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AN ANALOGUE OF HILBERT'S THEOREM 90 FOR THE RING OF ENTIRE FUNCTIONS

BY

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1. Introduction

Let us look at the following situation.

Let $\lambda \in C^q$ and $H(\lambda)$ be an n by n matrix whose entries are entire in λ. Regard λ as a row vector and suppose that there exists a linear transformation L: $C^q \to C^q$ such that

(I) L^m is the identity transformation,

(II) H satisfies the condition

$$H(\lambda)H(\lambda L)H(\lambda L^2) \cdots H(\lambda L^{m-1}) = I, \tag{1.1}$$

where I is the n by n identity matrix.

To compare with this situation, we shall state Theorem 90 of Hilbert.

THEOREM 90 OF HILBERT. *Let k be a field and K a finite cyclic extension of k. Denote by g a generator of* Gal(K/k). *Then*

$$N_{K/k}(a) \left(= ag(a)g^2(a) \dots g^{m-1}(a)\right) = 1 \tag{1.2}$$

for an element a of K if and only if

$$a = bg(b)^{-1} \tag{1.3}$$

for some $b \in K^*$ $(= K - \{0\})$, where m is the order of Gal(K/k) (*cf. D. Hilbert* [6]).

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As to (1.1), a result analogous to this theorem is known.

PROPOSITION 1.1. *Suppose that* (I) *is satisfied. Furthermore assume that*
(III) L^h *is not the identity transformation for* $0 < h < m$.
Let H be an n by n matrix whose entries are meromorphic in λ. *Then H satisfies condition* (1.1) *if and only if*

$$H(\lambda) = W(\lambda)W(\lambda L)^{-1} \tag{1.4}$$

for some matrix W whose entries are meromorphic in λ.

In this paper we shall prove a better result:

THEOREM 1.2. *Assume that L satisfies condition* (I). *Let* $H(\lambda)$ *be an n by n matrix whose entries are entire in* λ. *Then H satisfies condition* (1.1) *if and only if*
(i) $H(0)^m = L$; *and there exists an n by n matrix* $E(\lambda)$ *such that*
(ii) *the entries of E and* E^{-1} *are entire in* λ,
(iii) $H(\lambda) = E(\lambda)H(0)E(\lambda L)^{-1}$.

Note. In this theorem we do not assume (III) of Proposition 1.1.

The hardest part of the problem is to construct the matrix E so that the entries of E and E^{-1} are entire in λ. If $q = 1$ (i.e., λ is a scalar) and if (III) is also satisfied, we can construct such an E by using the W of Proposition 1.1 and Weierstrass' representation of an entire function as an ifinite product. Such a technique does not work for other cases. To overcome this difficulty we use the following result:

PROPOSITION 1.3. *Let* $(t, \lambda) = (t, \lambda_1, \ldots, \lambda_q) \in C^{q+1}$ *and* $F(t, \lambda)$ *be an n by n matrix whose entires are entire in* (t, λ). *Assume that there exists a linear transformation* L: $C^q \to C^q$ *such that*
(i) L^m *is the identity transformation*,
(ii) $F(t, \lambda)F(t, \lambda L)F(t, \lambda L^2)\ldots F(t, \lambda L^{m-1}) = I$.
Then, there exists an n by n matrix $A(t, \lambda)$ *such that*
(1) *the entries of* $A(t, \lambda)$ *are entire in* (t, λ),
(2) *the derivative of F with respect to t is given by*

$$(\partial F/\partial t)(t, \lambda) = A(t, \lambda)F(t, \lambda) - F(t, \lambda)A(t, \lambda L). \tag{1.5}$$

In fact, if we set $F(t, \lambda) = H(t\lambda)$, then condition (1.1) implies condition (ii) of Proposition 1.3. Now, define a matrix E by

$$dE/dt = A(t, \lambda)E, \quad E = I \text{ at } t = 0. \tag{1.6}$$

Then, the entries of E and E^{-1} are entire in (t, λ) and

$$H(t\lambda) = F(t, \lambda) = E(t, \lambda)F(0, \lambda)E(t, \lambda L)^{-1}. \tag{1.7}$$

Setting $t = 1$, we derive (iii) of Theorem 1.2 with $E = E(1, \lambda)$.

