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GENUS ACTIONS OF FINITE SIMPLE GROUPS

BY

ANDREW J. WOLDAR

1. Introduction

An action of a finite group G on a Riemann surface S is called a genus action provided G acts effectively and analytically on S but does not so act on any Riemann surface of lesser genus. The purpose of this paper is to prove:

THEOREM A. *Let G be a finite simple group (simple shall always mean simple nonabelian), $T(r, s, t)$ a Fuchsian triangle group, Δ a surface group, and S the closed Riemann surface induced from the short exact sequence*

$$1 \to \Delta \to T(r, s, t) \to G \to 1.$$

Then either
(i) *G is normal in* $\operatorname{Aut} S$, *the full group of automorphisms of S, or*
(ii) *G is isomorphic to $L_2(7)$ and $(r, s, t) = (3, 3, 7)$.*

THEOREM B. *Let G be a finite simple $(2, s, t)$-group with genus action on the Riemann surface S arising from the short exact sequence*

$$1 \to \Delta \to \Gamma \to G \to 1.$$

Then G is normal in $\operatorname{Aut} S$. *Moreover, if Γ is a triangle group, then* $\operatorname{Aut} S$ *embeds faithfully in* $\operatorname{Aut} G$.

Remark. The requirement in Theorem B that G be $(2, s, t)$-generated is far less restrictive than appearances would at first indicate. Indeed it is a long-standing conjecture that *every* finite simple group is so generated. In particular, the conjecture has been verified for the families of alternating and sporadic groups, among others (see, for example, [1], [2], [3], [4], [8], [15]). Concerning the requirement that Γ be a triangle group, this appears to be the case with

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overwhelming frequency. Indeed, there is no known example of a finite simple group in which Γ is non-triangular in its genus action.

Before proceeding, we give a brief organizational outline of the paper. In Section 2 we provide a somewhat philosophical justification for our interest in genus actions. Section 3 is devoted to preliminaries, including a proof that the genus action of a finite non-abelian (possibly non-simple) $(2, s, t)$-group covers the 2-sphere (Riemann sphere) with either three or four branch points. In Section 4 we present a proof of Theorem A, while in Section 5 we derive some useful generic results concerning genus action. A proof of Theorem B appears in Section 6.

2. Why genus action?

A natural question to ask is "What is the motivation for the theoretical development in this paper?" In particular, why are we interested in genus actions for finite simple groups?

One answer is that we seek a geometrical representation theory of finite simple groups as automorphism groups of Riemann surfaces. The genus actions play the role of irreducible representations in this theory. When these genus actions give rise to full automorphism groups of the related surfaces, our representation theory has a more natural form. This geometrical representation theory in turn has a faithful image in the ordinary integral representation theory of the group by means of the induced homology representation. We seek to relate these integral representations to the "number theory" of finite simple groups. Our genus homology representations are minimal integral representations which integrate in the sense of Schottky-Jung, and as such should have a special status.

As additional motivation, we cite the theorem of Greenberg [6] which asserts that given a finite group G and Riemann surface T, there exists a Riemann surface S such that G is the full automorphism group of S and $T = S/G$. Our results almost give Greenberg's theorem in the case where G is a finite simple group, S is a Riemann surface of least genus on which G acts, and T is the Riemann sphere.

3. Preliminaries

Let G be a finite (r, s, t)-group, i.e., a finite group having presentation given by

$$G = \langle g_1, g_2, g_3 | g_1^r = g_2^s = g_3^t = g_1 g_2 g_3 = 1, \text{etc.} \rangle,$$

with $r, s, t \geq 2$. Then G is the epimorphic image of the triangle group

$$T(r, s, t) = \langle A, B, C | A^r = B^s = C^t = ABC = 1 \rangle,$$

i.e., the group of orientation preserving symmetries of the appropriate plane P generated by rotations of $2\pi/r$, $2\pi/s$, and $2\pi/t$ respectively, about the vertices of a triangle having angles π/r, π/s, and π/t. The plane is spherical if

