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# ON A FORMULA FOR ALMOST-EVEN ARITHMETICAL FUNCTIONS

BY

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## Introduction

For an arithmetical function the property of being almost-even is a special case of limit-periodicity, which is itself a special case of almost-periodicity.

**1.1.** An arithmetical function $f$ is said to be *almost-periodic-B* (more precisely almost-periodic-$B^1$) if, given $\varepsilon > 0$, there exists a trigonometric polynomial $P$,

$$P(n) = \sum_{k=1}^{m} \lambda_k e(\alpha_k n), \quad \text{where } e(t) = \exp(2\pi i t),$$

such that

$$\limsup_{x \to \infty} \frac{1}{x} \sum_{n \le x} |P(n) - f(n)| \le \varepsilon. \tag{1}$$

This implies that $\sum_{n \le x} |f(n)| = O(x)$ and that, for each real $\alpha$,

$$\lim_{x \to \infty} \frac{1}{x} \sum_{n \le x} f(n) e(-\alpha n) \text{ exists, say is } C(f, \alpha).$$

The spectrum of $f$ is the (at most denumerable) subset $\operatorname{Sp} f$ of the quotient group $\mathbf{R}/\mathbf{Z}$ consisting of the residue-classes modulo 1 of those $\alpha$ for which $C(f, \alpha) \ne 0$.

The Fourier series of $f$ is the formal sum $\sum C(f, \alpha) e(\alpha n)$ extended to those $\alpha \in [0, 1[$ whose residue-class modulo 1 belongs to $\operatorname{Sp} f$.

The arithmetical function $f$ is said to be *limit-periodic-B* if, given $\varepsilon > 0$, there exists a *periodic* arithmetical function $P$ such that (1) holds.

Since a periodic arithmetical function can be expressed by a trigonometric polynomial, this implies that $f$ is almost-periodic-$B$. Its spectrum is contained in $\mathbf{Q}/\mathbf{Z}$ (i.e., $C(f, \alpha) = 0$ when $\alpha$ is irrational).

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It can be proved that the periodic function $P$ in (1) can be taken equal to

$$\sigma_N^{(f)}(n) = \sum_{k=0}^{N-1} C\left(f, \frac{k}{N}\right) e\left(\frac{k}{N} n\right)$$

where $N$ is suitably chosen.

**1.2.** Now, an arithmetical function $f$ is said to be *even modulo k* if $f(n)$ depends only upon $(k, n)$. It is said to be *even* if there exists a $k$ such that it is even modulo $k$.

The arithmetical function $f$ is said to be *almost-even-B* if, given $\varepsilon > 0$, there exists an even arithmetical function $g$ such that

$$\limsup_{x \to \infty} \frac{1}{x} \sum_{n \leq x} |g(n) - f(n)| \leq \varepsilon.$$

Since even arithmetical functions are obviously periodic, this implies that $f$ is limit-periodic-B.

It turns out that a limit-periodic-B arithmetical function is almost-even-B if and only if the following condition is satisfied:

(C) The Fourier coefficient $C(f, r)$ where the rational number $r$ is equal to $h/q$, with $q \in \mathbf{N}^*$ and $(h, q) = 1$, depends only upon $q$.

Condition (C) implies that, by grouping together the terms for which $q$ has the same value, the Fourier series for $f$ may be written in the form

$$\sum_{q=1}^{\infty} a_q c_q(n), \quad \text{where } c_q(n) \text{ is the Ramanujan sum } \sum_{\substack{1 \leq h \leq q \\ (h, q)=1}} e\left(\frac{h}{q} n\right).$$

This may be called the Ramanujan expansion of $f(n)$.

