alpha; z, y).$$

We apply the induction hypothesis to each term in the sum in (10) and obtain

$$
\begin{aligned}
(11)\qquad & \left|\sum_{0<|\alpha|<m} \frac{(x-y)^\alpha}{\alpha!} P_{m-|\alpha|}(b^\alpha; z, y)\right| \\
&\le \sum_{0<|\alpha|<m} \frac{1}{\alpha!}|x-y|^{|\alpha|}\left|P_{m-|\alpha|}(b^\alpha; z, y)\right| \\
&\le C_n \sum_{0<|\alpha|<m} \frac{1}{\alpha!}|x-y|^{|\alpha|}|z-y|^{m-|\alpha|} \\
&\quad\times \sum_{|\beta|=m-|\alpha|}\left(\frac{1}{|Q(z,y)|}\int_{Q(z,y)}|b^{\alpha+\beta}(s)|^q\,ds\right)^{1/q} \\
&\le C|x-y|^m \sum_{|\gamma|=m}\left(\frac{1}{|Q(z,y)|}\int_{Q(z,y)}|b^\gamma(s)|^q\,ds\right)^{1/q}
\end{aligned}
$$

It is easy to check that $Q(z, y) \subset Q(x, y)$ and $|Q(z, y)| \ge 2^{-n}|Q(x, y)|$. This permits us to replace $Q(z, y)$ by $Q(x, y)$ in (11) and obtain

$$
\begin{aligned}
(12)\qquad & \left|\sum_{0<|\alpha|<m} \frac{(x-y)^\alpha}{\alpha!} P_{m-|\alpha|}(b^\alpha; z, y)\right| \\
&\le C|x-y|^m \sum_{|\gamma|=m}\left(\frac{1}{|Q(x,y)|}\int_{Q(x,y)}|b^\gamma(s)|^q\,ds\right)^{1/q}
\end{aligned}
$$

Let S_{n-1} denote the unit sphere in $\mathbf{R}^n$. Let $z = x + |z-x|\omega$ where $\omega \in S_{n-1}$, and let $y = z + |y-z|\omega'$ where $\omega' \in S_{n-1}$. Then using the integral form of the remainder, we have

$$
\begin{aligned}
(13)\qquad & P_m(b; x, z) + P_m(b; z, y) \\
&= \sum_{|\alpha|=m} \frac{n}{\alpha!}\left(\omega^\alpha \int_0^{|x-z|} \rho^{m-1} b^\alpha(x+\rho\omega)\,d\rho\right. \\
&\qquad\left. + \omega'^\alpha \int_0^{|y-z|} \rho^{m-1} b^\alpha(z+\rho\omega')\,d\rho\right)
\end{aligned}
$$

Taking absolute values in (13) and averaging over appropriate ω's and (ω')'s on S_{n-1} (call these sets Ω and Ω') we have

$$(14)\qquad \begin{aligned}&|P_m(b;x,z)+P_m(b;z,y)|\\ &\le C\sum_{|\alpha|=m}\frac{m}{\alpha!}\left(\int_\Omega\int_0^{|x-y|}\rho^{m-1}|b^\alpha(x+\rho\omega)|\,d\rho\,d\omega\right.\\ &\qquad\left.+\int_{\Omega'}\int_0^{|x-y|}\rho^{m-1}|b^\alpha(y+\rho\omega')|\,d\rho\,d\omega'\right)\end{aligned}$$

By symmetry we consider only the first integral in (14) which we write as

$$(15)\qquad \begin{aligned}&\int_\Omega\int_0^{|x-y|}\rho^{m-n}|b^\alpha(x+\rho\omega)|\rho^{n-1}\,d\rho\,d\omega\\ &\le\left(\int_\Omega\int_0^{|x-y|}|b^\alpha(x+\rho\omega)|^q\rho^{n-1}\,d\rho\,d\omega\right)^{1/q}\\ &\quad\times\left(\int_\Omega\int_0^{|x-y|}\rho^{(m-n)q'+n-1}\,d\rho\,d\omega\right)^{1/q'}\\ &\le C\left(\int_{S_{n-1}}\int_0^{|x-y|}|b^\alpha(x+\rho\omega)|^q\rho^{n-1}\,d\rho\,d\omega\right)^{1/q}|x-y|^{m-n/q}\\ &\le C|x-y|^m\left(\frac{1}{|Q(x,y)|}\int_{Q(x,y)}|b^\alpha(\eta)|^q\,d\eta\right)^{1/q}\end{aligned}$$

Summing (15) over $|\alpha|=m$ and combining this with (12) the lemma now follows.

The good λ inequality

We now turn to the proof of the main estimate (8). Using a Whitney argument we write

$$\{C_*(a,b;f)(x)>\lambda\}$$

as a union of cubes $\{Q_j\}$ with mutually disjoint interiors and with distance from each Q_j to $\mathbf{R}^n\setminus\bigcup_j Q_j$ comparable to the diameter of Q_j. It now suffices to prove the main estimate for each Q_j. There exists a constant $C=C(n)$ such that for each j the cube $\tilde{Q}_j$ with the same center as Q_j but with $\operatorname{diam}\tilde{Q}_j=C(n)\operatorname{diam}Q_j$ intersects $\mathbf{R}^n\setminus\bigcup_j Q_j$. Thus for each j there exists a point $x_0=x_0(j)\in\tilde{Q}_i$ such that $C_*(a,b;f)(x_0)\le\lambda$.

We now fix a cube Q_j, and assume there exists a point $z=z(j)$ with

$$S^q_{m_1-1}(a)(z)S^q_{m_2-1}(b)(z)\Lambda_p f(z)\le\gamma\lambda.$$

(If no such point exists, the result is trivial for Q_j.) Let $\overline{Q}_j=\tilde{\tilde{Q}}_j$ and write

$f = f_1 + f_2$ where $f_1 = f\chi_{\overline{Q}_j}$. We now make appropriate estimates on f_1 and f_2 separately.