$$\frac{1}{r}+\frac{1}{s}+\frac{1}{t}>1,$$

euclidean if

$$\frac{1}{r}+\frac{1}{s}+\frac{1}{t}=1,$$

and hyperbolic if

$$\frac{1}{r}+\frac{1}{s}+\frac{1}{t}<1 \quad [9].$$

This leads to a short exact sequence

$$1 \to \Delta \to T(r,s,t) \to G \to 1$$

and an effective analytic action of G on the closed Riemann surface P/Δ. Moreover,

$$P/T(r,s,t)=S^2$$

(S^2 denotes the 2-sphere) and the branched covering $P/\Delta \to P/T(r,s,t)$ has 3 branch points of respective orders r, s, and t. By the Riemann-Hurwitz formula, we compute the genus $g(P/\Delta)$ of P/Δ to be

$$g(P/\Delta)=1+\frac{|G|}{2}\left(1-\frac{1}{r}-\frac{1}{s}-\frac{1}{t}\right).$$

Thus, if we further assume G is $(2,s,t)$-generated, we obtain

$$g\le 1+\frac{|G|}{2}\left(\frac{1}{2}-\frac{1}{s}-\frac{1}{t}\right),$$

where g is the least genus of any surface S which admits an effective and analytic action by G.

PROPOSITION 3.1. *Let G be a finite non-abelian $(2,s,t)$-group and let S be a Riemann surface of least genus on which G acts. Then $S/G=S^2$ and $\pi\colon S\to S/G$ has either 3 or 4 branch points.*

Proof. From above

$$2h - 2 + \sum\left(1 - \frac{1}{n_i}\right) \leq \frac{1}{2} - \frac{1}{s} - \frac{1}{t},$$

where h is the genus of S/G and n_i denotes the order of the branch point x_i of S $(1 \leq i \leq b)$. Clearly $h \leq 1$. If $h = 1$ then $b = 0$, whence G acts fixed point freely on S with orbit space the torus, a contradiction. Thus $h = 0$ and $b \leq 4$. As G cannot act as deck transformation group for the regular unbranched covering $S - \pi^{-1}(x_i) \to \mathbf{C}$, the result follows.

The following theorem, well known to the mathematicians of the previous century, is stated below without proof. (See, for example, [7].)

THEOREM 3.2. *Suppose G is a finite group acting effectively on a closed orientable surface S by orientation preserving homeomorphisms. If $g =$ genus(S/G) and there are b branch points of orders $n_1, \ldots, n_b$, then G has a presentation of the form*

$$\left\langle x_1, y_1, \ldots, x_g, y_g, e_1, \ldots, e_b \,\middle|\, \prod_{i=1}^{g} [x_i, y_i] e_1 \cdots e_b = e_1^{n_1} = \cdots = e_b^{n_b} = 1, \textit{etc.} \right\rangle.$$

As a consequence of Proposition 3.1 and Theorem 3.2, we are able to realize as a surface of least genus for a finite non-abelian $(2, s, t)$-group G the surface P/Δ induced from the short exact sequence

$$1 \to \Delta \to \Gamma \to G \to 1,$$

where Γ is a Fuchsian triangle group or quadrangle group. Thus a presentation for Γ is given by

$$\Gamma = \langle x_1, \ldots, x_k \,|\, x_1^{n_1} = \cdots = x_k^{n_k} = x_1 \cdots x_k = 1 \rangle$$

where $k = 3$ or 4. In general, we refer to the unordered k-tuple $(n_1, \ldots, n_k)$ as the signature of Γ.