2. Proof of Proposition 1.3

In case $m = 2$ (i.e., L^2 is the identity), we have $F(t, \lambda)F(t, \lambda L) = I$, and hence

$$K(t, \lambda)F(t, \lambda L) + F(t, \lambda)K(t, \lambda L) = 0, \tag{2.1}$$

where $K = \partial F/\partial t$. Set

$$B(t, \lambda) = K(t, \lambda)F(t, \lambda L) = K(t, \lambda)F(t, \lambda)^{-1}. \tag{2.2}$$

Then

$$B(t, \lambda L) = K(t, \lambda L)F(t, \lambda) = -F(t, \lambda L)K(t, \lambda).$$

Therefore

$$K(t, \lambda) = B(t, \lambda)F(t, \lambda) = -F(t, \lambda)B(t, \lambda L),$$

and

$$K(t, \lambda) = \tfrac{1}{2}[B(t, \lambda)F(t, \lambda) - F(t, \lambda)B(t, \lambda L)]. \tag{2.3}$$

Hence, setting

$$A(t, \lambda) = \tfrac{1}{2}B(t, \lambda) \left(= \tfrac{1}{2}K(t, \lambda)F(t, \lambda)^{-1}\right), \tag{2.4}$$

we derive (1.5). We can prove Proposition 1.3 for the general case in a similar manner. However, we shall provide here a much simpler proof.

Set

$$\begin{aligned} &\omega = \exp[2\pi i/m],\ \rho(\omega^k) = L^k \quad (k = 0, 1, \dots, m-1), \\ &F(t, \lambda; \omega^0) = I,\ F(t, \lambda; \omega) = F(t, \lambda), \\ &F(t, \lambda; \omega^k) = F(t, \lambda)F(t, \lambda L) \dots F(t, \lambda L^{k-1}) \\ &\qquad\qquad\qquad\qquad\qquad\qquad (k = 2, \dots, m-1). \end{aligned} \tag{2.5}$$

Then

$$F(t, \lambda; \omega^{k+h}) = F(t, \lambda; \omega^k) F(t, \lambda\rho(\omega^k); \omega^h) \qquad (k, h = 0, 1, \ldots, m-1). \tag{2.6}$$

Set

$$D(t, \lambda; \omega^k) = (\partial F/\partial t)(t, \lambda; \omega^k) F(t, \lambda; \omega^k)^{-1}. \tag{2.7}$$

Then

$$\begin{aligned} D(t, \lambda; \omega^{k+h}) \\ = D(t, \lambda; \omega^k) + F(t, \lambda; \omega^k) D(t, \lambda\rho(\omega^k); \omega^h) F(t, \lambda; \omega^k)^{-1}. \end{aligned} \tag{2.8}$$

Therefore, if we define A by

$$A(t, \lambda) = \frac{1}{n} \sum_{h=0}^{m-1} D(t, \lambda; \omega^h), \tag{2.9}$$

we have

$$A(t, \lambda) = D(t, \lambda; \omega^k) + F(t, \lambda; \omega^k) A(t, \lambda\rho(\omega^k)) F(t, \lambda; \omega^k)^{-1},$$

or

$$\begin{aligned} (\partial F/\partial t)(t, \lambda; \omega^k) = A(t, \lambda) F(t, \lambda; \omega^k) \\ - F(t, \lambda; \omega^k) A(t, \lambda\rho(\omega^k)). \end{aligned} \tag{2.10}$$

Setting $k = 1$, we derive (1.5).

3. A generalization of Proposition 1.3 and Theorem 1.2

We can generalize Proposition 1.3 further.

