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AN ALGEBRAIC FORMULA FOR THE GYSIN HOMOMORPHISM FROM G/B TO G/P

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The purpose of this note is to prove an algebraic formula for the Gysin homomorphism

$$\pi_* : H^\cdot(G/B; \mathbf{C}) \to H^\cdot(G/P; \mathbf{C})$$

associated to the canonical map $\pi : G/B \to G/P$ from a flag manifold to an algebraic homogeneous space. In addition, we obtain a formula for

$$\int_X : H^{2n}(X; \mathbf{C}) \to \mathbf{C}$$

when X is a homogeneous space of dimension n. Our formula shows π_* is a sum of residues, a consequence of the fact that one can compute the Gysin homomorphism f_* associated to a smooth connected surjective morphism $f: X \to Y$ between compact algebraic manifolds as a sum of residues on the zeros of a holomorphic vector field V on X with isolated zeros.

1. Statement of the main results

We begin with a list of notation:

G	a semi simple linear algebraic group
B	a Borel subgroup of G
H	a maximal torus in B with group of characters $X(H)$.
$\mathfrak{h} \subset \mathfrak{g}$	The Lie algebras of H and G
Δ	the root system of H in G, $\Delta \subset \mathfrak{h}^*$
Δ_+	the positive roots associated to B
Σ	a basis of Δ
Δ_θ	the set of positive roots that are linear combinations of roots in $\theta \subset \Sigma$.

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P_θ the parabolic containing B associated to $\theta \subset \Sigma$.
W the Weyl group of H in $G: W = N(H)/H$
W_θ the subgroup of W generated by all reflections σ_α $(\alpha \in \theta)$.

Suppose $R = \mathrm{Sym}(\mathfrak{h}^*)$ with the canonical W-action $w \cdot f(v) = f(w^{-1} \cdot v)$, where $w \cdot v$ denotes the usual action of W on $\mathfrak{h}$. A classical result of Borel says that the assignment

$$\alpha \in X(H) \to c_1(L_\alpha) \in H^2(G/B; \mathbf{C}),$$

where L_α is the line bundle on G/B canonically associated to α, induces algebra homomorphisms

$$\phi : R \to H^{\cdot}(G/B; \mathbf{C}) \quad \text{and} \quad \phi_\theta : R^{W_\theta} \to H^{\cdot}(G/P_\theta; \mathbf{C})$$

for any $\theta \subset \Sigma$ which induce ring isomorphisms

$$\Phi : R/RR_+^W \to H^{\cdot}(G/B; \mathbf{C}) \quad \text{and} \quad \Phi_\theta : R^{W_\theta}/R^{W_\theta}R_+^W \to H(G/P_\theta; \mathbf{C}).$$

$\mathcal{G} \subseteq W$ and $R_+^{\mathcal{G}} = \{ f \in R^{\mathcal{G}} | f(0) = 0\}$ where $R^{\mathcal{G}}$ is the ring of invariants of $\mathcal{G}$. Let d_θ denote $\prod_{\alpha \in \Delta_\theta} \alpha$, and R_{d_θ} denote the localization of R at d_θ. Consider the (symmetrization) operator $M_\theta : R \to R_{d_\theta}$ where

$$M_\theta(f) = \sum_{q \in W_\theta} \frac{\det(q) q \cdot f}{d_\theta}$$

We shall prove:

THEOREM 1. *In fact* (a) $M_\theta(R) \subseteq R^{W_\theta}$, *and* (b) *the diagram*

$$\begin{array}{ccc} R & \xrightarrow{\phi} & H^{\cdot}(G/B; \mathbf{C}) \\ {\scriptstyle M_\theta} \downarrow & & \downarrow {\scriptstyle \pi_*} \\ R^{W_\theta} & \xrightarrow{\phi_\theta} & H^{\cdot}(G/P_\theta; \mathbf{C}) \end{array} \tag{1}$$

commutes, where π_* *denotes the Gysin homomorphism. Consequently, the induced map* $\tilde{M}_\theta : R/RR_+^W \to R^{W_\theta}/R^{W_\theta}R_+^W$ *is the Gysin homomorphism.*

$\tilde{M}_\theta$ is well defined because M_θ is R^{W_θ} (hence R^W)-linear.
We also obtain the following integral formula for G/P.

