ILLINOIS JOURNAL OF MATHEMATICS
Volume 31 Number 2, Summer 1987

THE GROTHENDIECK GROUP OF A CLASSICAL ORDER OF FINITE LATTICE TYPE

BY

ALFRED WIEDEMANN

Let R be a complete Dedekind domain with quotient field K, and let Λ be an R-order in the separable K-algebra $A = K \otimes_R \Lambda$. *We assume throughout that* Λ *is of finite lattice type*, that is Λ has up to isomorphism only finitely many indecomposable lattices. In this note we provide an *explicit formula for the Grothendieck group* $G_0(\Lambda)$ *of* Λ and point out some first applications. The investigations leading to this result were initiated by some discussions with Sibylle Langkopf and K.W. Roggenkamp on the structure of the Grothendieck group of a Schurian order [8]. I am especially grateful to S. Langkopf for computing the Grothendieck groups of many Schurian orders of finite lattice type and pointing out that they all turned out to be torsionfree; a result which holds in general, as we shall see in (4.2).

1. Notations and the theorem

(1.1) Firstly, A being separable, we have a decomposition

$$A \cong \prod_{\nu=1}^{\sigma} (D_\nu)_{s_\nu},$$

of A into simple factors $(D_\nu)_{s_\nu}$ for $s_\nu \in \mathbf{N}$ and D_ν a finite dimensional skew field over K. We denote by Ω_ν the unique maximal R-order in D_ν and by Π_ν its radical. Moreover, we choose a maximal order $\Gamma = \prod_{\nu=1}^{\sigma}\Gamma_\nu$ in A containing Λ. Let L_ν be the indecomposable Γ_ν-lattice, and put $V_\nu = L_\nu/\Pi_\nu L_\nu$, $\nu = 1,\dots,\sigma$.

(1.2) Secondly, let $S_1,\dots,S_e$ be the non-isomorphic simple Λ-modules and let $P_1,\dots,P_e$ be their projective covers. Then let $v_{i\nu}$ be the multiplicity of S_i in a composition series of V_ν viewed as a Λ-module. With these multiplicities we form the integral vectors

$$v_\nu = (v_{1\nu}, v_{2\nu},\dots,v_{e\nu}) \in \mathbf{Z}^{(e)}, \quad \nu = 1,\dots,\sigma$$

and define U as subgroup of $\mathbf{Z}^{(e)}$ generated by the vectors $v_1,\dots,v_\sigma$.

Received March 14, 1985.

(1.3) THEOREM. *The Grothendieck group $G_0(\Lambda)$ is isomorphic to the direct sum $\mathbf{Z}^{(\sigma)} \oplus \mathbf{Z}^{(e)}/U$.*

Remark. In [2], M. Auslander and I. Reiten consider a commutative diagram of abelian groups involving an exact sequence isomorphic to

$$\mathbf{Z}^{(\sigma)} \to \mathbf{Z}^{(e)} \to G_0(\Lambda) \to \mathbf{Z}^{(\sigma)} \to 0,$$

showing that $G_0(\Lambda) \cong \mathbf{Z}^{(\sigma)} \oplus \mathbf{Z}^{(e)}/H$, H a subgroup of $\mathbf{Z}^{(e)}$ generated by σ elements.

(1.4) Obviously, the formula in the theorem is additive with respect to taking direct products of R-orders and also holds for a hereditary R-order Λ in a simple K-algebra, namely in this case we have $\sigma = 1$ and $U = \mathbf{Z}(1,\dots,1) \leq \mathbf{Z}^{(e)}$, and clearly

$$\mathbf{Z} \oplus \left(\mathbf{Z}^{(e)}/\mathbf{Z}(1,\dots,1)\right) \cong \mathbf{Z}^{(e)} \cong G_0(\Lambda).$$

Therefore, we assume from now on that Λ is two sided indecomposable, non-hereditary and has—up to isomorphism—n indecomposable left lattices which we number by $M_1 = P_1, \dots, M_e = P_e, M_{e+1}, \dots, M_n$.