THEOREM 3.1. *Let $t \in C$, $\lambda \in C^q$ and $\xi \in G$, where G is a compact group. Let ρ: $G \to GL_q(C)$ be a continuous map. Suppose that*

$$F: C \times C^q \times G \to GL_n(C) \tag{3.1}$$

is a continuous map such that

(i) *the entries of matrix* $F(t, \lambda; \xi)$ *are entire in* (t, λ) *for each fixed* $\xi \in G$,

(ii) *F satisfies the condition*

$$F(t, \lambda; \xi\eta) = F(t, \lambda; \xi)F(t, \lambda\rho(\xi); \eta) \tag{3.2}$$

for $t \in C$, $\lambda \in C^q$, $\xi \in G$ *and* $\eta \in G$.

Then, there exists an n by n matrix $A(t, \lambda)$ *such that*

(i) *the entries of A are entire in* (t, λ),

(ii) *the derivative of F with respect to t is given by*

$$(\partial F/\partial t)(t, \lambda; \xi) = A(t, \lambda)F(t, \lambda; \xi) - F(t, \lambda; \xi)A(t, \lambda\rho(\xi)). \tag{3.3}$$

In fact, if $\nu(\xi)$ is the normalized Haar measure on G with $\nu(G) = 1$, then A is given by

$$A(t, \lambda) = \int_G D(t, \lambda; \eta)\, d\nu(\eta), \tag{3.4}$$

where

$$D(t, \lambda; \xi) = (\partial F/\partial t)(t, \lambda; \xi)F(t, \lambda; \xi)^{-1}. \tag{3.5}$$

Note that (3.2) implies

$$D(t, \lambda; \xi\eta) = D(t, \lambda; \xi) + F(t, \lambda; \xi)D(t, \lambda\rho(\xi); \eta)F(t, \lambda; \xi)^{-1}. \tag{3.6}$$

Let us define an n by n matrix $E(t, \lambda)$ by

$$dE/dt = A(t, \lambda)E, \quad E = I \quad \text{at } t = 0. \tag{3.7}$$

Then, (3.3) implies that

$$F(t, \lambda; \xi) = E(t, \lambda)F(0, \lambda; \xi)E(t, \lambda\rho(\xi))^{-1}, \tag{3.8}$$

or

$$F(t, \lambda; \xi)E(t, \lambda\rho(\xi))F(0, \lambda; \xi)^{-1} = E(t, \lambda) \tag{3.8'}$$

Note that the entries of E and E^{-1} are entire in (t, λ). Thus we have proved the following theorem:

THEOREM 3.2. *Under the same assumptions as Theorem* 3.1, *there exists an n by n matrix* $E(t, \lambda)$ *such that*

(1) *the entries of E and* E^{-1} *are entire in* (t, λ),

(2) *F has the form* (3.8), *i.e.*,

$$F(t, \lambda; \xi) = E(t, \lambda)F(0, \lambda; \xi)E(t, \lambda\rho(\xi))^{-1}.$$

In the case when F does not depend on t, introducing t through the change of the variable replacing λ by $t\lambda$, we can prove the following theorem:

THEOREM 3.3. *Let $\lambda \in C^q$ and $\xi \in G$, where G is a compact group. Let $\rho: G \to GL_q(C)$ be a continuous map. Suppose that*

$$\Phi: C^q \times G \to GL_n(C)$$

is a continuous map such that
(i) *the entries of Φ are entire in λ for each fixed $\xi \in G$,*
(ii) *Φ satisfies the condition*

$$\Phi(\lambda; \xi\eta) = \Phi(\lambda; \xi)\Phi(\lambda\rho(\xi); \eta)$$

for $\lambda \in C^q$, $\xi \in G$ and $\eta \in G$.
Then there exists an n by n matrix $E(\lambda)$ such that
(1) *the entries of E and E^{-1} are entire in λ,*
(2) *Φ has the form $\Phi(\lambda; \xi) = E(\lambda)\Phi(0; \xi)E(\lambda\rho(\xi))^{-1}$.*

Remark 3.4. (i) In Theorem 3.1, it is not necessary to assume that

$$\rho(\xi\eta) = \rho(\xi)\rho(\eta) \quad \text{for } \xi \text{ and } \eta \in G. \tag{3.9}$$