THEOREM 2. *Let P be a parabolic in G, say $P = P_\theta$. Then for all $f \in R^{W_\theta}$,*

$$\int_{G/P} \Phi_\theta(\bar{f}) = \sum_{q \in W_\theta^\perp} \frac{\det(q) q \cdot f}{\prod_{\alpha \in \Delta \setminus \Delta_\theta} \alpha}$$

where $W_\theta^\perp$ denotes the set of all $w \in W$ so that $w \cdot \theta \subset \Delta_+$. In particular, if $\bar{f} = 0$ in $(R^{W_\theta}/R^{W_\theta}R_+^W)_m$, the residue classes of degree $m = \dim G/P$, for example if f is homogeneous of degree different from m, then

$$\sum_{q \in W_\theta^\perp} \frac{\det(q) q \cdot f}{\prod_{\alpha \in \Delta \setminus \Delta_\theta} \alpha} = 0$$

Similarly, one can give an explicit formula for

$$\pi_* : H^\cdot(G/P_{\theta'}; \mathbf{C}) \to H^\cdot(G/P_\theta; \mathbf{C})$$

for $\theta \subset \theta'$.

2. The Gysin homomorphism and zeros of holomorphic vector fields [3]

Assume that X is a compact algebraic manifold of dimension n with a holomorphic vector field V having isolated zeros Z. Consider Z as the variety with structure sheaf $\mathcal{O}_Z = \mathcal{O}_X / i(V)\Omega_X^1$. Recall that by [5] there exists a filtration

$$H^0(X; \mathcal{O}_Z) = F_n(V) \supset F_{n-1}(V) \supset \cdots \supset F_1(V) \supset F_0(V)$$

such that $F_i(V)F_j(V) \subset F_{i+j}(V)$ and graded ring isomorphisms

$$H^\cdot(X; \mathbf{C}) = \bigoplus H^{2i}(X; \mathbf{C}) \cong \bigoplus F_i(V)/F_{i+1}(V)$$

Now suppose $f: X \to Y$ is a smooth, connected, surjective morphism of relative dimension r between compact algebraic manifolds. Then there exists a unique holomorphic vector field V_* on Y so that $f_* V = V_*$. Moreover, V_* has isolated zeros and zero $(V_*) = f(Z)$. The main result of [3] is the definition of a residue morphism

$$\operatorname{Res}(V) = \operatorname{Res}_{X|Y}(V) : H^0(X; \mathcal{O}_Z) \to H^0(Y; \mathcal{O}_{f(Z)}),$$

where

$$\mathcal{O}_{f(Z)} = \mathcal{O}_Y / i(V_*)\Omega_Y^1,$$

having the properties that (a) $\mathrm{Res}(V)(F_p(V)) \subset F_{p-r}(V_*)$ and (b) the diagram

$$\begin{array}{ccc} F_p(V)/F_{p-1}(V) & \xrightarrow{\sim} & H^{2p}(X;\mathbf{C}) \\ \Big\downarrow{\scriptstyle \mathrm{Res}(V)} & & \Big\downarrow{\scriptstyle f_*} \\ F_{p-r}(V_*)/F_{p-r-1}(V_*) & \xrightarrow{\sim} & H^{2(p-r)}(Y;\mathbf{C}) \end{array} \tag{2}$$

is commutative. This amounts to the computation of f_* on Z.

We will now compute $\mathrm{Res}(V)$ when V has only simple zeros. Note first that V is tangent to the fibre $X_{\tilde{\zeta}}$ of f over a zero $\tilde{\zeta}$ of V_*. Letting $\tilde{V}$ denote the restriction of V to $X_{\tilde{\zeta}}$ note also that the Lie derivative $L(\tilde{V})_\zeta : T_\zeta(X_{\tilde{\zeta}}) \to T_\zeta(X_{\tilde{\zeta}})$ is an isomorphism for any zero ζ of $\tilde{V}$ since $L(\tilde{V})_\zeta = L(V)_\zeta | T_\zeta(X_{\tilde{\zeta}})$ and $L(V)_\zeta : T_\zeta(X) \to T_\zeta(X)$ is an isomorphism, ζ being a simple zero of V.