It is very easy to see that

$$a_q = \frac{1}{\varphi(q)} \lim_{x \to \infty} \frac{1}{x} \sum_{n \leq x} f(n) c_q(n).$$

**1.3.** Condition (C) obviously implies that, for every $N$, $\sigma_N^{(f)}(n)$ is of the form

$$\sum_{q/N} \lambda_q c_q(n).$$

On the other hand condition (C) is certainly satisfied if, for every $\varepsilon > 0$, $P(n)$ in (1) can be taken equal to a linear combination of Ramanujan sums (because, if $f$ is an almost-periodic-*B* arithmetical function and $\{f_\nu\}$ a sequence of almost-periodic-$B$ arithmetical functions such that

$$\lim_{\nu\to\infty}\left\{\limsup_{x\to\infty}\frac{1}{x}\sum_{n\le x}|f_\nu(n)-f(n)|\right\}=0,$$

then, for every real $\alpha$, $C(f,\alpha)=\lim_{\nu\to\infty}C(f_\nu,\alpha))$.

Thus the assertion that a limit-periodic-$B$ arithmetical function $f$ is almost-even-$B$ if and only if condition (C) is satisfied follows from the following fact:

Let $A$ be the vector space of arithmetical functions. The set of even arithmetical functions is the subspace of $A$ generated by the functions $c_q$. More precisely, for each positive integer $N$, the set $E_N$ of those arithmetical functions which are even modulo $N$ is the subspace of $A$ generated by the functions $c_q$ where $q/N$. This may be seen as follows.

Given the positive integer $N$ and a divisor $d$ of $N$, let

$$F_{N,d}(n)=\begin{cases}1 & \text{if } (N,n)=d,\\ 0 & \text{otherwise.}\end{cases}$$

If $N$ is fixed, then the functions $F_{N,d}$ where $d$ runs through the set of the divisors of $N$ is obviously a basis of the vector space $E_N$. So this space has dimension $\tau(N)$, the number of divisors of $N$.

On the other hand, for each $q$ dividing $N$, the function $c_q$ is even modulo $N$, for

$$c_q(n)=\sum_{d/(q,n)}d\mu\left(\frac{q}{d}\right)\quad\text{and}\quad (q,n)=(q,(N,n)).$$

The functions $c_q$ are linearly independent for

$$\lim_{x\to\infty}\frac{1}{x}\sum_{n\le x}c_{q_1}(n)c_{q_2}(n)=\begin{cases}0 & \text{if } q_1\ne q_2,\\ \varphi(q) & \text{if } q_1=q_2=q.\end{cases}$$

Therefore the $\tau(N)$ functions $c_q$ where $q/N$ form a basis of $E_N$.

**1.4.** The following result, due to A. Wintner,[1] is well known.

*Given an arithmetical function $f$, let $f'=f_*\mu$ (i.e., $f'(n)=\sum_{d/n}f(d)\mu(n/d)$). If*

$$\sum_{n=1}^{\infty}\frac{|f'(n)|}{n}<\infty,$$

*then $f$ is almost-even-$B$*[1] *and*

$$a_q=\sum_{n=1}^{\infty}\frac{f'(nq)}{nq}.$$

Here the series is obviously absolutely convergent.

[1] Eratosthenian averages, Baltimore, Maryland, 1943, Section 33.

One may raise the question whether the same formula (without absolute convergence) holds for any almost-even-$B$ arithmetical function.

We will prove here the following theorem which shows that the answer is yes.

THEOREM. *Let $f$ be an arithmetical function, and let $f' = f_* \mu$. Let $q$ be any positive integer.*

*Suppose that*

(i) $\sum_{n \le x} |f(n)| = O(x)$;

(ii) *For each positive integer $d$ dividing $q$,*

$$\lim_{x \to \infty} \frac{1}{x} \sum_{\substack{n \le x \\ (q, n) = d}} f(n) \text{ exists.}$$

*Then the series $\sum_{n=1}^{\infty} f'(nq)/nq$ converges and its sum is*

$$\frac{1}{\varphi(q)} \lim_{x \to \infty} \frac{1}{x} \sum_{n \le x} f(n) c_q(n).$$

The hypotheses of this theorem are certainly satisfied for all positive $q$ if $f$ is almost-periodic-B, not necessarily almost-even-$B^1$.