DEFINITION. Let G be a finite group whose action on a surface S is induced from the short exact sequence

$$1 \to \Delta \to \Gamma \to G \to 1.$$

We say the group K extends the action of G on S provided there exists a commutative row exact diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \Delta & \longrightarrow & \Lambda & \longrightarrow & K & \longrightarrow & 1 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 1 & \longrightarrow & \Delta & \longrightarrow & \Gamma & \longrightarrow & G & \longrightarrow & 1 \end{array}$$

such that:

(i) Λ is Fuchsian,
(ii) $|\Lambda : \Gamma|$ is finite, and
(iii) The vertical maps are (left to right) the identity map, inclusion map, and a group monomorphism respectively.

We observe that the existence of such a diagram implies that K acts effectively and analytically on S and that the action G inherits as a subgroup of K is consistent with its original action on S. By Aut S we shall mean the group of all automorphisms of S, i.e., all continuous bijective isometries which preserve the analytic structure of S. By a theorem of Schwarz, if S has genus at least 2 then Aut S has finite order, so that Aut S extends the action of G on S, and every extension K of G on S can be realized as a subgroup of Aut S in a natural way.

Suppose now that K extends the action of G on S, and let

$$\gamma = (n_1, \dots, n_k) \quad \text{and} \quad \lambda = (m_1, \dots, m_t)$$

denote the respective signatures of Γ and Λ. As the genus of S is unchanged by extension, we derive the following useful formula, obtained by equating the Euler characteristic at both levels of the appropriate commutative diagram:

$$|K|\left[2 - 2l - \sum\left(1 - \frac{1}{m_i}\right)\right] = |G|\left[2 - 2h - \sum\left(1 - \frac{1}{n_i}\right)\right].$$

(Here h denotes the genus of S/G and l that of S/K.) We shall use this formula later to establish an upper bound on the index $|K : G|$.

4. Proof of Theorem A

For the reader's convenience, we begin with a restatement of the theorem.

THEOREM A. *Let G be a finite simple group, $T(r, s, t)$ a Fuchsian triangle group, Δ a surface group, and S the surface induced from the short exact sequence*

$$1 \to \Delta \to T(r, s, t) \to G \to 1.$$

Then either

(i) *G is normal in* Aut S *or*

(ii) $G \cong L_2(7)$ *and* $(r, s, t) = (3,3,7)$.

Proof. Let K extend the action of G on S in accordance with the commutative row-exact diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \Delta & \longrightarrow & \Lambda & \longrightarrow & K & \longrightarrow & 1 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 1 & \longrightarrow & \Delta & \longrightarrow & T(r,s,t) & \longrightarrow & G & \longrightarrow & 1. \end{array}$$

As $T(r, s, t) \leq \Lambda$, we know Λ is a triangle group as well. All possible inclusions among triangle groups are given in [14]; as the normal inclusions clearly imply (i) of the theorem, we concern ourselves only with the non-normal inclusions found there. These appear in Table 1, γ and λ denoting the respective signatures of $\Gamma = T(r, s, t)$ and Λ.

Table 1

	γ	λ	$\lvert\Lambda : \Gamma\rvert$
A	$(7,7,7)$	$(2,3,7)$	24
B	$(2,7,7)$	$(2,3,7)$	9
C	$(3,3,7)$	$(2,3,7)$	8
D	$(4,8,8)$	$(2,3,8)$	12
E	$(3,8,8)$	$(2,3,8)$	10
F	$(9,9,9)$	$(2,3,9)$	12
G	$(4,4,5)$	$(2,4,5)$	6
H	$(n,4n,4n)$	$(2,3,4n)$	6
I	$(n,2n,2n)$	$(2,4,4n)$	4
J	$(3,n,3n)$	$(2,3,3n)$	4
K	$(2,n,2n)$	$(2,3,2n)$	3

We begin by analyzing the cases where $|\Lambda : \Gamma| > 10$, specifically A, D, and F of Table 1. From Singerman's classification of finitely maximal Fuchsian groups [14], and the fact that two triangle groups are conjugate in PGL(2, $\mathbf{C}$) if and only if they have the same signature, we see that in each of the cases corresponding to A, D, and F there exists an intermediate triangle group Γ' with signature γ' and indices as shown in Table 2.