However, condition (3.2) implies that

$$F(t, \lambda\rho(\xi\eta); \zeta) = F(t, \lambda\rho(\xi)\rho(\eta); \zeta) \quad \text{for } \xi, \eta \text{ and } \zeta \in G.$$

Hence, it would be convenient to verify (3.9) when we want to check condition (3.2).
(ii) Condition (3.2) can be relaxed in the following way:
The quantity

$$C = F(t, \lambda; \xi\eta)^{-1}F(t, \lambda; \xi)F(t, \lambda\rho(\xi); \eta) \tag{3.2'}$$

is independent of t.
In fact, (3.2′) also implies (3.6).
(iii) In Theorem 1.2, Condition (1.1) can be replaced by

$$H(\lambda)H(\lambda L)H(\lambda L^2)\dots H(\lambda L^{m-1}) = K, \tag{1.1'}$$

where K is an n by *n invertible constant* matrix.
In fact, utilizing (2.5) we can verify that

$$F(\lambda; \omega^{k+h})^{-1}F(\lambda; \omega^k)F(\lambda L^k; \omega^h) = \begin{cases} I & \text{if } \ 2 \le k + h \le n, \\ K & \text{if } \ n + 1 \le k + h \le 2n. \end{cases}$$

Hence we can apply Remark (ii). (Condition (i) of Theorem 1.2 should be replaced by $H(0)^m = K$.)

(iv) The requirement on the smoothness of the entries of $F(t, \lambda; \xi)$ may be relaxed. For example, we may assume that the entries of F and $\partial F/\partial t$ are continuous in $(t, \lambda; \xi)$ in a domain: $I \times \mathscr{U} \times G$, where I is a t-interval and $\mathscr{U}$ is an open set in the λ-space. Then the entries of A are also continuous in (t, λ). If we assume a continuous differentiability of the entries of F and $\partial F/\partial t$ with respect to λ, then the entries of A admit the same kind of smoothness. Those modifications are based on the observation concerning the differentiation of an integral of the type $\int_G g\,d\nu(\xi)$ with a function g which is smooth with respect to parameters.

(v) Theoem 3.1 can be extended to p-adic functions if G is finite. However, we do not know whether the matrix E defined by (3.7) is entire.

(vi) The matrices Φ and E of Theorem 3.3 satisfy condition (2) or

$$E(\lambda) = \Phi(\lambda; \xi) E(\lambda\rho(\xi)) \Phi(0: \xi)^{-1}. \tag{3.10}$$

We can interpret this relation in terms of automorphy factors and automorphic forms (cf. A. Borel [3]). In fact, if we define a map

$$\alpha: C^q \times G \to \operatorname{Aut}(M_n(C)) \tag{3.11}$$

by

$$\alpha(\lambda; \xi)[X] = \Phi(\lambda; \xi) X \Phi(0; \xi)^{-1} \tag{3.12}$$

where $M_n(C)$ is the vector space of n by n complex matrices, and $\lambda \in C^q$, $\xi \in G$ and $X \in M_n(C)$, then, since Φ satisfies the relation $\Phi(\lambda; \xi\eta) = \Phi(\lambda; \xi)\Phi(\lambda\rho(\xi); \eta)$, we have

$$\alpha(\lambda; \xi\eta) = \alpha(\lambda; \xi)\alpha(\lambda\rho(\xi); \eta) \tag{3.13}$$

and relation (3.10) can be written in the form

$$\alpha(\lambda: \xi)[E(\lambda\rho(\xi))] = E(\lambda). \tag{3.14}$$

This mans that α is an automorphy factor and that E is an automorphic form relative to α. For a given automorphy factor, automorphic functions are not unique. One of the most important problems in the study of automorphic functions is to find a basis for the space of automorphic functions. We shall investigate such a problem concerning the matrix E of Theorems 3.2 and 3.3, elsewhere.