**1.5.** We may remark that hypothesis (ii) is equivalent to:

(ii)′ For each positive integer $d$ dividing $q$,

$$\lim_{x \to \infty} \frac{1}{x} \sum_{n \le x} f(n) c_d(n) \text{ exists.}$$

In fact both condition (ii) and condition (ii)′ are equivalent to:

(ii)″ For every arithmetical function $g$ even modulo $q$,

$$\lim_{x \to \infty} \frac{1}{x} \sum_{n \le x} f(n) g(n) \text{ exists.}$$

This follows immediately from the above mentioned fact that the set of the functions $F_{q,d}$ where $d/q$ and the set of the functions $c_d$ where $d/q$ are bases of the vector space $E_q$.

**1.6.** The original proof of our theorem was rather complicated. The one that we give here is inspired by a proof which was communicated to us by Dr. A Hildebrand for the particular case when $q = 1$, namely the following result:

If $\sum_{n \le x} |f(n)| = O(x)$ and if $f$ has a mean value $M(f)$, then the series $\sum_{n=1}^{\infty} f'(n)/n$ converges and its sum is $M(f)$.

## 2. A basic lemma

The following lemma is essential for our proof.

LEMMA. *Let $\chi_q$ be the principal character modulo $q$, where $q$ is any positive integer.*

(i) $$\sum_{n<x} \frac{\mu(n)\chi_q(n)}{n} = O(e^{-\alpha\sqrt{\log x}}) \quad \textit{for some } \alpha \in ]0,1[;$$

(ii) *The series*

$$\sum_{n=1}^{\infty} \frac{\mu(n)\chi_q(n)\log n}{n}$$

*converges and its sum is* $-q/\varphi(q)$.

*Proof.* A classical proof, using the formula

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} = \frac{1}{\zeta(s)} \quad \text{for Re } s > 1$$

and an estimate of $|1/\zeta(s)|$, shows that there exists $\alpha > 0$ such that

$$M(x) = \sum_{n \le x} \mu(n) = O(xe^{-\alpha\sqrt{\log x}}).$$

A quite similar proof, using the formula

$$(2) \quad \sum_{n=1}^{\infty} \frac{\mu(n)\chi_q(n)}{n^s} = \frac{1}{L(s, \chi_q)} = \frac{1}{\zeta(s)} \prod_{p/q} \left(1 - \frac{1}{p^s}\right)^{-1} \quad (\text{Re } s > 1),$$

shows that there exists $\beta > 0$ such that

$$(3) \quad M_q(x) = \sum_{n \le x} \mu(n)\chi_q(n) = O(xe^{-\beta\sqrt{\log x}}).$$

This, with the equality

$$\sum_{x<n\le y} \frac{\mu(n)\chi_q(n)}{n} = \frac{M_q(y)}{y} - \frac{M_q(x)}{x} + \int_x^y \frac{M_q(t)}{t^2}\,dt \quad \text{for } 0 < x < y,$$

shows that the series

$$\sum_{n=1}^{\infty} \frac{\mu(n)\chi_q(n)}{n}$$

converges and that

$$\sum_{n>x} \frac{\mu(n)\chi_q(n)}{n} = O\left(\sqrt{\log x}\, e^{-\beta\sqrt{\log x}}\right) = O\left(e^{-\alpha\sqrt{\log x}}\right) \quad \text{for } 0 < \alpha < \beta.$$

Now it follows from (2) that

$$\sum_{n=1}^{\infty} \frac{\mu(n)\chi_q(n)}{n} = 0,$$

so that

$$\sum_{n\le x} \frac{\mu(n)\chi_q(n)}{n} = -\sum_{n>x} \frac{\mu(n)\chi_q(n)}{n}.$$

Similarly, (3) shows that the series

$$\sum_{n=1}^{\infty} \frac{\mu(n)\chi_q(n)\log n}{n}$$

converges, and the formula obtained by differentiation of (2) shows that its sum is $-q/\varphi(q)$.

## 3. Proof of the theorem

We now suppose that $f$ is an arithmetical function satisfying hypotheses (i) and (ii) of the theorem.