Denote by r and s the respective indices $|\Lambda : \Gamma'|$ and $|\Gamma' : \Gamma|$ with Γ, Γ', Λ as in Table 2. Then $r = |K : H|$ and $s = |H : G|$ where H is the image of Γ' under the epimorphism $\Lambda \to K$. Consider now the action of K on its H-cosets (i.e., the cosets of H in K) and let N denote the kernel of this action. Clearly K/N embeds in the symmetric group Σ_r on r letters and, as H stabilizes the trivial coset $1H$ under this action, H/N embeds in Σ_{r-1}. Thus $G/G \cap N \cong GN/N$ embeds in Σ_{r-1} as well. As G is simple, we must have $G \cap N = 1$ or

Table 2

	γ	γ'	$\lvert\Lambda : \Gamma'\rvert$	$\lvert\Gamma' : \Gamma\rvert$
A	$(7,7,7)$	$(3,3,7)$	8	3
D	$(4,8,8)$	$(2,8,8)$	6	2
F	$(9,9,9)$	$(3,3,9)$	4	3

Table 3

γ	λ	$r = \lvert\Lambda : \Gamma\rvert$	Possibilities for G
$(2,7,7)$	$(2,3,7)$	9	$A_5, A_6, A_7, A_8, L_2(7)$
$(3,3,7)$	$(2,3,7)$	8	$A_5, A_6, A_7, L_2(7)$
$(3,8,8)$	$(2,3,8)$	10	$A_5, A_6, A_7, A_8, A_9, L_2(7), L_2(8)$
$(4,4,5)$	$(2,4,5)$	6	A_5
$(n,4n,4n)$	$(2,3,4n)$	6	A_5
$(n,2n,2n)$	$(2,4,2n)$	4	None
$(3,n,3n)$	$(2,3,3n)$	4	None
$(2,n,2n)$	$(2,3,2n)$	3	None

G. If $G \cap N = G$, then $G \leq N$, whence we conclude $N = G$ or $N = H$ as $s = |H : G|$ is prime. But $N = G$ contradicts our assumption that $\Gamma \subseteq \Lambda$ is a non-normal inclusion, while $N = H$ implies $\Gamma' \lhd \Lambda$, again a contradiction (see C, H, J of Table 1). Thus $G \cap N = 1$ and we obtain a faithful embedding of G in Σ_{r-1}, i.e., G is isomorphic to a simple subgroup of Σ_7, Σ_5, or Σ_3 in accordance with Γ having signature $(7,7,7)$, $(4,8,8)$, and $(9,9,9)$ respectively. As neither Σ_5 nor Σ_3 possess elements of order 8 or 9, the latter two cases can never occur. We conclude that G is isomorphic to either $L_2(7)$ or A_7 and that $\Gamma = T(7,7,7)$.

But in either case, as $|H : G| = 3$, we see that H must itself embed in A_7. This is an obvious contradiction as copies of $L_2(7)$ in A_7 are maximal. Thus none of the three cases A, D, and F of Table 1 occurs.

The remaining cases are enumerated in Table 3. To treat them we no longer require the existence of an intermediate Fuchsian group. We merely note that the action of K on its G-cosets gives rise to a faithful embedding of K in Σ_r, and that under this embedding G gets mapped to a simple subgroup of A_{r-1}, the alternating group on $r - 1$ letters. All possibilities for G have been recorded in the table.