Note. In the case of Theorem 3.2, we define α by

$$\alpha(t, \lambda; \xi)[X] = F(t, \lambda; \xi) X F(0, \lambda; \xi)^{-1}. \tag{3.12'}$$

Then condition (3.2) implies that

$$\alpha(t; \lambda; \xi\eta) = \alpha(t, \lambda; \xi)\alpha(t, \lambda\rho(\xi); \eta) \tag{3.13'}$$

and relation (3.8′) can be written in the form

$$\alpha(t, \lambda; \xi)[E(t, \lambda\rho(\xi))] = E(t, \lambda). \tag{3.14'}$$

Hence, α is an automorphy factor and E is an automorphic form relative to α.

4. Results in terms of Galois-cohomology

Let X be a smooth complex anaytic manifold, E a vector bundle over X, and G a compact group acting continuously on the right on E by vector bundle automorphisms; i.e., there is given a homomorphism $G^0 \to \mathrm{Aut}(E)$. Let $\mathscr{R}$ denote $\Gamma(E, \mathscr{O}_E)$ (the ring of all global analytic functions on E) so that G acts on $\mathscr{R}$, i.e., there is a homomorphism ρ: $G \to \mathrm{Aut}(\mathscr{R})$. Thus G also acts on $GL_n(\mathscr{R})$. Now we can state an abstract version of Theorem 3.3 in terms of Galois-cohomology (cf. A. Grothendieck [5]).

THEOREM 4.1. $H^1(G, GL_n(\mathscr{R})) = H^1(G, GL_n(\Gamma(X, \mathscr{O}_X)))$.

Note. In case of Theorem 3.3, X consists of a point.

To prove Theorem 4.1, we introduce a new variable t through $(t, e) \to te$ $(e \in E)$. This replaces E by $C \times E$. Let $\Gamma(C \times E, \mathscr{O}_{C\times E})$ be denoted by $\mathscr{R}'$. Then G still acts on $C \times E$, and hence ρ can be extended to a homomorphism ρ': $G \to \mathrm{Aut}(\mathscr{R}')$. Further if we denote d/dt by $\mathscr{D}$, then we have $\mathscr{D}\rho'(\xi) = \rho'(\xi)\mathscr{D}$ for $\xi \in G$. Let $\mathscr{A} = GL_n(\mathscr{R}')$ and $\mathscr{F} = \mathscr{M}(\mathscr{R}')$. Now we can state an abstract version of Theorem 3.1.

THEOREM 4.2. *For every one cocycle f: $G \to \mathscr{A}$ there exists $b \in \mathscr{F}$ such that*

$$\mathscr{D}(f(\xi)) = bf(\xi) - f(\xi)\rho'(\xi)(b) \tag{4.1}$$

for all $\xi \in G$.

Note. A map f: $G \to \mathscr{A}$ is a one cocycle if

$$f(\xi\eta) = f(\xi)\rho'(\xi)(f(\eta)) \quad \text{for } \xi \in G \text{ and } \eta \in G.$$

To prove Theorem 4.2, we introduce on $\mathscr{F}$ a new structure of G-module by $(\xi, b) \to \xi \times b = f(\xi)\rho'(\xi)(b)f(\xi)^{-1}$. Let us denote this new G-module by $\mathscr{F}_f$. Set

$$h(\xi) = \mathscr{D}(f(\xi))f(\xi)^{-1}. \tag{4.2}$$

This is a map $G \to \mathscr{F}_f$. It is easily verified that

$$h(\xi\eta) = h(\xi) + \xi \times h(\eta). \tag{4.3}$$

This means that h is a one cocycle and hence defines an element in $H^1(G, \mathscr{F}_f)$.
Now the following lemma is the key to the proof of Theorem 4.2.

LEMMA 4.3. $H^1(G, \mathscr{F}_f) = \{0\}$.

To prove this, set

$$b = \int_G h(\eta)\, d\nu(\eta),$$

where ν is the normalized Haar measure on G with $\nu(G) = 1$. Then from (4.3) we derive

$$b = h(\xi) + \xi \times b. \tag{4.4}$$

Hence Lemma 4.3 follows immediately.

