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REAL ALGEBRAIC CURVES AND COMPLETE INTERSECTIONS

BY

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1. Introduction

In this paper real algebraic varieties and real algebraic morphisms are understood in the sense of Serre [10] (Serre considers algebraic varieties over an algebraically closed field but his basic definitions make sense over any field). In particular, we do not assume that algebraic varieties are irreducible. An algebraic variety $(X, \mathcal{O}_X)$ will be simply denoted by X if no confusion is possible. The set of singular points of X will be denoted by $\operatorname{Sing}(X)$. We say that a family $\{Y_i\}_{i=1,\dots,k}$ of subvarieties (not necessarily closed) of X is in general position if for each point x in the union $Y_1 \cup \cdots \cup Y_k$, the family $\{T_x(Y_i)\}_{i \in \Lambda(x)}$, where $\Lambda(x) = \{i | x \in Y_i\}$, of vector subspaces of $T_x(X)$ ($T_x(\cdot)$ is the Zariski tangent space at x) is in general position, i.e.,

$$\operatorname{codim} \bigcap_{i \in \Lambda(x)} T_x(Y_i) = \sum_{i \in \Lambda(x)} \operatorname{codim} T_x(Y_i).$$

A subvariety Y of X will be called an algebraic hypersurface if Y is of pure codimension 1 in X. By a real algebraic curve in X we shall mean a subvariety of X of pure dimension 1.

Any real algebraic variety can be endowed with the strong topology induced from the Euclidean topology on the reals. Unless otherwise explicitly specified we shall always consider the *strong topology*. However, the terms "closed subvariety" or "open subvariety" refer to the Zariski topology.

DEFINITION 1.1. *A real algebraic variety X of dimension n is said to be admissible if there exists a sequence of real algebraic morphisms $\pi_i: X_i \to X_{i-1}$, $i = 1, \dots, k$, such that*

(i) *X_0 is an affine nonsingular real algebraic variety diffeomorphic, as a C^∞ manifold, to the unit n-dimensional sphere S^n,*

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(ii) *π_i is the blowing up of X_{i-1} along a finite subset for all $i = 1, \dots, k$,*
(iii) *X is isomorphic to $X_k \setminus Y$, where Y is a closed subvariety of X_k, $\dim Y \leq n - 1$.*

If Y is a finite set, then X is called a strongly admissible variety.

This paper is devoted to proving the following result.

THEOREM 1.2. *Let X be an admissible n-dimensional variety, $n \geq 3$, and let C be an algebraic closed (not necessarily irreducible) curve in X. Then there exist algebraic closed irreducible hypersurfaces $H_1, \dots, H_{n-1}$ in X such that $C = H_1 \cap \cdots \cap H_{n-1}$, $\mathrm{Sing}(H_i) \subset \mathrm{Sing}(C)$ for all $i = 1, \dots, n-1$ and the family of nonsingular hypersurfaces $\{H_i \setminus \mathrm{Sing}(C)\}_{i=1,\dots,n-1}$ is in general position. Moreover, if X is strongly admissible and C is compact, then H_i can be chosen connected and compact.*

The most obvious and important examples of strongly admissible varieties are S^n, the real affine space $\mathbf{R}^n$ and the real projective space $\mathbf{R}P^n$. Observe that $\mathbf{R}^n$ is isomorphic, via the stereographic projection, to S^n with one point removed and $\mathbf{R}P^n$ is isomorphic to the variety obtained by blowing up S^n at a single point.

It should be mentioned that Theorem 1.2 is false, even for nonsingular algebraic curves in affine or projective spaces, over every algebraically closed field (cf. for instance [8]).

2. Proof of Theorem 1.2

Let X be an affine real algebraic variety and let Y and Z be closed subvarieties of X with Z contained in Y. We shall regard the blowing up $B(Y, Z)$ of Y along Z as a closed subvariety of the blowing up $B(X, Z)$ of X along Z.

It will be convenient to identify S^n with $S^n \times \{0\}$ in S^{n+1}.