(1.5) We denote by $K_0(\text{mod}\,\Lambda, 0)$ the quotient of the abelian group freely generated by all the isomorphism classes $[M]$ for M a Λ-lattice, modulo the subgroup generated by all relations of the form $[M] + [M'] - [M \oplus M']$. Then $K_0(\text{mod}\,\Lambda, 0)$ is free and may be identified with the free abelian group having $[M_i]$, $i = 1, \dots, n$ as generators. Recall that the Grothendieck group $G_0(\Lambda)$ of Λ is per definition the quotient of $K_0(\text{mod}\,\Lambda, 0)$ modulo the subgroup generated by all the relations of the form $[M'] + [M''] - [M]$ for each short exact sequence $0 \to M' \to M \to M'' \to 0$ of Λ-lattices [5]. Note that $G_0(\Lambda) = K_0(\text{mod}\,\Lambda)$ in the notation of [2].

(1.6) For each $i = e + 1, \dots, n$ let

$$0 \to N_i \to E_i \to M_i \to 0$$

be the almost split sequence of M_i [1] giving rise to the relation

$$\mu_i = [M_i] + [N_i] - [E_i] \quad \text{in } K_0\,(\text{mod}\,\Lambda, 0).$$

Since Λ is of finite lattice type, we know by a result due to Auslander and Butler [2], [4] that $G_0(\Lambda)$ is the quotient of $K_0(\text{mod}\,\Lambda, 0)$ modulo the subgroup generated by all the relations μ_i:

$$G_0(\Lambda) \cong K_0(\text{mod}\,\Lambda, 0)/\langle \mu_i,\ i = e + 1, \dots, n\rangle.$$

Additionally for $i = 1, \ldots, e$ we put

$$\rho_i = [P_i] - [\mathrm{rad}_\Lambda P_i].$$

2. Review of the Igusa-Todorov algorithm for orders

In this section we recall the procedure and one of the main results of the integral version [10] of the algorithm invented by Igusa and Todorov for the computation of the preprojective partition of a representation finite artin algebra [3], [6].

(2.1) We start with the integral $(n + e) \times n$-matrix

$$\mathbf{G}^0 = \left(g_{ji}^0\right)_{1 \le j \le n+e,\, 1 \le i \le n},$$

where the integers g_{ji}^0 are defined through the following equations in $K_0(\mathrm{mod}\,\Lambda, 0)$:

$$\rho_i = \sum_{j=1}^{n} g_{ji}^0 [M_j] \quad \text{for } 1 \le i \le e,$$

$$\mu_i = \sum_{j=1}^{n} g_{ji}^0 [M_j] \quad \text{for } e + 1 \le i \le n,$$

and

$$g_{n+k,l}^0 = \begin{cases} 1 & \text{if } k = l \\ 0 & \text{if } k \ne l \end{cases} \qquad \text{for } 1 \le k \le e,\ 1 \le l \le n.$$

Then $\mathbf{G}^0$ has the form

$$\mathbf{G}^0 = \begin{array}{c} \\ P_1 \\ \vdots \\ P_e \\ M_{e+1} \\ \vdots \\ M_n \\ n+1 \\ \vdots \\ n+e \end{array} \begin{array}{c} P_1 \cdots P_e \quad M_{e+1} \cdots M_n \\ \left[\begin{array}{c|c} & \\ * & * \\ & \\ \hline \begin{matrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{matrix} & 0 \end{array}\right] \end{array}$$

Since $K_0(\mathrm{mod}\ \Lambda, 0)$ is free abelian on the generators $[M_i]$, $i = 1, \ldots, n$ we may identify $K_0(\mathrm{mod}\ \Lambda, 0)$ with the space of integral columns of length n by assigning

$$[M_i] \mapsto \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \ldots i, \quad i = 1, \ldots, n.$$

Via this identification we consider the columns of the upper $n \times n$-part of the following integral $(n+e) \times n$-matrices $\mathbf{G}^{\iota}$, $\iota = 0, 1, \ldots, m, *$ as elements of $K_0(\mathrm{mod}\ \Lambda, 0)$.

The following facts will be important for us.