5. An example

A traditional method for solving differential equations which goes back to Riemann's treatment of hypergeometric differential equation has 5 stages:

(1) classification of differential equations by means of suitable transformations;
(2) identification of invariants under such transformations in terms of solutions;
(3) construction of a standard equation representing an equivalence class in terms of its invariants;
(4) computation of invariants for a given equation;
(5) reduction of a given equation to a standard equation.

Since 1970 a German-American School (W. Jurkat (Ulm-Syracuse), D.A. Lutz (Milwaukee-San Diego), W. Balser (Ulm) et al) has done extensive work on the classification of meromorphic differential equations (cf. W. Jurkat [8]). In particular they identified the invariants in terms of monodromy matrices and Stokes multipliers. Inspired by the German-American group, a French School (B. Malgrange (Grenoble), J.-P. Ramis (Strasbourg) et al) described the space of invariants in terms of certain cohomology groups related with differential equations.

Now, "computation of invariants" has become a point of interest. Such a computation may be carried out in many ways. This problem is essentially related with the computation of monodromy matrices and Stokes multipliers. These quantities may be computed numerically, if a differential equation is given. *We are interested in studying these quantities as functions of suitable parameters*. In a local study, this leads us to a perturbation theory (regular and/or singular: cf. W. Balser [2]), or a deformation theory such as recent work on isomonodromic deformations (cf. Flaschka-Newell [4], Jimbo-Miwa-Ueno [7], T. Kimura [10], K. Okamoto [12] and Römer-Schröder [13]) and isoformal deformations (cf. Babbitt-Varadalajan [1]). A motivation of our researches is to study some special but important cases (cf. Y. Sibuya [14]).

Precisely speaking we study solutions of

$$(\delta^2 - p\delta)y - P(x)y = 0 \quad (\delta = xd/dx), \tag{5.1}$$

where

$$P(x) = x^m + \sum_{h=1}^{m-1} a_h x^{m-h}, \tag{5.2}$$

p is an integer such that $0 \leq p \leq m-1$, and a_h $(h = 1, \dots, m-1)$ are parameters. We assume that $x = 0$ *is an apparent singular point*. This assumption implies that a_{m-p} is a certain polynomial in other parameters a_h $(h \neq m-p)$ for each pair (m, p): for example,

$$\begin{aligned} &a_{m-1} = 0 \ (p = 1), \quad a_{m-2} = a_{m-1}^2 \ (p = 2), \\ &a_{m-3} = a_{m-1}a_{m-2} - \tfrac{1}{4}a_{m-1}^3 \ (p = 3), \text{ etc.} \end{aligned} \tag{5.3}$$

We shall denote by a the vector $(a_1, \dots, a_{m-p-1}, a_{m-p+1}, \dots, a_{m-1}) \in C^{m-2}$.

If $p = 1$, Equation (5.1) becomes

$$d^2y/dx^2 - Q(x)y = 0, \tag{5.4}$$

where

$$Q(x) = x^{m-2} + \sum_{h=1}^{m-2} a_h x^{m-h-2}. \tag{5.5}$$

Asymptotic solutions of Equation (5.4) with (5.5) were studied in Y. Sibuya [14]. Many results in this book can be extended to equation (5.1).

PROPOSITION 5.1. *There exist two linearly independent solutions of equation* (5.1):

$$\begin{aligned} \varphi_1(x, a) &= 1 + \sum_{h=1}^{p-1} \varphi_{1h}(a)x^h + \sum_{h=p+1}^{\infty} \varphi_{1h}(a)x^h, \\ \varphi_2(x, a) &= x^p + \sum_{h=p+1}^{\infty} \varphi_{2h}(a)x^h, \end{aligned} \tag{5.6}$$

which are unique and entire in (x, a).

This result is an application of the method of G. Frobenius to (5.1) at $x = 0$. Note that $x = 0$ is an apparent singular point.