The f_1 estimate. We first note that

$$C_*(a, b; f)(x) \leq \sum_{j=1}^{4} C_*^j(a, b; f)(x)$$

where

$$C_*^1(a, b; f)(x) = \sup_{\varepsilon>0}\left|\int_{|x-y|>\varepsilon} \frac{P_{m_1-1}(a; x, y)P_{m_2-1}(b; x, y)\Omega(x-y)f(y)\,dy}{|x-y|^{n+M-2}}\right|,$$

$$C_*^2(a, b; f)(x) = \sum_{|\alpha|=m_1-1}\frac{1}{\alpha!} \times \sup_{\varepsilon>0}\left|\int_{|x-y|>\varepsilon} \frac{P_{m_2-1}(b; x, y)(x-y)^{\alpha}\Omega(x-y)a^{\alpha}(y)f(y)\,dy}{|x-y|^{n+M-2}}\right|$$

$$C_*^3(a, b; f)(x) = \sum_{|\beta|=m_2-1}\frac{1}{\beta!} \times \sup_{\varepsilon>0}\left|\int_{|x-y|>\varepsilon} \frac{P_{m_1-1}(a; x, y)(x-y)^{\beta}\Omega(x-y)b^{\beta}(y)f(y)\,dy}{|x-y|^{n+M-2}}\right|$$

$$C_*^4(a, b; f)(x) = \sum_{\substack{|\alpha|=m_1-1\\|\beta|=m_2-1}}\frac{1}{\alpha!\beta!} \times \sup_{\varepsilon>0}\left|\int_{|x-y|>\varepsilon} \frac{(x-y)^{\alpha+\beta}\Omega(x-y)a^{\alpha}(y)b^{\beta}(y)f(y)\,dy}{|x-y|^{n+M-2}}\right|$$

We note that if a and b have derivatives of orders $m_1 - 1$ and $m_2 - 1$ respectively in $L^q(\mathbf{R}^n)$, q sufficiently large, then

$$(17) \qquad \|C_*^j(a, b; f)\|_r \leq C\Big(\sum_{|\alpha|=m_1-1}\|a^{\alpha}\|_q\Big)\Big(\sum_{|\beta|=m_2-1}\|b^{\beta}\|_q\Big)\|f\|_p$$

for $1 \leq j \leq 4$ where $1 > 1/r = 1/p + 2/q$. For $j = 1, 2, 3$, this follows from [2] while for $j = 4$, the result follows from standard Calderon-Zygmund theory (see [S]).

We now choose a C_0^{∞} function φ such that $\varphi(x) \equiv 1$ for $x \in \overline{Q}_j$, $\varphi(x) \equiv 0$ for $x \notin \overline{\overline{Q}}_j$, $|\varphi(x)| \leq 1$ for all x, and for any multiindex α with $|\alpha| \leq M$,

$$|\varphi^{\alpha}(x)| \leq C\big(\operatorname{diam}\overline{\overline{Q}}_j\big)^{-|\alpha|}.$$

We note that C is independent of j. We now define

$$(18)\quad a_\varphi(y) \equiv P_{m_1-1}\left(a(\cdot) - \sum_{|\alpha|=m_1-1} \frac{1}{\alpha!} m_{Q_j}(a^\alpha)(\cdot)^\alpha; y, z\right)\varphi(y),$$

$$b_\varphi(y) \equiv P_{m_2-1}\left(b(\cdot) - \sum_{|\beta|=m_2-1} \frac{1}{\beta!} m_{Q_j}(b^\beta)(\cdot)^\beta; y, z\right)\varphi(y).$$

We note that a_φ and b_φ have support in $\overline{\overline{Q_j}}$. We now estimate the derivatives of order $m_1 - 1$ of a_φ. Let γ be a multiindex of order $m_1 - 1$. Then

(19)

$$a_\varphi^\gamma(y) = \sum_{\gamma=\mu+\nu} C_{\mu,\nu}\left\{ P_{m_1-1}^{(\mu)}\left(a(\cdot) - \sum_{|\alpha|=m_1-1} \frac{1}{\alpha!} m_{Q_j}(a)(\cdot)^\alpha; y, z\right)\varphi^{(\nu)}(y)\right\}$$

$$= \sum_{\gamma=\mu+\nu} C_{\mu,\nu} P_{m_1-1-|\mu|}\left(\frac{\partial^\mu}{\partial y^\mu}\left(a(\cdot) - \sum_{|\alpha|=m_1-1} \frac{1}{\alpha!} m_{Q_j}(a)(\cdot)^\alpha\right); y, z\right)$$

$$\times \varphi^{(\nu)}(y)$$

From the lemma we have

$$(20)\quad \left| P_{m_1-1-|\mu|}\left(\frac{\partial^\mu}{\partial y^\mu}\left(a(\cdot) - \sum_{|\alpha|=m_1-1} \frac{1}{\alpha!} m_{Q_j}(a)(\cdot)^\alpha\right); y, z\right)\right|$$

$$\le C|y-z|^{m_1-1-|\mu|}$$

$$\times \sum_{|\eta|=m_1-1}\left(\frac{1}{|Q(y,z)|}\int_{Q(y,z)} \left|a^{(\eta)}(x) - m_{Q_j}(a^{(\eta)})\right|^q dx\right)^{1/q}$$

$$\le C|y-z|^{m_1-1-|\mu|} S_{m_1-1}^q(a)(z).$$

From the assumptions on φ, we have $|\varphi^{(\nu)}(y)| \le C|y-z|^{-|\nu|}$. Combining this with (20) we have for $|\gamma| = m_1 - 1$,

$$(21)\quad \left|a_\varphi^{(\gamma)}(y)\right| \le \sum_{\gamma=\mu+\nu} C_{\mu,\nu} S_{m_1-1}^q(a)(z)$$

$$\le C S_{m_1-1}^q(a)(z).$$

Since a_φ has support in $\overline{\overline{Q_j}}$, we finally obtain, for $|\gamma| = m_1 - 1$,

$$(22)\quad \|a_\varphi^\gamma\|_q \le C S_{m_1-1}^q(a)(z)\left|\overline{\overline{Q_j}}\right|^{1/q}.$$