LEMMA 2.1. *Let F be a finite subset of $S^3 \subset S^n$, $n \geq 3$. Then the homomorphisms*

(2.1.1) $\pi_1(B(S^3, F)) \to \pi_1(B(S^n, F))$,
(2.1.2) $H^1(B(S^n, F), \mathbf{Z}/2\mathbf{Z}) \to H^1(B(S^3, F), \mathbf{Z}/2\mathbf{Z})$,

of the fundamental groups and cohomology groups, respectively, induced by the inclusion $B(S^3, F) \subset B(S^n, F)$, are isomorphisms. Moreover:

(2.1.3) *every element in $H^2(B(S^3, F), \mathbf{Z}/2\mathbf{Z})$ is a square, with respect to the cup product, of an element in $H^1(B(S^3, F), \mathbf{Z}/2\mathbf{Z})$;*
(2.1.4) *the restriction (modulo 2) homomorphism $H^2(B(S^3, F), \mathbf{Z}) \to H^2(B(S^3, F), \mathbf{Z}/2\mathbf{Z})$ is bijective.*

Proof. Elementary exercise.

Given a topological space X, a continuous real vector bundle ξ over X and a section s of ξ, we shall denote by $s^{-1}(0)$ the set of zeros of s. The k-th Stiefel-Whitney characteristic class of ξ will be denoted by $w_k(\xi)$. If ξ is oriented, then $e(\xi)$ will denote the Euler characteristic class of ξ.

Now we are ready to prove the main technical result.

LEMMA 2.2. *Let $\pi: B(S^n, F) \to S^n$ be the blowing up of S^n, $n \geq 3$, along a finite subset F. Let M be a compact C^∞ one-dimensional submanifold of $B(S^n, F)$ and let $K_1, \dots, K_{n-1}$, K be finite, mutually disjoint subsets of $B(S^n, F) \setminus M$. Then there exist compact connected C^∞ hypersurfaces $N_1, \dots, N_{n-1}$ in $B(S^n, F) \setminus K$ such that the family $\{N_i\}_{i=1,\dots,n-1}$ is in general position, $M = N_1 \cap \cdots \cap N_{n-1}$ and N_i contains K_i for all $i = 1, \dots, n-1$.*

Proof. Without any loss of generality, we may assume that F is a subset of $S^3 \subset S^n$. We claim that there exists a C^∞ diffeomorphism of $B(S^n, F)$ transforming M onto a submanifold of $B(S^3, F)$. By (2.1.1), given a C^∞ embedding $a: S^1 \to B(S^n, F)$, one can find a C^∞ embedding $b: S^1 \to B(S^n, F)$ homotopic to a whose image is contained in $B(S^3, F)$. Since every connected component of M is C^∞ diffeomorphic to S^1, there exists a C^∞ embedding $e: M \to B(S^n, F)$ homotopic to the inclusion map $M \to B(S^n, F)$ with the image contained in $B(S^3, F)$. Now the claim follows from [6], p. 183, Exercise 10. Therefore, we shall assume in the proof that M is contained in $B(S^3, F)$.

Since $B(S^3, F)$ is an orientable manifold (recall that $B(S^3, F)$ is diffeomorphic to the connected sum of S^3 and k copies of $\mathbf{R}P^3$, where k is the cardinality of F), the normal vector bundle of M in $B(S^3, F)$ is trivial. It follows that there exist a C^∞ real vector bundle ξ over $B(S^3, F)$ and a C^∞ section s of ξ such that rank $\xi = 2$, ξ is orientable, s is transverse to the zero section of ξ and $M = s^{-1}(0)$. Indeed, one can find a C^∞ map

$$f = (f_1, f_2): B(S^3, F) \to \mathbf{R}^2$$

and a neighborhood U of M in $B(S^3, F)$ such that $f|U$ is transverse to 0 in $\mathbf{R}^2$ and

$$M = f^{-1}(0) \cap U.$$

The set $f^{-1}(0)$ can be written as $f^{-1}(0) = M_1 \cup M_2$, where $M_1 = M$ and M_2 is a closed subset of $B(S^3, F)$ disjoint from M. Let $U_1 = B(S^3, F) \setminus M_2$ and $U_2 = B(S^3, F) \setminus M_1$. Since the functions f_1 and f_2 have no common zero on $U_1 \cap U_2$, there exist C^∞ functions $g_1, g_2: U_1 \cap U_2 \to \mathbf{R}$ such that det $g_{12} = 1$