(2.2) Performing column operations with the first e columns, that is adding to each column multiples of the first e columns, we reach in a first step the matrix $\mathbf{G}^1 = (g^1_{ji})$ of the form

$$\mathbf{G}^1 = \begin{array}{c} \\ P_1 \\ \vdots \\ P_e \\ M_{e+1} \\ \vdots \\ M_n \\ n+1 \\ \vdots \\ n+e \end{array} \begin{array}{c} \begin{array}{cc} P_1 \ldots P_e & M_{e+1} \ldots M_n \end{array} \\ \left[\begin{array}{c|c} \begin{matrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{matrix} & 0 \\ \hline * & * \\ \hline * & * \end{array}\right] \end{array}$$

Note that we used here the hypothesis that Λ is indecomposable and non-hereditary.

(2.3) We now follow the procedure described in [10] and—using column operations only—finally reach after a suitable permutation of the columns and of the first n rows an integral matrix $\mathbf{G}^m$ of the form

$$\mathbf{G}^m = \left(\begin{array}{c|c} \begin{matrix}1 & & 0\\ & \ddots & \\ 0 & & 1\end{matrix} & 0 \\ \hline * & \begin{matrix} \boxed{0} & & & & & 0\\ & \ddots & & & & \\ & & \boxed{0} & & & \\ & & & \begin{array}{|cccc|}\hline 1 & & & -1\\ -1 & \ddots & & 0 \\ & \ddots & \ddots & \\ 0 & & -1 & 1\\ \hline\end{array} & & \\ & & & & \ddots & \\ 0 & & & & & \begin{array}{|cccc|}\hline 1 & & & -1\\ -1 & \ddots & & 0 \\ & \ddots & \ddots & \\ 0 & & -1 & 1\\ \hline\end{array} \end{matrix} \\ \hline * & * \end{array}\right)$$

There is a natural one to one correspondence between the diagonal blocks of type

$$\boxed{0} \qquad \text{or} \qquad \begin{array}{|cccc|}\hline 1 & & & -1\\ -1 & \ddots & & 0 \\ & \ddots & \ddots & \\ 0 & & -1 & 1\\ \hline\end{array}$$

in $\mathbf{G}^m$ and the simple factors of A induced by the fact that the lattices labelling the columns corresponding to each one of these blocks are exactly the indecomposable lattices of a hereditary R-order in the associated simple factor of A.

(2.4) By further obvious column operations we replace each block of type

$$\begin{array}{|cccc|}\hline 1 & & & -1\\ -1 & \ddots & & 0 \\ & \ddots & \ddots & \\ 0 & & -1 & 1\\ \hline\end{array}$$

in the diagonal by a block of the form

$$\boxed{\begin{matrix} 1 & & & \\ & \ddots & & 0 \\ 0 & & 1 & \\ -1 & & -1 & 0 \end{matrix}}$$

Thus, again after a suitable permutation of the rows and columns we finally reach the matrix $\mathbf{G}^* = (g^*_{ji})$ of the form

$$\mathbf{G}^* = \begin{array}{c} \\ \\ \\ L_1 \\ \vdots \\ L_\sigma \\ n+1 \\ \vdots \\ n+e \end{array} \begin{array}{c} \begin{array}{cc} \phantom{\begin{matrix} 1 & \ddots & 1 \end{matrix}} & L_1 \ldots L_\sigma \end{array} \\ \left(\begin{array}{ccc|ccc} 1 & & 0 & & & \\ & \ddots & & & 0 & \\ 0 & & 1 & & & \\ \hline & & & & & \\ & * & & & 0 & \\ & & & & & \\ \hline & & & v_{11} & \cdots & v_{1\sigma} \\ & * & & \vdots & & \vdots \\ & & & v_{e1} & \cdots & v_{e\sigma} \end{array}\right) \end{array}$$

where $L_1, \ldots, L_\sigma$ are the indecomposable Λ-lattices labelling now the last σ columns, and we have put $v_{i\nu} = g^*_{n+i,\, n-\sigma+\nu}$.

(2.5) Then by [10, Theorem 1] each one of the Λ-lattices L_ν, $\nu = 1, \ldots, \sigma$ is an indecomposable lattice over a maximal order Γ_ν in one of the simple factors of A: say $\Gamma_\nu \subset (D_\nu)_{s_\nu}$ with the notation of (1.1). Moreover, $v_{i\nu}$ is the multiplicity of S_i in a composition series of $L_\nu/\Pi_\nu L_\nu$ as a Λ-module.