Since V and, consequently, V_* have only simple zeros and $f(Z) = \mathrm{zero}(V_*)$, it follows easily that

$$H^0(X;\mathcal{O}_Z) \cong \mathbf{C}^Z \quad \text{and} \quad H^0(Y;\mathcal{O}_{f(Z)}) \cong \mathbf{C}^{f(Z)},$$

where $\mathbf{C}^A$ denotes the ring of complex valued functions on A. Thus we need to compute $\mathrm{Res}(V)(s)(\tilde{\zeta})$ for any $s \in H^0(X;\mathcal{O}_Z)$ and $\tilde{\zeta} \in f(Z)$.

PROPOSITION 1. *With the above notation*,

$$\mathrm{Res}(V)(s)(\tilde{\zeta}) = \sum_{\zeta \in f^{-1}(\tilde{\zeta}) \cap Z} \frac{s(\zeta)}{\det L(\tilde{V})_\zeta}$$

Proof. By assumption, X has local coordinates (u, v) near any $\zeta \in f^{-1}(\tilde{\zeta}) \cap Z$ so that $\zeta = (0,0)$ and $f(u, v) = u$. In terms of these coordinates,

$$V = \sum_{i=1}^{n-r} a_i(u)\frac{\partial}{\partial u_i} + \sum_{j=1}^{r} b_j(u,v)\frac{\partial}{\partial v_j}$$

and

$$\tilde{V} = \sum_{j=1}^{r} b_j(0,v)\frac{\partial}{\partial v_j}$$

Now by definition of $\mathrm{Res}(V)$ (cf. [3]),

$$\mathrm{Res}(V)(s)(\tilde{\zeta}) = \sum_{\zeta \in f^{-1}(\tilde{\zeta}) \cap Z} \mathrm{Res}\begin{bmatrix} s(\zeta)\, dv_1 \wedge \cdots \wedge dv_r \\ b_1(0,v), \ldots, b_r(0,v) \end{bmatrix}_{v=0}.$$

Writing

$$b_i(0, v) = \sum_{j=1}^{r} \lambda_{ij} v_j + \text{higher order terms}$$

and using the fact that $\det(\lambda_{ij}) \neq 0$ and the standard algorithm for computing Grothendieck residues, it follows that

$$\operatorname{Res}(V)(s)(\tilde{\zeta}) = \sum_{\zeta} \frac{s(\zeta)}{\det(\lambda_{ij})} = \sum_{\zeta} \frac{s(\zeta)}{\det L(\tilde{V})_{\zeta}}$$

which proves the proposition.

3. The residue morphism on G/B

Recall that a vector $v \in \mathfrak{h}$ is said to be regular if $(G/B)^{\exp(tv)} = (G/B)^H$. Equivalently, v is regular if $\alpha(v) \neq 0$ for any $\alpha \in \Delta$. If v is regular, the holomorphic vector field V on G/B generated by v has zero set $Z = \{w \cdot B \mid w \in W\} \cong W$, where $w \cdot B$ denotes the Borel gBg^{-1} where $g \in N(H)$ is a representative of ω. Moreover, all zeros of V are simple. Thus $H^0(G/B; \mathcal{O}_Z) \cong \mathbf{C}^W$. Let $P_\theta \supset B$ denote the standard parabolic corresponding to $\theta \subset \Sigma$ and V_* the vector field $\pi_* V$ on G/P_θ where $\pi: G/B \to G/P_\theta$ is the natural map. Clearly,

$$H^0(G/P_\theta; \mathcal{O}_{\pi(Z)}) \cong \mathbf{C}^{W/W_\theta},$$

due to the fact that $\pi: Z \to \operatorname{zero}(\pi_* V)$ induces the natural map $W \to W/W_\theta$.

We will first compute $\operatorname{Res}(V)$.

PROPOSITION 2. *For any $s \in \mathbf{C}^W$ and $\bar{w} \in W/W_\theta$,*

$$\operatorname{Res}(V)(s)(\bar{w}) = \sum_{q \in W_\theta} \frac{\det(q) s(wq)}{\prod_{\alpha \in -\Delta_\theta} (w \cdot \alpha(v))}$$

Proof. By Proposition 1, it suffices to show that for any $q \in W_\theta$,

$$\det L(\tilde{V})_{wq} = \frac{1}{\det(q)} \Big(\prod_{\alpha \in -\Delta_\theta} w \cdot \alpha(v) \Big)$$

where $\tilde{V}$ denotes the restriction of V to $\pi^{-1}(wP_\theta) = w \cdot P_\theta / B$.