**3.1.** By hypothesis (i) there exists $K > 0$ such that

(4) $$\sum_{n\le x} |f(n)| \le Kx \quad \text{for every positive } x.$$

**3.2.** We now make the following remark.

For each divisor $d$ of $q$ set

$$m_d = \lim_{x\to\infty} \frac{1}{x} \sum_{\substack{n\le x\\ (q,n)=d}} f(n).$$

If $\sigma$ is any real number $> 1$, then, as $x$ tends to infinity,

$$\sum_{\substack{x<n\leq\sigma x\\(q,n)=d}} \frac{f(n)}{n} \text{ tends to } m_d \log\sigma.$$

*Proof.* Let $\Phi(t) = \sum_{n\leq t,(q,n)=d} f(n)$. We have $|\Phi(t)| \leq Kt$ for every $t > 0$, and $\Phi(t)/t$ tends to $m_d$ as $t$ tends to infinity.

For $x \geq (\sigma - 1)^{-1}$ we also have

$$\sum_{\substack{x<n\leq\sigma x\\(q,n)=d}} \frac{f(n)}{n} = \frac{\Phi(\sigma x)}{\sigma x} - \frac{\Phi(x)}{x} + \int_x^{\sigma x} \frac{\Phi(t)}{t^2}\, dt.$$

As $x$ tends to infinity, $\Phi(x)/x$ and $\Phi(\sigma x)/\sigma x$ tend to $m_d$. Furthermore we have

$$\int_x^{\sigma x} \frac{\Phi(t)}{t^2}\, dt = \int_1^{\sigma} \frac{\Phi(xu)}{xu^2}\, du.$$

As

$$\left|\frac{\Phi(xu)}{xu^2}\right| \leq \frac{K}{u} \quad \text{for every positive } x$$

and $\Phi(xu)/xu^2$ tends to $m_d/u$ as $x$ tends to infinity, this tends to

$$\int_1^{\sigma} \frac{m_d}{u}\, du = m_d \log\sigma.$$

**3.3.** Now, for $x \geq 1$, we have

$$\sum_{n\leq x} \frac{f'(qn)}{qn} = \sum_{n\leq x} \frac{1}{qn}\left(\sum_{d/qn} f(d)\mu\left(\frac{qn}{d}\right)\right) = \frac{1}{q}\sum_{\substack{n\leq x\\d/qn}} \frac{f(d)}{n}\mu\left(\frac{qn}{d}\right).$$

In the last sum we will group together the terms for which $(q, d)$ has the same value. The latter must be a divisor of $q$. Let $\delta$ be any divisor of $q$ and let $q' = q/\delta$. Then $(q, d)$ is equal to $\delta$ if and only if $d = \delta d'$ where $(q', d') = 1$. When it is so, $d$ divides $qn$ if and only if $d'/n$, that is $n = md'$. Now $n = md'$ gives $qn/d = mq'$. Thus we obtain

$$\begin{aligned}\sum_{n\leq x} \frac{f'(qn)}{qn} &= \frac{1}{q}\left(\sum_{\delta q'=q}\left(\sum_{\substack{md'\leq x\\(q',d')=1}} \frac{f(\delta d')}{md'}\right)\mu(mq')\right)\\ &= \frac{1}{q}\sum_{\delta q'=q}\left(\sum_{md'\leq x} \frac{f(\delta d')}{md'}\chi_{q'}(d')\mu(mq')\right),\end{aligned}$$

where $\chi_{q'}$ is the principal character modulo $q'$.

Using the fact that $\mu(mq') = \mu(q')\mu(m)\chi_{q'}(m)$ this gives

$$\sum_{n\le x} \frac{f'(qn)}{qn} = \frac{1}{q} \sum_{\delta q'=q} \mu(q')\left( \sum_{md'\le x} \mu(m)\chi_{q'}(md')\frac{f(\delta d')}{md'}\right).$$

We may rewrite this formula in the form

$$\sum_{n\le x} \frac{f'(qn)}{qn} = \frac{1}{q} \sum_{\delta/q} \mu(q')G_\delta(x), \tag{5}$$

where $q' = q/\delta$ and

$$G_\delta(x) = \sum_{mn\le x} \mu(m)\chi_{q'}(mn)\frac{f(\delta n)}{mn}.$$

Thus, to prove the convergence of the series $\sum_{n=1}^{\infty} f'(qn)/qn$, it is sufficient to show that, for each divisor $\delta$ of $q$, $G_\delta(x)$ tends to a finite limit as $x$ tends to infinity.