(2.6) Note that if L'_ν is another irreducible Λ-lattice and a lattice over another maximal order Γ'_ν in $(D_\nu)_{s_\nu}$, then the multiplicity of S_i in the composition series both of $L_\nu/\Pi_\nu L_\nu$ and $L'_\nu/\Pi_\nu L'_\nu$ are the same. This easily follows from a theorem of Brauer, Swan, Strooker [7, VII 3.4.]. Therefore, the integral vectors v_ν, $\nu = 1, \ldots, \sigma$ indeed are independent of the particular choice of Γ in (1.1).

(2.7) Without loss of generality we assume from now on that the $L_\nu = M_{n-\sigma+\nu}$ for $\nu = 1, \ldots, \sigma$ are labelling the last σ columns of all the matrices $\mathbf{G}^\iota$. An analysis of the algorithm in [10] shows that each one of the columns of $\mathbf{G}^*$ labelled by L_ν is the sum of the column of $\mathbf{G}^0$ also labelled by L_ν and certain multiples of the first $n - \sigma$ columns of $\mathbf{G}^0$. The fact that the first n entries of each one of these last σ columns of $\mathbf{G}^*$ are zero and the remaining ones are $v_{1\nu}, \ldots, v_{e\nu}$ gives rise to the following equation in $K_0(\mathrm{mod}\ \Lambda, 0)$:

$$\sum_{i=1}^{e} v_{i\nu}\rho_i + \sum_{i=e+1}^{n-\sigma} w_{i\nu}\mu_i + \mu_{n-\sigma+\nu} = 0$$

for certain integers $w_{i\nu}$ and for all $\nu = 1, \ldots, \sigma$.

3. Proof of the theorem

Let

$$\langle \rho_i \rangle = \langle \rho_i;\ i = 1, \ldots, e \rangle, \quad \langle \mu_i \rangle = \langle \mu_i;\ i = e+1, \ldots, n \rangle$$

and

$$\langle \rho_i, \mu_i \rangle = \langle \rho_i \rangle + \langle \mu_i \rangle$$

be the subgroups of $K_0(\mathrm{mod}\ \Lambda, 0)$ being generated by the indicated elements in $K_0(\mathrm{mod}\ \Lambda, 0)$.

Since $G_0(\Lambda) \cong K_0(\mathrm{mod}\ \Lambda, 0)/\langle \mu_i \rangle$, we have the following exact sequence of abelian groups:

$$0 \to \frac{\langle \rho_i, \mu_i \rangle}{\langle \mu_i \rangle} \to G_0(\Lambda) \to \frac{K_0(\mathrm{mod}\ \Lambda, 0)}{\langle \rho_i, \mu_i \rangle} \to 0.$$

By the shape of $\mathbf{G}^*$ in (2.4) it is clear that $\langle \rho_i, \mu_i \rangle$ is a pure subgroup of $K_0(\mathrm{mod}\ \Lambda, 0)$ of rank $n - \sigma$. Therefore, $K_0(\mathrm{mod}\ \Lambda, 0)/\langle \rho_{\mathrm{i}}, \mu_{\mathrm{i}} \rangle \cong \mathbf{Z}^{(\sigma)}$, the above sequence splits, and we get

$$G_0(\Lambda) \cong \mathbf{Z}^{(\sigma)} \oplus \frac{\langle \rho_i \rangle}{\langle \rho_i \rangle \cap \langle \mu_i \rangle}.$$

Secondly, the shape of $\mathbf{G}^1$ in (2.2) shows that $\langle \rho_{\mathrm{i}} \rangle$ is free abelian on the generators $\rho_1, \ldots, \rho_e$. Now consider an element $x \in \langle \rho_i \rangle \cap \langle \mu_i \rangle$. There are integers $\alpha_i(x)$, $\beta_i(x)$ satisfying

$$x = \sum_{i=1}^{e} \alpha_i(x)\rho_i = \sum_{i=e+1}^{n} -\beta_i(x)\mu_i.$$

Therefore, the integral vector $(\alpha_1(x), \ldots, \alpha_e(x), \beta_{e+1}(x), \ldots, \beta_n(x))$ represents a solution of the linear equation

$$\sum_{i=1}^{e} Y_i\rho_i + \sum_{i=e+1}^{n} Y_i\mu_i = 0$$

in the variables $Y_1, \ldots, Y_n$. Since the columns ρ_i and μ_i form the upper $n \times n$-part of $\mathbf{G}^0$, and since this matrix has rank $n - \sigma$ the solution space of this equation is free abelian of rank σ.