on $U_1 \cap U_2$, where

$$g_{12} = \begin{pmatrix} f_1 & g_1 \\ f_2 & g_2 \end{pmatrix}.$$

Now let ξ be the C^∞ vector bundle over $B(S^3, F)$ corresponding to the open covering $\{U_1, U_2\}$ of $B(S^3, F)$ and the transition function g_{12}. Then the maps

$$U_1 \to \mathbf{R}^2, \qquad x \to (f_1(x), f_2(x)),$$
$$U_2 \to \mathbf{R}^2, \qquad x \to (1, 0)$$

determine the C^∞ section s of ξ. It is easy to see that ξ and s satisfy all requirements. By (2.1.2) and (2.1.3), there exists an element z in $H^1(B(S^n, F), \mathbf{Z}/2\mathbf{Z})$ such that

$$w_2(\xi) = i^*(z) \cup i^*(z),$$

where $i^* : H^1(B(S^n, F), \mathbf{Z}/2\mathbf{Z}) \to H^1(B(S^3, F), \mathbf{Z}/2\mathbf{Z})$ is the homomorphism induced by the inclusion $B(S^3, F) \subset B(S^n, F)$. Let γ be a continuous real line bundle over $B(S^n, F)$ with $w_1(\gamma) = z$ [7]. Note that $w_2((\gamma \oplus \gamma)|B(S^3, F)) = w_2(\xi)$. Fix any orientations on ξ and $\gamma \oplus \gamma$. By (2.1.4) and [9], p. 99, $e((\gamma \oplus \gamma)|B(S^3, F)) = e(\xi)$. It follows that the vector bundles $(\gamma \oplus \gamma)|B(S^3, F)$ and ξ are C^0 isomorphic [7]. Clearly, we may assume that γ is a C^∞ real line bundle and the vector bundles $(\gamma \oplus \gamma)|B(S^3, F)$ and ξ are C^∞ isomorphic. It follows that there exists a C^∞ section $t = (t_1, t_2)$ of $\gamma \oplus \gamma$ such that $t|B(S^3, F)$ is transverse to the zero section of $(\gamma \oplus \gamma)|B(S^3, F)$ and $M = t^{-1}(0) \cap B(S^3, F)$. Moreover, t can be chosen with $t_j|B(S^3, F)$ transverse to the zero section of $\gamma|B(S^3, F)$ for $j = 1, 2$. By a standard transversality argument, there exists a C^∞ section $u = (u_1, u_2)$ of $\gamma \oplus \gamma$ such that u is transverse to the zero section of $\gamma \oplus \gamma$, u is equal to t in a neighborhood of $B(S^3, F)$ and u_j is transverse to the zero section of γ for $j = 1, 2$. By cutting small disks in the connected components of $u_j^{-1}(0)$ and joining the remaining sets by tubes, one can construct a compact connected C^∞ hypersurface P_j of $B(S^n, F)$. Moreover, this construction can be performed in such a way that P_j is transverse to $B(S^3, F)$, P_1 is transverse to P_2,

$$M = P_1 \cap P_2 \cap B(S^3, F)$$

and P_j represents the same homology class in $H_{n-1}(B(S^n, F), \mathbf{Z}/2\mathbf{Z})$ as $u_j^{-1}(0)$ for $j = 1, 2$ (this is obvious if $n \geq 4$ since one can cut out disks in the connected components of $u_j^{-1}(0)$ and attach tubes outside $B(S^3, F)$; if $n = 3$, then one uses the fact that $u_1^{-1}(0)$ is transverse to $u_2^{-1}(0)$ and that the complement of any compact connected C^∞ surface in $B(S^3, F)$ has at most two connected components). The last condition implies that the C^∞ real line bundle over $B(S^n, F)$ corresponding, in the standard way, to the C^∞ hypersurface P_j is C^∞ isomorphic to γ. Thus there exists a C^∞ section v_j of γ such that v_j is transverse to the zero section of γ and $P_j = v_j^{-1}(0)$. Note that the restriction $(v_1, v_2)|B(S^3, F)$ is transverse to the zero section of $(\gamma \oplus \gamma)|B(S^3, F)$.