LEMMA. *The set of eigenvalues of* $L(\tilde{V})_{wq}$ *is* $\{wq \cdot \alpha(v) | \alpha \in -\Delta_\theta\}$.

Proof. It suffices to consider the case where $G = w \cdot P_\theta$ and to show that the eigenvalues for V at $w \in W$ are of the form $(w \cdot \alpha)(v)$ where $\alpha \in -\Delta_+$. Since $(w \cdot \alpha)(v) = \alpha(w^{-1} \cdot v)$, it even suffices to show that the set of eigenvalues of V at $e = B$ is $\{\alpha(v) | \alpha \in -\Delta_+\}$. To do this, recall that the natural map $G \to G/B$ induces an isomorphism

$$T_B(G/B) \cong \bigoplus_{\alpha \in -\Delta_+} \mathfrak{g}_\alpha$$

where $\mathfrak{g}_\alpha$ is the tangent space at e to the unique 1-parameter subgroup $u_\alpha : \mathbf{C} \to \mathfrak{g}$ such that $hu_\alpha(z)h^{-1} = u_\alpha(\alpha(h)z)$ for $h \in H$ and $z \in \mathbf{C}$. Let $y_\alpha \in \mathfrak{g}_\alpha$ denote

$$\left.\frac{d}{dz}\right|_{z=0} u_\alpha(z).$$

Then

$$\begin{aligned}
[V, y_\alpha] &= \left.\frac{d}{dt}\right|_{t=0} \mathrm{Ad}(\exp tv)\, y_\alpha \\
&= \left.\frac{d}{dt}\right|_{t=0} \left.\frac{d}{dz}\right|_{z=0} (\exp tv)\, u_\alpha(z)(\exp(-tv)) \\
&= \left.\frac{d}{dt}\right|_{t=0} \left.\frac{d}{dz}\right|_{z=0} u_\alpha(t\alpha(v)z) \\
&= \alpha(v)\, y_\alpha
\end{aligned}$$

This completes the proof of the lemma.

Hence,

$$\det L(\tilde{V})_{wq} = \prod_{\alpha \in -\Delta_\theta} wq \cdot \alpha(v) = \det(q) \prod_{\alpha \in -\Delta_\theta} w \cdot \alpha(v).$$

Since $\det(q) = \pm 1$, we are done with the proof of Proposition 2.

4. The proofs

We will begin by proving Theorem 1, employing induction on degree. Thus, let $R_p = \{f \in R | f \text{ is homogeneous of degree } p\}$. Since $M_\theta(1) = 0$ (it is well known that for a finite reflection group G, $\Sigma_{q \in G} \det(q) = 0$) the result is trivial for $p = 0$. Letting $n = \dim G/B$ and $l = \dim P_\theta/B$, we assume that the theorem holds when $f \in R_{p-1}$ where $p \leq n$. Since

$$H^0(G/P_\theta; \mathcal{O}_{\pi(Z)}) \cong \mathbf{C}^{W/W_\theta},$$

which is induced from the algebra homomorphism $\Psi : R \to \mathbf{C}^W$ determined by the condition that

$$\Psi(\alpha)(w) = -w \cdot \alpha(v)$$

for all $\alpha \in \mathfrak{h}^*$, where v is the regular element of $\mathfrak{h}$ generating V. By [1], Ψ_θ induces an isomorphism

$$\overline{\Psi}_\theta : R_p^{W_\theta}/(R^{W_\theta}R_0^W)_p \to F_p(\pi_* V)/F_{p-1}(\pi_* V)$$

and by (2), there exists a unique map M_θ^* such that the following diagram commutes:

$$\text{(3)} \qquad \begin{array}{ccc} R_p/(RR_0^W)_p & \xrightarrow{\quad\overline{\Psi}\quad} & F_p(V)/F_{p-1}(V) \\ {\scriptstyle M_\theta^*}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{Res}(V)} \\ R_{p-l}^{W_\theta}/(R^{W_\theta}R_+^W)_{p-l} & \xrightarrow{\quad\overline{\Psi}_\theta\quad} & F_{p-l}(\pi_*V)/F_{p-l-1}(\pi_*V) \end{array}$$