Similarly, for $|\gamma| = m_2 - 1$ we have the estimate

$$(23)\quad \|b_\varphi^\gamma\|_q \le C S_{m_2-1}^q(b)(z)\left|\overline{\overline{Q_j}}\right|^{1/q}.$$

It is easy to see that

$$\|f_1\|_p \le \Lambda_p f(z)\left|\overline{\overline{Q_j}}\right|^{1/p}. \tag{24}$$

We observe that for $y \in \overline{Q}_j$,

(25)

$$\begin{aligned} P_{m_1}(a; x, y) &= P_{m_1}\left(P_{m_1-1}(a; (\cdot), z)\varphi(\cdot); x, y\right) \\ &= P_{m_1}\left(P_{m_1-1}\left(a(t) - \sum_{|\alpha|=m_1-1} \frac{1}{\alpha!} m_{Q_j}(a^{(\alpha)})t^\alpha; (\cdot), z\right)\varphi(\cdot); x, y\right) \\ &= P_{m_1}\left(a_\varphi(\cdot); x, y\right), \end{aligned}$$

and likewise, $P_{m_2}(b; x, y) = P_{m_2}(b_\varphi(\cdot); x, y)$. It now follows that for $x \in Q_j$,

$$C_*(a, b; f_1)(x) = C_*(a_\varphi, b_\varphi; f_1)(x).$$

Thus from (17), (22), (23), and (24) we now have

$$\begin{aligned} (26)\quad &\left|x \in Q_j \colon C_*(a, b; f_1)(x) > \beta\lambda\right| \\ &= \left|x \in Q_j \colon C_*(a_\varphi, b_\varphi; f_1)(x) > \beta\lambda\right| \\ &\le \sum_{j=1}^{4} \left|x \in Q_j \colon C_*^j(a_\varphi, b_\varphi, f_1)(x) > \frac{\beta}{4}\lambda\right| \\ &\le \left(\frac{\beta\lambda}{4}\right)^{-r} \sum_{j=1}^{4} \left\|C_*^j(a_\varphi, b_\varphi; f_1)\right\|_r^r \\ &\le C\left(\frac{\beta\lambda}{4}\right)^{-r} \left(\left(\sum_{|\alpha|=m_1-1} \|a_\varphi^\alpha\|q\right)\left(\sum_{|\beta|=m_2-1} \|b_\varphi^\beta\|q\right)\|f_1\|p\right)^r \\ &\le C\left(\frac{\beta\lambda}{4}\right)^{-r} \left\{S_{m_1-1}^q(a)(z) S_{m_2-1}^q(b)(z)\Lambda_p f(z)\right\}\left|\overline{\overline{Q}}_j\right|^{r(2/q+1/p)} \\ &\le C\left(\frac{\beta\lambda}{4}\right)^{-r} (\gamma\lambda)^r \left|\overline{\overline{Q}}_j\right| \\ &\le C\left(\frac{\gamma}{\beta}\right)^r |Q_j|. \end{aligned}$$

This completes the estimate on f_1.

The f_2 estimate for $\varepsilon \approx$ diam $\tilde{Q}_j$. Let $K = K(n)$ be a large positive integer depending only on n. The estimate for f_2 is split into the two cases

$$\operatorname{diam} \tilde{Q}_j \le \varepsilon \le K(n)\operatorname{diam} \tilde{Q}_j \quad\text{and}\quad \varepsilon > K(n)\operatorname{diam} \tilde{Q}_j.$$

The case of $\varepsilon < \operatorname{diam} \tilde{Q}_j$ is ignored since f_2 has support outside $\overline{Q_j}$. Let

$$a_j(x) = a(x) - \sum_{|\alpha|=m_1-1} \frac{1}{\alpha!} m_{Q_j}(a)x^\alpha$$

and $b_j(x) = b(x) - \sum_{|\beta|=m_2-1}(1/\beta!)m_{Q_j}(b)x^\beta$. We now set

$$(27)\qquad k(x, y) = \frac{P_{m_1}(a_j; x, y)P_{m_2}(b_j; x, y)\Omega(x-y)}{|x-y|^{n+M-2}}.$$

We now choose x_0 in $\tilde{Q}_j$ with $x_0 \in \mathbf{R}^n \setminus \cup Q_j$ and for $x \in Q_j$ we write

$$(28)\qquad \begin{aligned} |C_\varepsilon(a, b; f_2)(x)| &= |C_\varepsilon(a_j, b_j; f_2)(x)| \\ &\le \left| \int_{|x-y|>\varepsilon} (k(x, y) - k(x_0, y)) f_2(y)\, dy \right| \\ &\quad + \left| \int_{\varepsilon<|x-y|\le C\varepsilon} k(x, y) f_2(y)\, dy \right| \\ &\quad + \left| \int_{\varepsilon<|x_0-y|<C\varepsilon} k(x_0, y) f_2(y)\, dy \right| \\ &\quad + \left| \int_{|x_0-y|>\varepsilon} k(x_0, y) f_2(y)\, dy \right|. \end{aligned}$$

In the last integral we write $f_2 = f - f_1$ and incorporate the f_1 part of this integral into the third integral after enlarging the region of integration to

$$\operatorname{diam} \tilde{Q}_j \le |x_0 - y| \le K(n) \operatorname{diam} \tilde{Q}_j.$$