**3.4.** We now introduce a fixed $\lambda \ge e^{1/4}$ and in the formula which defines $G_\delta(x)$ we separate the terms for which $n \le x/\lambda$ and those for which $n > x/\lambda$. We thus obtain, for $x > \lambda$,

$$\begin{aligned} G_\delta(x) &= \sum_{n\le x/\lambda} \frac{f(\delta n)\chi_{q'}(n)}{n}\left( \sum_{m\le x/n} \frac{\mu(m)\chi_{q'}(m)}{m}\right) \\ &\quad + \sum_{\substack{x/\lambda<n\le x\\ mn\le x}} \mu(m)\chi_{q'}(mn)\frac{f(\delta n)}{mn}, \\ &= \sum_1 + \sum_2, \text{ say.} \end{aligned} \tag{6}$$

**3.4.1.** By the lemma of §2 there exist $\alpha \in ]0,1[$ and $C > 0$ such that

$$\left| \sum_{m\le X} \frac{\mu(m)\chi_{q'}(m)}{m} \right| \le Ce^{-\alpha\sqrt{\log X}} \quad \text{for every } X \ge 1.$$

So

$$\left|\sum_1\right| \le C \sum_{n<x/\lambda} \frac{|f(\delta n)|}{n} e^{-\alpha\sqrt{\log(x/n)}}.$$

Setting $\Psi_\delta(t) = \sum_{n\le t} |f(\delta n)|$ we have

$$\sum_{n\le x/\lambda} \frac{|f(\delta n)|}{n} e^{-\alpha\sqrt{\log(x/n)}} = \Psi_\delta\left(\frac{x}{\lambda}\right)\frac{\lambda}{x}e^{-\alpha\sqrt{\log \lambda}}$$

$$- \int_1^{x/\lambda} \Psi_\delta(t)\frac{d}{dt}\left(\frac{e^{-\alpha\sqrt{\log(x/t)}}}{t}\right) dt.$$

As $0 \leq \Psi_\delta(t) \leq \delta K t$ by (4) and

$$\left|\frac{d}{dt}\left(\frac{e^{-\alpha\sqrt{\log(x/t)}}}{t}\right)\right| = \frac{e^{-\alpha\sqrt{\log(x/t)}}}{t^2}\left|1 - \frac{\alpha}{2\sqrt{\log(x/t)}}\right|$$
$$\leq \frac{e^{-\alpha\sqrt{\log(x/t)}}}{t^2} \quad \text{for } 1 \leq t \leq \frac{x}{\lambda},$$

this yields

$$\sum_{n\leq x/\lambda} \frac{|f(\delta n)|}{n} e^{-\alpha\sqrt{\log(x/n)}} \leq \delta K\left(e^{-\alpha\sqrt{\log\lambda}} + \int_1^{x/\lambda} \frac{e^{-\alpha\sqrt{\log(x/t)}}}{t}\,dt\right).$$

The change of variable $t = xe^{-u^2}$ gives

$$\int_1^{x/\lambda} \frac{e^{-\alpha\sqrt{\log(x/t)}}}{t}\,dt = 2\int_{\sqrt{\log\lambda}}^{\sqrt{\log x}} ue^{-\alpha u}\,du,$$

whence

$$\int_1^{x/\lambda} \frac{e^{-\alpha\sqrt{\log(x/t)}}}{t}\,dt \leq 2\int_{\sqrt{\log\lambda}}^{\infty} ue^{-\alpha u}\,du = 2\left(\frac{\sqrt{\log\lambda}}{\alpha} + \frac{1}{\alpha^2}\right)e^{-\alpha\sqrt{\log\lambda}}.$$

We finally obtain

$$\left|\sum_1\right| \leq C\delta K e^{-\alpha\sqrt{\log\lambda}}\left(1 + \frac{2\sqrt{\log\lambda}}{\alpha} + \frac{2}{\alpha^2}\right) = g_1(\lambda),\ \text{say}. \tag{7}$$

Note that $g_1(\lambda)$ tends to zero as $\lambda$ tends to infinity.