Now suppose that $f \in R_p$. Then

$$\begin{aligned} \mathrm{Res}(V)\overline{\Psi}(f)(\overline{w}) &= \sum_{q \in W_\theta} \frac{\det(q)\overline{\Psi}(f)(wq)}{\prod_{\alpha \in -\Delta_\theta} w \cdot \alpha(v)} \\ &= (-1)^p \sum_{q \in W_\theta} \frac{\det(q)(wq \cdot f)(v)}{\prod_{\alpha \in \Delta_\theta} (-w \cdot \alpha(v))} \\ &= (-1)^{p-l} \frac{1}{d_\theta(w^{-1} \cdot v)} \left(\sum_{q \in W_\theta} \det(q) q \cdot f \right)(w^{-1} \cdot v). \end{aligned}$$

Since for $g \in R_{p-l}^{W_\theta}$, $\Psi_\theta(g)(\overline{w}) = (-1)^{p-l} g(w^{-1} \cdot v)$, it follows, by composing with $\overline{\Psi}_\theta^{-1}$ to get M_θ^*, that for each $f \in R_p$ there exists a $g \in (RR_+^W)_p$ so that

$$M_\theta^*(f+g) = \sum_{q \in W_\theta} \frac{\det(q) q \cdot (f+g)}{d_\theta} = M_\theta(f+g)$$

is a polynomial. However, by the fact that M_θ is R^W-linear, the induction assumption implies $M_\theta(g) \in R^{W_\theta}$. Consequently, $M_\theta(f) \in R^{W_\theta}$ and part (a) is verified. But part (b) now follows immediately from the above computation, in view of the commutativity of (2) and (3).

We will next prove Theorem 2. First consider the commutative diagram

$$\begin{array}{ccc} H^{2n}(G/B;\mathbf{C}) & \xrightarrow{\pi_*} & H^{2(n-l)}(G/P;\mathbf{C}) \\ & \int_{G/B} \searrow \quad \swarrow \int_{G/P} & \\ & \mathbf{C} & \end{array}$$

Now by Theorem 1, $\int_{G/B}$ is the map

$$M_\Sigma(f) = \sum_{w \in W} \frac{\det(w) w \cdot f}{\prod_{\alpha \in \Delta_+} \alpha}$$

while π_* is given by $M_\theta(f)$. Since π_* is surjective, the result follows from the fact that

$$\sum_{r \in W_\theta^\perp} \frac{\det(r) r}{\prod_{\alpha \in \Delta_+ \setminus \Delta_\theta} \alpha} (M_\theta(f)) = \sum_{r \in W_\theta^\perp} \sum_{q \in W_\theta} \frac{\det(rq) rq}{\prod_{\alpha \in \Delta_+} \alpha} = M_\Sigma(f)$$

since every $w \in W$ has a unique decomposition $w = rq$ where $r \in W_\theta^\perp$ and $q \in W_\theta$.

5. Closing remarks

1. In fact, one can prove somewhat more about the Gysin homomorphism. Namely, if τ denotes the element of W_θ of maximal length, then $M_\theta = A_\tau$, where, for any $w \in W$, A_w denotes the differential operator employed in [4] and [6] to study $H^\cdot(G/B; C)$. In case $\theta = \Sigma$, this assertion is *Lemmé* 4 of [6]. Similarly, if w_0 is the element of W of maximal length, then $\int_{G/P_\theta}$ is A_{w_1}, where $w_1 = w_0\tau$. This is due to the fact that $A_{w_0} = A_{w_1}A_\tau$ since $w_0 = w_1\tau$ is a reduced expression for w_0.

2. In the case of $G = SL_n$, B is the group of upper triangular matrices in G, and P the parabolic subgroup of G containing B such that G/P is the Grassmann manifold $G_{k,n}$ of k-planes in $\mathbf{C}^n$. One can use the Gysin homomorphism of $\pi: SL_n/B \to G_{k,n}$ to give a direct derivation of the basic theorems on the structure of $H^\cdot(G_{k,n})$ (i.e., the basis theorem, Pieri's formula, and the determinantal formula). This approach has been carried out in [2].

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