Here $K(n)$ is chosen to be large enough to insure that the ball centered at x_0 with radius $K(n)\operatorname{diam} \tilde{Q}_j$ contains $\overline{Q_j}$. We finally obtain

$$(29)\qquad \begin{aligned} |C_\varepsilon(a, b; f)(x)| &\le \left| \int_{|x-y|>\varepsilon} [k(x, y) - k(x_0, y)] f_2(y)\, dy \right| \\ &\quad + \int_{R(x)} |k(x, y) f(y)|\, dy + \int_{R(x_0)} |k(x_0, y) f(y)|\, dy \\ &\quad + |C_\varepsilon(a, b; f)(x_0)| \end{aligned}$$

where $R(\cdot) = \operatorname{diam} \tilde{Q}_j \le |\cdot - y| \le k(n)\operatorname{diam} \tilde{Q}_j$. The last integral is bounded by λ since $x_0 \notin \cup Q_j$. The middle integrals are error terms which we will estimate later. We now deal with the first integral in (29). We have

$$(30)\qquad \begin{aligned} &k(x, y) - k(x_0, y) \\ &= P_{m_1}(a_j; x, y)P_{m_2}(b_j; x, y)\left[\frac{\Omega(x-y)}{|x-y|^{n+M-2}} - \frac{\Omega(x_0-y)}{|x_0-y|^{n+M-2}}\right] \\ &\quad + \left[P_{m_1}(a_j; x, y) - P_{m_1}(a_j; x_0, y)\right] P_{m_2}(b_j; x, y)\frac{\Omega(x_0-y)}{|x_0-y|^{n+M-2}} \\ &\quad + P_{m_1}(a_j; x_0, y)\left[P_{m_2}(b_j; x, y) - P_{m_2}(b_j; x_0, y)\right]\frac{\Omega(x_0-y)}{|x_0-y|^{n+M-2}} \\ &\equiv k_1(x, x_0, y) + k_2(x, x_0, y) + k_3(x, x_0, y). \end{aligned}$$

For $|x-y|>\varepsilon$, standard arguments imply

$$(31)\qquad \left|\frac{\Omega(x-y)}{|x-y|^{n+M-2}}-\frac{\Omega(x_0-y)}{|x_0-y|^{n+M-2}}\right|\le C\frac{|x-x_0|}{|x-y|^{n+M-1}}.$$

We now write

$$(32)\qquad \left|\int_{|x-y|>\varepsilon}k_1(x,x_0,y)f(y)\,dy\right|\le\sum_{\mu=1}^{4}\left|\int_{|x-y|>\varepsilon}k_{1\mu}(x,x_0,y)f(y)\,dy\right|$$

where

$$(33)\qquad \begin{aligned} k_{11}(x,x_0,y)&=P_{m_1-1}(a_j;x,y)P_{m_2-1}(b_j;x,y)\\ &\quad\times\left[\frac{\Omega(x-y)}{|x-y|^{n+M-2}}-\frac{\Omega(x_0-y)}{|x_0-y|^{n+M-2}}\right],\\ k_{12}(x,x_0,y)&=\left(\sum_{\alpha=m_1-1}\frac{(x-y)^\alpha}{\alpha!}a_j^{(\alpha)}(y)\right)P_{m_2-1}(b_j;x,y)\\ &\quad\times\left[\frac{\Omega(x-y)}{|x-y|^{n+M-2}}-\frac{\Omega(x_0-y)}{|x_0-y|^{n+M-2}}\right],\\ k_{13}(x,x_0,y)&=P_{m_1-1}(a_j;x,y)\left(\sum_{\beta=m_2-1}\frac{(x-y)^\beta}{\beta!}b_j^{(\beta)}(y)\right)\\ &\quad\times\left[\frac{\Omega(x-y)}{|x-y|^{n+M-2}}-\frac{\Omega(x_0-y)}{|x_0-y|^{n+M-2}}\right],\\ k_{14}(x,x_0,y)&=\left(\sum_{\alpha=m_1-1}\frac{(x-y)^\alpha}{\alpha!}a_j^{(\alpha)}(y)\right)\left(\sum_{\beta=m_2-1}\frac{(x-y)^\beta}{\beta!}b_j^{(\beta)}(y)\right)\\ &\quad\times\left[\frac{\Omega(x-y)}{|x-y|^{n+M-2}}-\frac{\Omega(x_0-y)}{|x_0-y|^{n+M-2}}\right]. \end{aligned}$$

We estimate these integrals separately. We have using the lemma and (31):

(34)

$$\begin{aligned} &\left|\int_{|x-y|>\varepsilon}k_{11}(x,x_0,y)f(y)\,dy\right|\\ &\quad\le C\operatorname{diam}(\tilde{Q}_j)\int_{|x-y|>\varepsilon}\left|\frac{P_{m_1-1}(a_j;x,y)}{|x-y|^{m_1-1}}\right|\left|\frac{P_{m_2-1}(b_j;x,y)}{|x-y|^{m_2-1}}\right|\frac{|f(y)|}{|x-y|^{n+1}}\,dy \end{aligned}$$

$$\leq C\operatorname{diam}(\tilde{Q}_j)$$
$$\times \int_{|x-y|>\varepsilon}\left(\sum_{\alpha=m_1-1}\left(\frac{1}{|Q(x,y)|}\int_{Q(x,y)}\left|a^{(\alpha)}-m_{Q_j}(a^{(\alpha)})\right|^q\right)^{1/q}\right)$$
$$\times\left(\sum_{\beta=m_2-1}\left(\frac{1}{|Q(x,y)|}\int_{Q(x,y)}\left|b^{(\beta)}-m_{Q_j}(b^{(\beta)})\right|^q\right)^{1/q}\right)\frac{|f(y)|}{|x-y|^{n+1}}\,dy$$