**3.4.2.** Now, since the conditions $x/\lambda < n \leq x$ and $mn \leq x$ are equivalent to $m < \lambda$ and $x/\lambda < n \leq x/m$, we have

$$\sum_2 = \sum_{m<\lambda} \frac{\mu(m)\chi_{q'}(m)}{m}\left(\sum_{x/\lambda<n\leq x/m} \frac{\chi_{q'}(n)f(\delta n)}{n}\right). \tag{8}$$

We remark that

$$\sum_{x/\lambda<n\leq x/m} \frac{\chi_{q'}(n)f(\delta n)}{n} = \delta \sum_{\substack{\delta x/\lambda<\delta n\leq\delta x/m \\ (n,q')=1}} \frac{f(\delta n)}{\delta n} = \delta \sum_{\substack{\delta x/\lambda<n'\leq\delta x/m \\ (n',q)=\delta}} \frac{f(n')}{n'},$$

for the integers $n'$ which satisfy $(n', q) = \delta$ are the integers $\delta n$ where $(n, q') = 1$.

It follows, by the remark of §3.2, that for each $m$, as $x$ tends to infinity,

$$\sum_{x/\lambda < n \le x/m} \frac{\chi_{q'}(n) f(\delta n)}{n} \text{ tends to } \delta m_\delta \log \frac{\lambda}{m}.$$

Therefore, by (8), as $x$ tends to infinity we have

$$\sum_2 \text{ tends to } \delta m_\delta \sum_{m<\lambda} \frac{\mu(m)\chi_{q'}(m)}{m} \log \frac{\lambda}{m} = g_2(\lambda), \text{ say.}$$

Also

$$g_2(\lambda) \text{ tends to } \frac{q'}{\varphi(q')} \delta m_\delta$$

as $\lambda$ tends to infinity, for

$$g_2(\lambda) = \delta m_\delta \left\{ \left( \sum_{m<\lambda} \frac{\mu(m)\chi_{q'}(m)}{m} \right) \log \lambda - \sum_{m<\lambda} \frac{\mu(m)\chi_{q'}(m) \log m}{m} \right\}$$

and, by the lemma of §2,

$$\sum_{m<\lambda} \frac{\mu(m)\chi_{q'}(m)}{m} = O\left(e^{-\alpha\sqrt{\log \lambda}}\right) \quad \text{where } \alpha > 0,$$

and

$$\sum_{m<\lambda} \frac{\mu(m)\chi_{q'}(m) \log m}{m} \text{ tends to } -\frac{q'}{\varphi(q')}.$$

**3.5.** By (6) and (7) we have

$$\left|G_\delta(x) - \frac{q'}{\varphi(q')}\delta m_\delta\right| \le g_1(\lambda) + \left|\sum_2 - g_2(\lambda)\right| + \left|g_2(\lambda) - \frac{q'}{\varphi(q')}\delta m_\delta\right|.$$

As $\Sigma_2$ tends to $g_2(\lambda)$ as $x$ tends to infinity this gives

$$\limsup_{x\to\infty} \left|G_\delta(x) - \frac{q'}{\varphi(q')}\delta m_\delta\right| \le g_1(\lambda) + \left|g_2(\lambda) - \frac{q'}{\varphi(q')}\delta m_\delta\right|.$$

This holds for every $\lambda \ge e^{1/4}$. Since the right-hand side tends to zero as $\lambda$ tends to infinity, this shows that

$$G_\delta(x) \text{ tends to } \frac{q'}{\varphi(q')}\delta m_\delta$$

as $x$ tends to infinity. It follows by (5) that the series $\sum_{n=1}^{\infty} f'(qn)/qn$ converges and that

$$\sum_{n=1}^{\infty} \frac{f'(qn)}{qn} = \frac{1}{q} \sum_{\delta q'=q} \frac{q'\mu(q')}{\varphi(q')} \delta m_\delta = \sum_{\delta q'=q} \frac{\mu(q')}{\varphi(q')} m_\delta .$$

**3.6.** To complete the proof of our theorem it remains to show that

$$\lim_{x\to\infty} \frac{1}{x} \sum_{n\le x} f(n) c_q(n) = \varphi(q) \sum_{\delta q'=q} \frac{\mu(q')}{\varphi(q')} m_\delta . \tag{9}$$

(We already know by the remark of §1.5 that the limit exists).