Let $L_i = \{(x_1, \dots, x_{n+1}) \in S^n | x_{i+2} = 0\}$ and let $P_i = B(L_i, F)$, $i = 3, \dots, n-1$. Note that P_i is a compact connected C^∞ hypersurface in $B(S^n, F)$, the family $\{P_i\}_{i=3,\dots,n-1}$ is in general position and $B(S^3, F) = P_3 \cap \cdots \cap P_{n-1}$. Let γ_i be a C^∞ real line bundle over $B(S^n, F)$ and let v_i be a C^∞ section of γ_i such that v_i is transverse to the zero section of γ_i and $P_i = v_i^{-1}(0)$ for $i = 3, \dots, n-1$. Clearly, the section $v = (v_1, \dots, v_{n-1})$ of $\gamma_1 \oplus \cdots \oplus \gamma_{n-1}$, where $\gamma_1 = \gamma_2 = \gamma$, is transverse to the zero section of $\gamma_1 \oplus \cdots \oplus \gamma_{n-1}$ and $M = s^{-1}(0)$. By a transversality argument, there exists a section $w = (w_1, \dots, w_{n-1})$ of $\gamma_1 \oplus \cdots \oplus \gamma_{n-1}$, arbitrarily close to v in the C^∞ topology, such that w vanishes on M, the section w_i is transverse to the zero section of γ_i, $Q_i = w_i^{-1}(0)$ is disjoint from K and the family $\{Q_i\}_{i=1,\dots,n-1}$ of C^∞ hypersurfaces in $B(S^n, F)$ is in general position. We can assume that each Q_i is connected and $M = Q_1 \cap \cdots \cap Q_{n-1}$ provided that w is sufficiently close to v. Let $F_1, \dots, F_{n-1}$ be mutually disjoint subsets of $B(S^n, F) \setminus (M \cup K)$ such that F_i is contained in Q_i and F_i has the same cardinality as K_i, $i = 1, \dots, n-1$. Now one can find a C^∞ diffeomorphism

$$\varphi : B(S^n, F) \to B(S^n, F)$$

such that φ is the identity in a neighborhood of $M \cup K$ and $\varphi(F_i) = K_i$. It suffices to set $N_i = \varphi(Q_i)$ for $i = 1, \dots, n-1$.

Let X be an affine real algebraic variety. An algebraic real vector bundle ξ over X is said to be strongly algebraic if there exists an algebraic vector bundle η over X such that the direct sum $\xi \oplus \eta$ is algebraically isomorphic to the product vector bundle $X \times \mathbf{R}^k$ for some k [2], [4].

Proof of Theorem 1.2. We may assume that X is an open subvariety of a compact admissible variety Y. Let D be the Zariski closure of C in Y.

It follows from [5] that there exist a sequence of real algebraic morphisms $\pi_j : Y_j \to Y_{j-1}$ and a sequence of real algebraic curves D_j, $j = 1, \dots, k$, such that $Y_0 = Y$, π_1 is the blowing up of Y_0 along $\mathrm{Sing}(D)$, D_1 is the Zariski closure of $\pi_1^{-1}(D \setminus \mathrm{Sing}(D))$ in Y_1 and for each $j = 1, \dots, k-1$, the curve D_j is contained in Y_j, π_{j+1} is the blowing up of Y_j along $\mathrm{Sing}(D_j)$, D_{j+1} is the Zariski closure of $\pi_{j+1}^{-1}(D_j \setminus \mathrm{Sing}(D_j))$ in Y_{j+1}, D_k is nonsingular and $\pi^{-1}(\mathrm{Sing}(D)) \cap D_k$ is a finite set, where $\pi = \pi_1 \circ \cdots \circ \pi_k$. Let $Z = Y_k$ and $E = D_k$. Clearly, Z is a compact admissible variety. It follows that Z is diffeomorphic to $B(S^n, F)$ for some finite subset F of S^n. Let $K_1, \dots, K_{n-1}$ be finite, mutually disjoint subsets of