Replacing $m_{Q_j}(a^\alpha)$ by $m_{Q(x,y)}(a^\alpha)$ and likewise for $b^{(\beta)}$ and then using an estimate from [3, p. 695, Lemma 2.2] we obtain

$$(35)\quad \left|\int_{|x-y|>\varepsilon} k_{11}(x,x_0,y)\,dy\right| \leq CS^q_{m_1-1}(a)(z)S^q_{m_2-1}(b)(z)\operatorname{diam}(\tilde{Q}_j)$$
$$\times\int_{|x-y|>\varepsilon}\frac{\left(1+\log\left(\frac{|x-y|}{\operatorname{diam}\tilde{Q}_j}\right)\right)^2|f(y)|}{|x-y|^{n+1}}\,dy.$$

The last integral can be estimated as

$$(36)\quad \operatorname{diam}(\tilde{Q}_j)\sum_{\nu=1}^{\infty}\int_{2^\nu\varepsilon<|x-y|<2^{\nu+1}\varepsilon}\frac{\left(1+\log\left(\frac{|x-y|}{\operatorname{diam}\tilde{Q}_j}\right)\right)^2|f(y)|}{|x-y|^{n+1}}\,dy$$
$$\leq \sum_{\nu=1}^{\infty}(\nu+2)^2\frac{\operatorname{diam}(\tilde{Q}_j)}{(2^\nu\varepsilon)^{n+1}}\int_{|x-y|<2^{\nu+1}\varepsilon}|f(y)|\,dy$$
$$\leq C\sum_{\nu=1}^{\infty}\frac{(\nu+2)^2}{2^\nu}\left(\frac{1}{(2^{\nu+1}\varepsilon)^n}\int_{|x-y|<2^{\nu+1}\varepsilon}|f(y)|^p\,dy\right)^{1/p}$$
$$\leq C'\Lambda_p f(z).$$

Thus we obtain

$$(37)\quad \left|\int_{|x-y|>\varepsilon}k_{11}(x,x_0,y)f(y)\,dy\right| \leq CS^q_{m_1-1}(a)(z)S^q_{m_2-1}(b)(z)\Lambda^p f(z)$$
$$\leq C\gamma\lambda.$$

For the next integral, for $q > p'$ we have

(38)

$$\left|\int_{|x-y|>\varepsilon}k_{12}(x,x_0,y)\,dy\right|$$
$$\leq \sum_{|\alpha|=m_1-1}\frac{1}{\alpha!}$$
$$\times\int_{|x-y|>\varepsilon}\frac{\left|a^{(\alpha)}(y)-m_{Q_j}(a^{(\alpha)})\right|\left|P_{m_2-1}(b_j;x,y)\right||x-x_0||f(y)|\,dy}{|x-y|^{n+m_2}}$$

$$\leq \sum_{|\alpha|=m_1-1} \frac{\operatorname{diam} \tilde{Q}_j}{\alpha!} \int_{|x-y|>\varepsilon} \left| \frac{P_{m_2-1}(b_j; x, y)}{|x-y|^{m_2-1}} \right|$$

$$\times \frac{\left|a^{(\alpha)}(y) - m_{Q_j}(a^{(\alpha)})\right| |f(y)|}{|x-y|^{n+1}} \, dy$$

$$\leq \sum_{|\alpha|=m_1-1} \frac{\operatorname{diam}(\tilde{Q}_j)}{\alpha!} S^q_{m_2-1}(b)(z)$$

$$\times \sum_{\nu=1}^{\infty} \int_{2^\nu \varepsilon < |x-y| < 2^{\nu+1}\varepsilon} \frac{\left(1 + \log \frac{|x-y|}{\operatorname{diam} \tilde{Q}_j}\right) \left|a^\alpha(y) - m_{Q_j}(a^{(\alpha)})\right| |f(y)| \, dy}{|x-y|^{n+1}}$$

$$\leq C \sum_{|\alpha|=m_1-1} \frac{1}{\alpha!} S^q_{m_2-1}(b)(z) \sum_{\nu=1}^{\infty} \frac{(\nu+2)}{2^\nu}$$

$$\times \left(\frac{1}{(2^{\nu+1}\varepsilon)^n} \int_{|x-y|<2^{\nu+1}\varepsilon} \left|a^\alpha(y) - m_{Q_j}(a^{(\alpha)})\right| |f(y)| \, dy \right)$$

$$\leq C \sum_{|\alpha|=m_1-1} \frac{1}{\alpha!} S^q_{m_2-1}(b)(z) \sum_{\nu=1}^{\infty} \frac{(\nu+2)}{2^\nu}$$

$$\times \left(\frac{1}{(2^{\nu+1}\varepsilon)^n} \int_{|x-y|<2^{\nu+1}\varepsilon} \left|a^\alpha(y) - m_{Q_j}(a^{(\alpha)})\right|^q dy \right)^{1/q}$$

$$\times \left(\frac{1}{(2^{\nu+1}\varepsilon)^n} \int_{|x-y|<2^{\nu+1}\varepsilon} |f(y)|^p \, dy \right)^{1/p}$$

$$\leq C \sum_{|\alpha|=m_1-1} \frac{1}{\alpha!} S^q_{m_2-1}(b)(z) \Lambda_p f(z)$$

$$\times \sum_{\nu=1}^{\infty} \frac{(\nu+2)}{2^\nu} \left(\frac{1}{|2^{\nu+1}\tilde{Q}_j|} \int_{2^{\nu+1}\tilde{Q}_j} \left|a^{(\alpha)}(y) - m_{Q_j}(a^{(\alpha)})\right|^q dy \right)^{1/q}$$

$$\leq C \sum_{|\alpha|=m_1-1} \frac{1}{\alpha!} S^q_{m_2-1}(b)(z) \Lambda_p f(z)$$

$$\times \sum_{\nu=1}^{\infty} \nu \frac{(\nu+2)}{2^\nu} \left(\frac{1}{|2^{\nu+1}\tilde{Q}_j|} \int_{2^{\nu+1}\tilde{Q}_j} \left|a^{(\alpha)}(y) - m_{2^{\nu+1}\tilde{Q}_j}(a^{(\alpha)})\right|^q dy \right)^{1/q}$$

$$\leq C S^q_{m_1-1}(a)(z) S^q_{m_2-1}(b)(z) \Lambda_p f(z)$$

$$\leq C\gamma\lambda.$$

A completely similar argument will give us the same estimate for $|\int_{|x-y|>\varepsilon} k_{13}(x, x_0, y) f(y)\, dy|$. Finally, we note that the same estimate for

$|\int_{|x-y|>\varepsilon} k_{14}(x, x_0, y) f(y)\, dy|$ follows by an argument analogous to the one above except that both the a and b terms must be treated in a manner similar to the way in which the a terms were treated above. The details are left to the reader.