Case 1. $\gamma = (2,7,7)$. We first observe that G cannot be isomorphic to either A_5 or A_6 as neither of these groups possess elements of order 7. Suppose G is isomorphic to A_7. As $|K : G| = 9$ this implies K is isomorphic to a subgroup of index 8 in Σ_9. As no such group exists we obtain a contradiction. Finally suppose G is isomorphic to $L_2(7)$, in which case K has order $2^3 \cdot 3^3 \cdot 7$. Let M be maximal normal in K. As K is a $(2,3,7)$-group and 2, 3, and 7 are relatively prime, it follows that K/M is a simple $(2,3,7)$-group, whence $K/M \cong L_2(7)$. Thus K embeds in a 3-local subgroup of Σ_9. As no 3-local of Σ_9 contains a copy of $L_2(7)$, we obtain the final contradiction.

Case 2. $\gamma = (3,3,7)$. Once again G contains elements of order 7, so cannot be isomorphic to A_5 or A_6. Suppose G is isomorphic to A_7. As $|K:G| = 8$ in this case, we see that K is isomorphic to a subgroup of index 2 in Σ_8. Thus $K \cong A_8$, a contradiction as A_8 is not $(2,3,7)$-generated [4].

Case 3. $\gamma = (3,8,8)$. This case is easily handled as none of the groups listed possess elements of order 8.

Case 4. $\gamma = (4,4,5)$. Not possible as A_5 contains no elements of order 4.

Case 5. $\gamma = (n,4n,4n)$. Not possible as A_5 contains no elements of order 4.
Clearly no other cases can arise, so the theorem is proved.

Remark. The exceptional case in the theorem actually occurs. The associated diagram is given by

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \Delta & \longrightarrow & T(2,3,7) & \longrightarrow & \operatorname{Aut} S & \longrightarrow & 1 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 1 & \longrightarrow & \Delta & \longrightarrow & T(3,3,7) & \longrightarrow & G & \longrightarrow & 1 \end{array}$$

where $\operatorname{Aut} S$ is isomorphic to the holomorph $\operatorname{Hol}(E_8)$ of an elementary abelian group of order 8.

5. Extensions of genus actions

In this section we specialize to genus actions of G. Our first result gives conditions under which the genus action of a finite hyperbolic group cannot be properly extended.

PROPOSITION 5.1. *Let G be a finite hyperbolic group whose genus action arises from the short exact sequence*

$$1 \to \Delta \to \Gamma \to G \to 1$$

where Γ is a triangle group. Suppose G is the epimorphic image of a group K which extends the action of G on S. Then $K = G$.

Proof. By hypothesis, there exists a commutative row exact diagram of the form

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \Delta & \longrightarrow & \Lambda & \longrightarrow & K & \longrightarrow & 1 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 1 & \longrightarrow & \Delta & \longrightarrow & \Gamma & \longrightarrow & G & \longrightarrow & 1. \end{array}$$

As Γ is a triangle group by hypothesis, Λ is triangular as well. Denote by (n_1, n_2, n_3) and (m_1, m_2, m_3) the respective signatures of Γ and Λ. We thereby obtain from Section 3 the equation

$$|G|\left(2 - \sum\left(1 - \frac{1}{n_i}\right)\right) = |K|\left(2 - \sum\left(1 - \frac{1}{m_i}\right)\right).$$

Clearly K is an (m_1, m_2, m_3)-group. By assumption there exists a group epimorphism ρ: $K \to G$; let m_i^* denote the order of $\rho(x)$ where $x \in K$ has order m_i. Then G is an (m_1^*, m_2^*, m_3^*)-group and, as a genus action gives rise to the largest possible Euler characteristic, we obtain

$$|G|\left(2 - \sum\left(1 - \frac{1}{m_i^*}\right)\right) \le |K|\left(2 - \sum\left(1 - \frac{1}{m_i}\right)\right).$$

As $m_i^* \le m_i$, it immediately follows that

$$|G|\left(2 - \sum\left(1 - \frac{1}{m_i}\right)\right) \le |K|\left(2 - \sum\left(1 - \frac{1}{m_i}\right)\right).$$

Finally, as G is hyperbolic, we observe that the quantity $2 - \Sigma(1 - 1/m_i^*)$ is necessarily negative. This implies that $2 - \Sigma(1 - 1/m_i)$ is negative as well and division yields $|K| \le |G|$. As K extends G the result follows.