On the other hand, the formula in (2.7) provides the σ linear independent solutions

$$\Big(v_{1\nu}, \ldots, v_{e\nu}, w_{e+1,\nu}, \ldots, w_{n-\sigma,\nu}, 0, \ldots, 0, \underset{\substack{\vdots \\ n-\sigma+\nu}}{1}, 0, \ldots, 0\Big), \qquad \nu = 1, \ldots, \sigma$$

which obviously form a $\mathbf{Z}$-basis of the solution space for the above equation. This shows that x is an integral linear combination of the σ elements $\sum_{i=1}^{e} v_{i\nu}\rho_i$, $\nu = 1, \ldots, \sigma$. If we now identify $\langle \rho_i \rangle$ with $\mathbf{Z}^{(e)}$ by

$$\rho_i \mapsto \Big(0, \ldots, 0, \underset{\substack{\vdots \\ i}}{1}, 0, \ldots, 0\Big) \in \mathbf{Z}^{(e)},$$

then $\langle \rho_i \rangle \cap \langle \mu_i \rangle$ is generated by the integral vectors

$$v_\nu = (v_{1\nu}, \ldots, v_{e\nu}) \in \mathbf{Z}^{(e)}.$$

Together with the observation in (2.6) this finishes the proof of the theorem.

4. Applications and a remark

We keep all the notation of the introduction and Section 1. In particular, Λ *will always be an R-order of finite lattice type.*

(4.1) First we consider the case when A is simple, that is $A = (D)_s$ for $s \in \mathbf{N}$ and D is a finite dimensional skew field over K with maximal R-order Ω and $\Pi = \operatorname{Rad} \Omega$. By the theorem we have

$$G_0(\Lambda) \cong \mathbf{Z} \oplus \mathbf{Z}^{(e)}/\mathbf{Z}v_1$$

where $v_1 = (v_{11}, \ldots, v_{e1}) \neq 0$ is defined as in (1.2). Hence

$$G_0(\Lambda) \cong \mathbf{Z}^{(e)} \oplus \mathbf{Z}/(\gcd(v_{11}, \ldots, v_{e1})).$$

In particular, *the torsion part of $G_0(\Lambda)$ is cyclic and $G_0(\Lambda)$ has rank e.* Since the $n - e$ almost split sequences provide the $n - e$ relations μ_i, $i = e + 1, \ldots, n$, this shows that *the relations μ_i are linearly independent in* $K_0(\operatorname{mod} \Lambda, 0)$. This is one of the main results in Chapter 2 of [2].

(4.2) We keep the hypotheses of (4.1). Let $P_1, \dots, P_e$ be the non-isomorphic indecomposable projective Λ-lattices. We now additionally assume that $\mathrm{End}_\Lambda(P_i)/\mathrm{Rad}\,\mathrm{End}_\Lambda(P_i)$ and Ω/Π are isomorphic for $i = 1, \dots, e$.

If T denotes the simple A-module and $KP_i \cong T^{(r_i)}$, $i = 1, \dots, e$, then

$$v_1 = (v_{11}, \dots, v_{e1}) = (r_1, \dots, r_e)$$

and therefore

$$G_0(\Lambda) \cong \mathbf{Z}^{(e)} \oplus \mathbf{Z}/(\gcd(r_1, \dots, r_e)).$$

In particular, if Λ is "tiled", that is, $\mathrm{End}_\Lambda(P_i) \cong \Omega$ for all i, *then* $G_0(\Lambda)$ *is free of rank* e.