Returning now to (30), we now deal with the second term involving $k_2(x, x_0, y)$. A similar argument will apply to the last term involving $k_3(x, x_0, y)$. We use the following identity (see [1]).

$$(39)\quad P_{m_1}(a_j; x, y) - P_{m_1}(a_j; x_0, y)$$
$$= P_{m_1}(a_j; x, x_0) + \sum_{0<|\alpha|<m_1} \frac{(x - x_0)^{\alpha}}{\alpha!} P_{m_1-|\alpha|}\left(a_j^{(\alpha)}; x_0, y\right)$$
$$= P_{m_1-1}(a_j; x, x_0) + \sum_{0<|\alpha|<m_1-1} \frac{(x - x_0)^{\alpha}}{\alpha!} P_{m_1-|\alpha|}\left(a_j^{(\alpha)}; x_0, y\right)$$
$$+ \sum_{|\alpha|=m_1-1} \frac{(x - x_0)^{\alpha}}{\alpha!} a_j^{(\alpha)}(y).$$

Each of the three terms above can now be used to write $k_2(x, x_0, y)$ as

$$k_{21}(x, x_0, y) + k_{22}(x, x_0, y) + k_{23}(x, x_0, y).$$

We now estimate each of the corresponding integrals separately. We have

$$(40)$$
$$\left|\int_{|x-y|>\varepsilon} k_{21}(x, x_0, y) f_2(y)\, dy\right|$$
$$= \left|\int_{|x-y|>\varepsilon} \frac{P_{m_1-1}(a_j; x, x_0) P_{m_2}(b_j; x, y) \Omega(x_0 - y) f(y)\, dy}{|x_0 - y|^{n+M-2}}\right|$$
$$\leq C \left|\int_{|x-y|>\varepsilon} \frac{P_{m_1-1}(a_j; x, x_0) P_{m_2-1}(b_j; x, y) \Omega(x_0 - y) f(y)\, dy}{|x - y|^{n+M-2}}\right|$$
$$+ C \sum_{|\beta|=m_2-1} \frac{1}{\beta!}$$
$$\times \left|\int_{|x-y|>\varepsilon} \frac{P_{m_1-1}(a_j; x, x_0) b_j^{(\beta)}(y)(x - y)^{\beta} \Omega(x_0 - y) f(y)\, dy}{|x - y|^{n+M-2}}\right|$$

The first integral in (40) is majorized by

$$(41)\quad C|x-x_0|\int_{|x-y|>\varepsilon}\left|\frac{P_{m_1-1}(a_j;x,x_0)}{|x-x_0||x-y|^{m_1-2}}\right|\left|\frac{P_{m_2-1}(b_j;x,y)}{|x-y|^{m_2-1}}\right|\frac{|f(y)|\,dy}{|x-y|^{n+1}}$$

$$\le CS^q_{m_1-1}(a)(z)S^q_{m_2-1}(b)(z)\Lambda_p f(z)$$

$$\le C\gamma\lambda.$$

Each integral in the sum in (40) is majorized by

$$(42)\qquad C|x-x_0|\int_{|x-y|>\varepsilon}\left|\frac{P_{m_1-1}(a_j;x,x_0)}{|x-x_0||x-y|^{m_1-2}}.\right|\frac{|b_j^{(\beta)}(y)||f(y)|\,dy}{|x-y|^{n+1}}$$

$$\le CS^q_{m_1-1}(a)(z)S^q_{m_2-1}(b)(z)\Lambda_p f(z)$$

$$\le C\gamma\lambda.$$

We note that each of the above estimates follows from the same type of argument used in estimating the integral with k_{12} above. We now estimate the integral with $k_{22}(x, x_0, y)$. We have

(43)

$$\left|\int_{|x-y|>\varepsilon}k_{22}(x,x_0,y)f_2(y)\,dy\right|$$

$$\le C\sum_{0<|\alpha|<m_1}\frac{1}{\alpha!}$$

$$\times\left|\int_{|x-y|>\varepsilon}\frac{P_{m_1-|\alpha|}\left(a_j^{(\alpha)};x_0,y\right)P_{m_2-1}(b_j;x,y)(x-x_0)^\alpha\Omega(x_0-y)f(y)\,dy}{|x-y|^{n+M-2}}\right|$$

$$+C\sum_{0<|\alpha|<m_1}\sum_{|\beta|=m_2-1}\frac{1}{\alpha!\beta!}$$

$$\times\left|\int_{|x-y|>\varepsilon}\frac{P_{m_1-|\alpha|}\left(a_j^{(\alpha)};x_0,y\right)(x-x_0)^\alpha(x-y)^\beta\Omega(x_0-y)b_j^{(\beta)}(y)f(y)\,dy}{|x-y|^{n+M-2}}\right.$$

$$\equiv I'_{22}+I''_{22}.$$

We now estimate I'_{22} by breaking up the first remainder. We have

(44)

$$I'_{22} \le C \sum_{0<|\alpha|<m_1} \frac{1}{\alpha!}|x-x_0| \times \int_{|x-y|>\varepsilon} \left| \frac{P_{m_1-|\alpha|-1}(a_j^{(\alpha)}; x_0, y)}{|x_0-y|^{m_1-|\alpha|-1}} \right| \left| \frac{P_{m_2-1}(b_j; x, y)}{|x-y|^{m_2-1}} \right| \frac{|f(y)|\,dy}{|x-y|^{n+1}}$$
$$+ C \sum_{0<|\alpha|<m_1} \frac{1}{\alpha!} \sum_{|\gamma|=m_1-|\alpha|-1} \frac{1}{\gamma!}|x-x_0| \times \int_{|x-y|>\varepsilon} \left| \frac{P_{m_2-1}(b_j; x, y)}{|x-y|^{m_2-1}} \right| \frac{|a_j^{(\alpha+\gamma)}(y)|\,|f(y)|\,dy}{|x-y|^{n+1}}.$$