PROPOSITION 5.2. *Equation* (5.1) *admits a solution* $\varphi(x, a)$ *such that*

(i) φ *is entire in* (x, a),

(ii) φ *admits an asymptotic representation*

$$\varphi = x^{-b(a)+(2p-m)/4}\left[1 + O(x^{-1/2})\right] \exp[-E(x, a)] \tag{5.7}$$

as $x \to \infty$ *in* $|\arg x| < 3\pi/m$, *where*

$$\left[1 + \sum_{k=1}^{m-1} a_k x^{-k}\right]^{1/2} = 1 + \sum_{h=1}^{\infty} b_h(a) x^{-h},$$

$$E(x, a) = (2/m)x^{m/2} + \sum_{1 \le h < m/2} (2/(m - 2h)) b_h(a) x^{(m-2h)/2},$$

and

$$b(a) = \begin{cases} 0 & \text{if } m \text{ is odd}, \\ b_{m/2}(a) & \text{if } m \text{ is even}. \end{cases}$$

Condition (ii) determines the solution φ *uniquely.*

This is a simple modification of Theorem 6.1 of Y. Sibuya [14] (see also F.E. Mullin [11]).

Set

$$\omega = \exp(2\pi i/m), \quad G(a) = \left(\omega a_1, \dots, \omega^k a_k, \dots, \omega^{m-1} a_{m-1}\right), \tag{5.8}$$

and

$$f_k(x, a) = \varphi\left(\omega^{-k} x, G^{-k}(a)\right). \tag{5.9}$$

PROPOSITION 5.3. *For every integer* k, f_k *is a solution of equation* (5.1) *and*

$$\begin{aligned} f_k &= (\omega^{-k} x)^{-(-1)^k b(a) + (2p-m)/4}\left[1 + O(x^{-1/2})\right] \\ &\quad \times \exp\left[-(-1)^k E(x, a)\right], \\ \delta f_k &= (\omega^{-k} x)^{-(-1)^k b(a) + (2p+m)/4}\left[-1 + O(x^{-1/2})\right] \\ &\quad \times \exp\left[-(-1)^k E(x, a)\right] \end{aligned} \tag{5.10}$$

as $x \to \infty$ *in* $|\arg x - 2\pi k/m| < 3\pi/m$.

This result follows from (5.9), Proposition 5.2 and

$$b_k(G(a)) = \omega^k b_k(a) \quad (k = 1,2,\ldots). \tag{5.11}$$

PROPOSITION 5.4. *For the solutions* φ_1 *and* φ_2 *we have*

$$\varphi_1(\omega x, G(a)) = \varphi_1(x, a), \quad \varphi_2(\omega x, G(a)) = \omega^p \varphi_2(x, a). \tag{5.12}$$

This result is a consequence of the uniqueness of φ_1 and φ_2.

Set

$$\Phi_k(x, a) = \begin{bmatrix} f_k & f_{k+1} \\ \delta f_k & \delta f_{k+1} \end{bmatrix}, \quad W_k(x, a) = \det \Phi_k(x, a). \tag{5.13}$$

PROPOSITION 5.5. *For every integer* k, *we have*

$$\Phi_k(x, a) = \Phi_0(\omega^{-k}x, G^{-k}(a)) \tag{5.14}$$

and

$$W_k(x, a) = 2x^p \omega^{-(-1)^k b(a) - kp - (2p-m)/4}. \tag{5.15}$$

This result follows from (5.11) and Proposition 5.3.

Set

$$\Phi_k(x, a) = \Phi_{k+1}(x, a) S_k(a). \tag{5.16}$$

The matrices S_k are called Stokes multipliers.

PROPOSITION 5.6. *For every integer* k, *we have*

$$S_k(a) = \begin{bmatrix} C_k(a) & 1 \\ \tilde{C}_k(a) & 0 \end{bmatrix}, \tag{5.17}$$

where

$$C_k(a) = (W_{k+1})^{-1} \begin{bmatrix} f_k & f_{k+2} \\ \delta f_k & \delta f_{k+2} \end{bmatrix}, \tag{5.18}$$

$$\tilde{C}_k(a) = -(W_{k+1})^{-1} W_k = -\omega^{-(-1)^k 2b(a) + p}, \tag{5.19}$$

and

$$S_k(a) = S_0(G^{-k}(a)). \tag{5.20}$$

These results can be verified by simple computations.