$$Z \setminus \left(E \cup \pi^{-1}(Y \setminus X)\right)$$

such that K_i has a common point with $\pi^{-1}(x)$ for all $i = 1, \dots, n-1$ and x in $\mathrm{Sing}(C)$. By Lemma 2.2, there exist compact connected C^∞ hypersurfaces $N_1, \dots, N_{n-1}$ in Z such that the family $\{N_i\}_{i=1,\dots,n-1}$ is in general position, $E = N_1 \cap \cdots \cap N_{n-1}$ and N_i contains K_i for $i = 1, \dots, n-1$. Moreover, we may assume that N_i is disjoint from $\pi^{-1}(Y \setminus X)$ if $Y \setminus X$ is a finite set and C is compact. Indeed, since $\pi^{-1}(Y \setminus X) \cap E$ is a finite set disjoint from $\pi^{-1}(C)$ and E is nonsingular, the set $\pi^{-1}(Y \setminus X) \cap E$ is actually empty. Now we can blow down $\pi^{-1}(Y \setminus X)$. More precisely, we can find a C^∞ manifold Z' and a C^∞ surjective map $p: Z \to Z'$ such that the set $K = p(\pi^{-1}(Y \setminus X))$ has exactly l points, where l is the cardinality of $Y \setminus X$, and p induces a diffeomorphism for $Z \setminus \pi^{-1}(Y \setminus X)$ onto $Z' \setminus K$. Note that $p(E)$ is a compact C^∞ submanifold of $Z' \setminus K$. It suffices to apply Lemma 2.2 to Z', $p(E)$, $p(K_1), \dots, p(K_{n-1})$ and K.

Now for each $i = 1, \dots, n-1$, we can choose a C^∞ real line bundle γ_i over Z and a C^∞ section s_i of γ_i such that s_i is transverse to the zero section of γ_i and $N_i = s_i^{-1}(0)$. Notice that Z is an affine real algebraic variety (cf. for instance [1] or [4]). Moreover, every element in the homology group $H_{n-1}(Z, \mathbf{Z}/2\mathbf{Z})$ can be represented by an algebraic closed hypersurface of Z. It follows that every C^∞ real line bundle over Z is C^∞ isomorphic to a strongly algebraic vector bundle [3], [4], [11]. Thus we can assume that all γ_i are strongly algebraic line bundles. We claim that for each $i = 1, \dots, n-1$, there exists an algebraic section u_i of γ_i vanishing on $E \cup K_i$ and arbitrarily close to s_i in the C^∞ topology. Indeed, by definition of a strongly algebraic vector bundle, there exist global algebraic sections $v_1, \dots, v_r$ of γ_i such that for each point x in Z the vectors $v_1(x), \dots, v_r(x)$ generate the fiber of γ_i over x. Clearly, s_i can be written as $s_i = h_1 v_1 + \cdots + h_r v_r$, where $h_q: Z \to \mathbf{R}$ are C^∞ functions vanishing on $E \cup K_i$. Now the Weierstrass approximation theorem (cf. [12], p. 54, for the relative version) implies the existence of u_i. Set $G_i = u_i^{-1}(0)$. If u_i is sufficiently close to s_i, then $G_1, \dots, G_{n-1}$ are nonsingular algebraic hypersurfaces in Z which are in general position and satisfy $E = G_1 \cap \cdots \cap G_{n-1}$. Each G_i is connected, hence irreducible being nonsingular, and has a nonempty intersection with $\pi^{-1}(x)$ for all x in $\mathrm{Sing}(C)$. Moreover, if $Y \setminus X$ is a finite set and C is compact, then G_i is disjoint from $\pi^{-1}(Y \setminus X)$. Let $H_i = X \cap \pi(G_i)$ for $i = 1, \dots, n-1$. Since

$$H_i = X \cap \left(\pi(G_i) \cup \mathrm{Sing}(D)\right),$$

H_i is a closed algebraic hypersurface in X. All other claims about $H_1, \dots, H_{n-1}$ are obvious.

Remark 2.3. It follows from the proof given above that any compact C^∞ one-dimensional submanifold of a strongly admissible variety is isotopic to a nonsingular algebraic curve which is a complete intersection.

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