By familiar arguments, each of these integrals is bounded by

$$S^q_{m_1-1}(a)(z) S^q_{m_2-1}(b)(z) \Lambda_p f(z) \le c\gamma\lambda.$$

We now estimate I''_{22} again by breaking up the first remainder. We have

(45)

$$I''_{22} \le C \sum_{\substack{0<|\alpha|<m_1\\ |\beta|=m_2-1}} \frac{1}{\alpha!\beta!}|x-x_0| \times \int_{|x-y|>\varepsilon} \left| \frac{P_{m_1-|\alpha|-1}(a_j^{(\alpha)}; x_0, y)}{|x_0-y|^{m_1-|\alpha|-1}} \right| \frac{|b_j^{(\beta)}(y)|\,|f(y)|\,dy}{|x-y|^{n+1}}$$
$$+ C \sum_{\substack{0<|\alpha|<m_1\\ |\beta|=m_2-1}} \frac{1}{\alpha!\beta!} \sum_{|\gamma|=m_1-|\alpha|-1} \frac{1}{\gamma!}|x-x_0| \times \int_{|x-y|>\varepsilon} \frac{|a_j^{(\alpha+\gamma)}(y)|\,|b_j^{(\beta)}(y)|\,|f(y)|}{|x-y|^{n+1}}\,dy.$$

Again by familiar arguments each of these integrals is bounded by

$$C S^q_{m_1-1}(a)(z) S^q_{m_2-1}(b)(z) \Lambda_p f(z) \le c\gamma\lambda.$$

This completes the estimate of the integral in (43) involving $k_{22}(x, x_0, y)$. The same estimate holds for the integral involving $k_{23}(x, x_0, y)$ by the above argument with the roles of a and b interchanged. This now completes the estimate of the first term in (29). To summarize, we have shown that

$$\left|\int_{|x-y|>\varepsilon}[k(x,y)-k(x_0,y)]f_2(y)\,dy\right|\leq C\gamma\lambda. \tag{46}$$

To complete the f_2 estimate for $\operatorname{diam}\tilde{Q}_j\leq\varepsilon\leq K(n)\operatorname{diam}(\tilde{Q}_j)$ it only remains to estimate the error terms in (29). We will estimate the first error term while the second one is handled similarly. We have

$$\begin{aligned}(47)\quad &\int_{R(x)}|k(x,y)f(y)|\,dy\\ &=\int_{R(x)}\frac{|P_{m_1}(a_j;x,y)P_{m_2}(b_j;x,y)\Omega(x-y)f(y)|}{|x-y|^{n+M-2}}\,dy\\ &\leq\int_{R(x)}\left|\frac{P_{m_1-1}(a_j;x,y)}{|x-y|^{m_1-1}}\right|\left|\frac{P_{m_2-1}(b_j;x,y)}{|x-y|^{m_2-1}}\right|\frac{|f(y)|}{|x-y|^n}\,dy\\ &\quad+\sum_{|\alpha|=m_1-1}\frac{1}{\alpha!}\int_{R(x)}\left|\frac{P_{m_2-1}(b_j;x,y)}{|x-y|^{m_2-1}}\right|\frac{|a_j^{(\alpha)}(y)||f(y)|}{|x-y|^n}\,dy\\ &\quad+\sum_{|\beta|=m_2-1}\frac{1}{\beta!}\int_{R(x)}\left|\frac{P_{m_1-1}(a_j;x,y)}{|x-y|^{m_1-1}}\right|\frac{|b_j^{(\beta)}(y)||f(y)|}{|x-y|^n}\,dy\\ &\quad+\sum_{\substack{|\alpha|=m_1-1\\|\beta|=m_2-1}}\frac{1}{\alpha!\beta!}\int_{R(x)}\frac{|a_j^{(\alpha)}(y)||b_j^{(\beta)}(y)||f(y)|\,dy}{|x-y|^n}\end{aligned}$$

We note that for $y\in R(x)$,

$$1\leq\frac{K(n)\operatorname{diam}(Q_j)}{|x-y|}.$$

Thus each error term is increased by multiplying by $K(n)\operatorname{diam}(Q_j)$ on the outside and increasing the exponent of $|x-y|$ under $|f(y)|$ by 1. After doing so each integral can be estimated by familiar arguments used earlier. Each error term is majorized by

$$CK(n)S^q_{m_1-1}(a)(z)S^q_{m_2-1}(b)(z)\Lambda_pf(z).$$

It now follows that the entire error term is majorized by $C(n)\gamma\lambda$. In summary we have shown that for any $x \in Q_j$,

$$\sup_{\varepsilon \approx \operatorname{diam} \tilde{Q}_j} |C_\varepsilon(a, b; f_2)(x)| \le C\gamma\lambda + \lambda. \tag{48}$$