PROPOSITION 5.7. *The matrices* S_k $(k = 0, \ldots, m-1)$ *satisfy the relation*

$$S_{m-1}(a)S_{m-2}(a)\ldots S_2(a)S_1(a)S_0(a) = I, \tag{5.21}$$

where I *is the* 2 *by* 2 *identity matrix.*

This result follows from the fact that $\Phi_{m+k} = \Phi_k$ (cf. (5.14)). Note that the monodromy group of equation (5.1) is trivial, since $x = 0$ is an apparent singular point.

We are interested in the meaning of relation (5.21). Utilizing (5.20) we can write (5.21) in the form

$$S_0(G^{-m+1}(a))S_0(G^{-m+2}(a))\ldots S_0(G^{-1}(a))S_0(a) = I, \tag{5.22}$$

or

$$S_0(a)S_0(G(a))\ldots S_0(G^{m-2}(a))S_0(G^{m-1}(a)) = I. \tag{5.23}$$

Theorem 1.2 applies to (5.23). Hence, there exists a 2 by 2 matrix $E(a)$ such that

(i) the entries of E and E^{-1} are entire in a,
(ii) S_0 has the form

$$S_0(a) = E(a)S_0(0)E(G(a))^{-1}. \tag{5.24}$$

On the other hand, let us look at relation (5.16). Setting $k = 0$ and utilizing (5.14) we derive

$$\Phi_0(x, a) = \Phi_0(\omega^{-1}x, G^{-1}(a))S_0(a)$$

or

$$S_0(a) = \Phi_0(\omega^{-1}x, G^{-1}(a))^{-1}\Phi_0(x, a). \tag{5.25}$$

Utilizing the two linearly independent solutions φ_1 and φ_2 (cf. Proposition 5.1), we write the two solutions f_0 and f_1 as linear combinations of φ_1 and φ_2, i.e.,

$$\Phi_0(x, a) = \Phi(x, a)\Gamma(a), \tag{5.26}$$

where

$$\Phi(x, a) = \begin{bmatrix} \varphi_1 & \varphi_2 \\ \delta\varphi_1 & \delta\varphi_2 \end{bmatrix}$$

and the entries of Γ and Γ^{-1} are entire in a. The matrix $\Gamma(a)$ is called a central connection matrix. Note that

$$\Phi_0(\omega^{-1}x, G^{-1}(a)) = \Phi(x, a)\begin{bmatrix} 1 & 0 \\ 0 & \omega^{-p} \end{bmatrix}\Gamma(G^{-1}(a)) \tag{5.27}$$

(cf. (5.12)). Therefore, from (5.25), (5.26) and (5.27), we derive

$$S_0(a) = \Gamma(G^{-1}(a))^{-1}\begin{bmatrix} 1 & 0 \\ 0 & \omega^{p} \end{bmatrix}\Gamma(a) \tag{5.28}$$

and hence

$$S_0(0) = \Gamma(0)^{-1}\begin{bmatrix} 1 & 0 \\ 0 & \omega^{p} \end{bmatrix}\Gamma(0). \tag{5.29}$$

Thus we have

$$S_0(a) = F(G^{-1}(a))^{-1}S_0(0)F(a), \tag{5.30}$$

where

$$F(a) = \Gamma(0)^{-1}\Gamma(a). \tag{5.31}$$

(Note that the entries of F and F^{-1} are entire in a.) This means that the matrix E of (5.24) and the matrix $\tilde{F} = F(G^{-1}(a))^{-1}$ satisfy the same relation with S_0 (i.e., (5.24) and (5.30)). However, E and $\tilde{F}$ were constructed through two totally different processes. We shall investigate the relation between E and $\tilde{F}$ more carefully elsewhere. For doing this, we hope that it would be helpful to regard Stokes multipliers S_k as an automorphy factor and E and $\tilde{F}$ as associated automorphic forms (cf. Remark 3.4 (vi)).

We strongly believe that the study of Stokes multipliers as functions of suitably chosen parameters will lead us to a theory similar to that of automorphy factors and automorphic functions (cf., also, Jurkat-Zwiesler [9]).

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