**3.6.1.** We have

$$\begin{aligned}
\frac{1}{x} \sum_{n\le x} f(n) c_q(n) &= \frac{1}{x} \sum_{n\le x} f(n) \Bigg( \sum_{d/q,\, n)} d\mu\Big(\frac{q}{d}\Big)\Bigg) \\
&= \frac{1}{x} \sum_{\substack{n\le x \\ d/(q,n)}} f(n)\, d\mu\Big(\frac{q}{d}\Big) \\
&= \frac{1}{x} \sum_{\delta/q} \Bigg( \sum_{\substack{n\le x \\ (q,n)=\delta \\ d/\delta}} f(n)\, d\mu\Big(\frac{q}{d}\Big) \Bigg) \\
&= \sum_{\delta/q} \Bigg\{ \Bigg( \sum_{d/\delta} d\mu\Big(\frac{q}{d}\Big)\Bigg) \Bigg( \frac{1}{x} \sum_{\substack{n\le x \\ (q,n)=\delta}} f(n) \Bigg) \Bigg\}.
\end{aligned}$$

This shows that

$$\lim_{x\to\infty} \frac{1}{x} \sum_{n\le x} f(n) c_q(n) = \sum_{\delta/q} \Bigg( \sum_{d/\delta} d\mu\Big(\frac{q}{d}\Big)\Bigg) m_\delta .$$

**3.6.2.** To obtain (9) it suffices to show that, for each divisor $\delta$ of $q$,

$$\sum_{d/\delta} d\mu\Big(\frac{q}{d}\Big) = \varphi(q) \frac{\mu(q')}{\varphi(q')} \quad \text{where } q' = \frac{q}{\delta}. \tag{10}$$

We have

$$\mu\Big(\frac{q}{d}\Big) = \mu\Big(\frac{\delta q'}{d}\Big) = \mu\Big(\frac{\delta}{d}\Big) \mu(q') \chi_{q'}\Big(\frac{\delta}{d}\Big).$$

So

$$\sum_{d/\delta} d\mu\left(\frac{q}{d}\right) = \mu(q') \sum_{d/\delta} d\mu\left(\frac{\delta}{d}\right)\chi_{q'}\left(\frac{\delta}{d}\right).$$

Let $h = i_*(\mu\chi_{q'})$, where $i(n) = n$ for every $n$. We have

$$\sum_{d/\delta} d\mu\left(\frac{\delta}{d}\right)\chi_{q'}\left(\frac{\delta}{d}\right) = h(\delta).$$

$h$ is multiplicative and, for $p$ prime and $r \geq 1$,

$$h(p^r) = p^r - p^{r-1}\chi_{q'}(p) = \begin{cases} p^r & \text{if } p/q', \\ p^r\left(1 - \frac{1}{p}\right) & \text{if } p \nmid q'. \end{cases}$$

It follows that

$$\sum_{d/\delta} d\mu\left(\frac{\delta}{d}\right)\chi_{q'}\left(\frac{\delta}{d}\right) = \delta \prod_{\substack{p/\delta \\ p \nmid q'}} \left(1 - \frac{1}{p}\right),$$

so that

$$\sum_{d/\delta} d\mu\left(\frac{q}{d}\right) = \mu(q')\delta \prod_{\substack{p/\delta \\ p \nmid q'}} \left(1 - \frac{1}{p}\right). \tag{11}$$

On the other hand we have

$$\varphi(q) = q\prod_{p/q}\left(1 - \frac{1}{p}\right) = \delta q'\left\{\prod_{p/q'}\left(1 - \frac{1}{p}\right)\right\}\left\{\prod_{\substack{p/\delta \\ p \nmid q'}}\left(1 - \frac{1}{p}\right)\right\}$$

$$= \delta\varphi(q') \prod_{\substack{p/\delta \\ p \nmid q'}} \left(1 - \frac{1}{p}\right).$$

This with (11) gives (10).

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