The f_2 estimate for $\varepsilon > K(n)\operatorname{diam} \tilde{Q}_j$. Let Q_j^ε denote the cube with sides parallel to the axes with the same center as Q_j and diameter ε. Let

$$a_\varepsilon(x) = a(x) - \sum_{|\alpha| = m_1 - 1} \frac{1}{\alpha!} m_{Q_j^\varepsilon}(a) x^\alpha$$

and let $b_\varepsilon(x)$ be defined in a similar manner. Let

$$k_\varepsilon(x, y) = \frac{P_{m_1}(a_\varepsilon; x, y) P_{m_2}(b_\varepsilon; x, y) \Omega(x - y)}{|x - y|^{n+M-2}}. \tag{49}$$

Proceeding as before we have

$$\begin{aligned} |C_\varepsilon(a, b; f)(x)| \le & \left| \int_{|x-y|>\varepsilon} [k_\varepsilon(x - y) - k_\varepsilon(x_0, y)] f(y)\, dy \right| \\ & + \int_{\varepsilon \le |x-y| \le C\varepsilon} |k_\varepsilon(x, y) f(y)|\, dy \\ & + \int_{\varepsilon \le |x_0 - y| \le C\varepsilon} |k_\varepsilon(x_0, y) f(y)|\, dy \\ & + \left| \int_{|x_0 - y| > \varepsilon} k_\varepsilon(x_0, y) f(y) \right| dy. \end{aligned} \tag{50}$$

The last integral in (50) is $\le \lambda$ since $x_0 \notin \bigcup_j Q_j$. The same estimates hold for the error terms since $\operatorname{diam} Q_\varepsilon = \varepsilon$. For the first term we must be careful since $|x - x_0|$ and $\operatorname{diam} Q_\varepsilon$ are no longer comparable. In particular we no longer have the estimate

$$\left| P_{m_1 - 1}(a_\varepsilon; x, x_0) \right| \le C|x - x_0|^{m_1 - 1} S^q_{m_1 - 1}(a_\varepsilon)(z)$$

since the ratio of $|Q_\varepsilon|$ to $|Q(x, x_0)|$ may become unbounded. To deal with such terms we select a point x_ε such that $|x - x_\varepsilon| = 2\varepsilon$. Then $\varepsilon < |x_0 - x_\varepsilon| < 3\varepsilon$. We then write

$$\begin{aligned} P_{m_1 - 1}(a_\varepsilon; x, x_0) = & P_{m_1 - 1}(a_\varepsilon; x, x_\varepsilon) - P_{m_1 - 1}(a_\varepsilon; x_0, x_\varepsilon) \\ & - \sum_{0 < |\alpha| < m_1 - 1} \frac{(x - x_0)^\alpha}{\alpha!} P_{m_1 - 1 - |\alpha|}(a_\varepsilon; x_0, x_\varepsilon) \end{aligned} \tag{51}$$

The appropriate estimates now hold for each of these terms. For example, the first term in the first integral in (40) can be estimated as follows:

(52)

$$\begin{aligned}&\left|\int_{|x-y|>\varepsilon}\frac{P_{m_1-1}(a_\varepsilon;x,x_\varepsilon)P_{m_2-1}(b_\varepsilon;x,y)\Omega(x_0-y)f(y)\,dy}{|x-y|^{n+M-2}}\right|\\&\le C|x-x_\varepsilon|\int_{|x-y|>\varepsilon}\left|\frac{P_{m_1-1}(a_\varepsilon;x,x_\varepsilon)}{|x-x_\varepsilon|^{m_1-1}}\right|\left|\frac{P_{m_2-1}(b_\varepsilon;x,y)}{|x-y|^{m_2-1}}\right|\frac{|f(y)|}{|x-y|^{n+1}}\,dy\\&\le 2CS^q_{m_1-1}(a)(z)\left(\varepsilon\int_{|x-y|>\varepsilon}\left|\frac{P_{m_2-1}(b_\varepsilon;x,y)}{|x-y|^{m_2-1}}\right|\frac{|f(y)|}{|x-y|^{n+1}}\,dy\right)\\&\le 2CS^q_{m_1}(a)(z)S^q_{m_1}(b)(z)\Lambda_pf(z)\\&\le C'\gamma\lambda.\end{aligned}$$

All the remaining estimates can be handled similarly. Ultimately, for $\varepsilon > K(n)\mathrm{diam}(\tilde{Q}_j)$, we obtain

$$\left|\int_{|x-y|>\varepsilon}[k_\varepsilon(x-y)-k_\varepsilon(x_0,y)]f(y)\,dy\right|\le C\gamma\lambda. \tag{53}$$

From (50) it now follows that

$$\sup_{\varepsilon>K(n)\mathrm{diam}(\tilde{Q}_j)}|C_\varepsilon(a,b;f_2)(x)|\le\lambda+C\gamma\lambda. \tag{54}$$

The estimates on f_2 now yield the pointwise estimate $C_*(a,b;f_2)(x)\le\lambda+C\gamma\lambda$ for all $x\in Q_j$.

Conclusion

We now choose γ_0 such that $C\gamma_0<1$ where C is the constant in (54). Then by (26) with $\beta=1$, we have, for $\gamma<\gamma_0$,

(55)

$$\begin{aligned}&\left|\{x\in Q_j\colon C_*(a,b;f)(x)>3\lambda,\ S^q_{m_1}(a)(x)S^q_{m_2}b(x)\Lambda_pf(x)\le\gamma\lambda\}\right|\\&\le\left|\{x\in Q_j\colon C_*(a,b;f_1)(x)>2\lambda-C\gamma\lambda\}\right|\\&\quad+\left|\{x\in Q_j\colon C_*(a,b;f_2)(x)>\lambda+C\gamma\lambda\}\right|\\&\le\left|\{x\in Q_j\colon C_*(a,b;f_1)(x)>\lambda\}\right|\\&\le C\gamma^r\left|\bar{\bar{Q}}_j\right|\\&\le C'\gamma^r|Q_j|\end{aligned}$$

This establishes the good λ inequality and completes the proof of the theorem.

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