(4.3) Let R be the p-adic completion of the algebraic integers in a number field. If Λ is a block of RG, the group ring of a finite group G with coefficients in R then a result due to Swan [9] says that

$$G_0(\Lambda) \cong G_0(A) \cong \mathbf{Z}^{(\sigma)}.$$

On the other hand $G_0(\Lambda) \cong \mathbf{Z}^{(\sigma)} \oplus \mathbf{Z}^{(e)}/\langle v_1, \dots, v_\sigma\rangle$. *Therefore,*

$$\mathbf{Z}^{(e)} = \langle v_1, \dots, v_\sigma\rangle;$$

in particular, $\sigma \geq e$, and *the greatest common divisor of the determinants of all regular* $e \times e$- *minors of the matrix*

$$(v_{i\nu})_{1 \leq i \leq e,\ 1 \leq \nu \leq \sigma}$$

as defined in (1.2) *is one.*

(4.4) If gldim $\Lambda < \infty$, then it is known that $G_0(\Lambda) \cong \mathbf{Z}^{(e)}$; in particular, this forces $\sigma \leq e$. In this case our theorem implies that

$$\mathbf{Z}^{(e)}/\langle v_1, \dots, v_\sigma\rangle \cong \mathbf{Z}^{(e-\sigma)},$$

and *therefore the* $v_1, \dots, v_\sigma$ *are free generators of a pure subgroup of* $\mathbf{Z}^{(e)}$.

(4.5) Using the whole information provided by the matrix $\mathbf{G}^*$, for each Λ-lattice M it is very easy to describe its class $[M]$ in $G_0(\Lambda)$ explicitly as element of $\mathbf{Z}^{(\sigma)} \oplus \mathbf{Z}^{(e)}/U$: Assume that $M \cong M_j$ is indecomposable and labels the j-th column of $\mathbf{G}^*$.

If $j \leq n - \sigma$, then $[M_j]$ is represented by the element

$$((g^*_{n-\sigma+1,j}, \dots, g^*_{nj}), (g^*_{n+1,j}, \dots, g^*_{n+e,j}) + U))$$

in $\mathbf{Z}^{(\sigma)} \oplus \mathbf{Z}^{(e)}/U$. Clearly, $[L_\nu]$ is represented by the element

$$\left(\left(0,\ldots,0,\underset{\overset{\vdots}{\nu}}{1},0,\ldots,0\right),0+U\right).$$

For $j \leq n-\sigma$, this follows immediately from the fact that the largest Γ-sublattice $\mathrm{tr}_\Gamma(M_j)$ of M_j is isomorphic to

$$\bigoplus_{\nu=1}^{\sigma} L_\nu^{(g^*_{n-\sigma+\nu,j})},$$

and the numbers $g^*_{n+1,j},\ldots,g^*_{n+e,j}$ are the multiplicities of $S_1,\ldots,S_e$ as composition factors of the factor module $M_j/\mathrm{tr}_\Gamma(M_j)$ [10, Section 4, (6)].

References

1. M. Auslander, "Existence theorems for almost split sequences" in *Ring Theory II*, Proc. Sec. Oklahoma Conf. 1975, Marcel Dekker, 1977, pp. 1–44.
2. M. Auslander and I. Reiten, *Grothendieck groups of algebras and orders*, J. Pure Appl. Algebra, vol. 39 (1986), pp. 1–51.
3. M. Auslander and S.O. Smalø, *Preprojective modules over artin algebras*, J. Algebra, vol. 66 (1980), pp. 61–122.
4. M.C.R. Butler, "Grothendieck groups and almost split sequences" in *Integral representations and applications*, Springer Lecture Notes in Math., vol. 882, 1981, pp. 357–368.
5. C. Curtis and I. Reiner, *Methods of representation theory*, Vol. I, Wiley, New York, 1981.
6. K. Igusa and G. Todorov, *Preprojective partitions and the determinant of the Hom matrix*, Proc. Amer. Math. Soc., vol. 94 (1985), pp. 189–197.
7. K.W. Roggenkamp, *Lattices over orders II*, Springer Lecture Notes in Math., vol. 142, 1970.
8. K.W. Roggenkamp and A. Wiedemann, *Auslander-Reiten quivers of Schurian orders*, Comm. Alg., vol. 12 (1984), pp. 2525–2578.
9. R.G. Swan, *The Grothendieck ring of a finite group*, Topology, vol. 2 (1963), pp. 85–110.
10. A. Wiedemann, *An integral version of the Igusa-Todorov algorithm*, J. London Math. Soc., (2), vol. 32 (1985), pp. 75–87.

Universität Stuttgart
Stuttgart, West Germany