PROPOSITION 5.2. *Let G be a finite simple group with genus action on the surface S arising from*

$$1 \to \Delta \to \Gamma \to G \to 1$$

where Γ is a triangle group. Suppose further that G is normal in a group K which extends this action. Then $C_K(G) = 1$. In particular, if G is normal in Aut S *we have $C_{\mathrm{Aut}\, S}(G) = 1$ and* Aut S *embeds faithfully in* Aut G.

Proof. It is a classical result that the only finite simple (non-abelian) group which occurs among the spherical and euclidean groups is the spherical group A_5. But A_5 is clearly maximal in its action on the sphere, (otherwise A_5 would embed in a cyclic or dihedral group). Thus $K = G$ in this case and $C_K(G) =$

$Z(G) = 1$ as required. We may therefore assume G is hyperbolic. Suppose x is an arbitrary non-identity element of $C_K(G)$. As $Z(G) = 1$, x is an element of $K - G$ whence $G\langle x\rangle \cong G \times \langle x\rangle$ properly extends the genus action of G. But G is the epimorphic image of $G\langle x\rangle$ under the projection $gx^i \mapsto g$. This contradicts Proposition 5.1, and we conclude that $C_K(G) = 1$ as claimed. As the automizer $N_K(G)/C_K(G)$ of G in K is isomorphic to a subgroup of Aut G and as $N_K(G) = K$ by assumption, we conclude that K embeds faithfully in Aut G. Choosing K equal to Aut S now completes the proof.

6. Proof of Theorem B

We begin with a series of technical lemmas. The reader is reminded that for us simple always means simple non-abelian.

LEMMA 6.1. *Let G be a simple subgroup of A_{19}, the alternating group on 19 letters. Then G is isomorphic to one of the following groups*:

$$A_n (5 \le n \le 19),\ L_2(7),\ L_2(8),\ L_2(11),$$
$$L_2(13),\ L_2(16),\ L_2(17),\ L_3(3),\ M_{11},\ M_{12}.$$

Proof. This is immediate from a classification of Sims of primitive permutation groups of small degree [13].

LEMMA 6.2. *Let G be a finite noneuclidean group with genus action on the surface S induced by the sequence*

$$1 \to \Delta \to \Gamma \to G \to 1.$$

Assume further that the genus g corresponding to this action satisfies

$$g < 1 + \frac{|G|}{12}.$$

Then Γ is a triangle group.

Proof. Trivial.

LEMMA 6.3. *Let G be a simple subgroup of A_{19} with genus action induced by the sequence*

$$1 \to \Delta \to \Gamma \to G \to 1.$$

Then G is normal in any group K which extends this action.

Proof. We first show that Γ must be a triangle group. This has been accomplished for the alternating groups [4] and for the two-dimensional

Table 4

G	(r, s, t)	g^*	$1 + \frac{\lvert G\rvert}{12}$
$L_3(3)$	$(2,3,13)$	253	469
M_{11}	$(2,4,11)$	631	661
M_{12}	$(2,3,10)$	3169	7921

projective linear groups [8], [5], [12]. The remaining subgroups G of A_{19} are given in Lemma 6.1 as $L_3(3)$, M_{11}, and M_{12}. In each case we note from [11] that G is an (r, s, t)-group, and we calculate the corresponding genus g^* of P/Δ induced from

$$1 \to \Delta \to T(r, s, t) \to G \to 1$$

by the formula

$$g^* = 1 + \frac{|G|}{2}\left(\frac{1}{r} + \frac{1}{s} + \frac{1}{t} - 1\right).$$

We list our findings in Table 4 below. As $g^* < 1 + |G|/12$ in each case, we conclude that $g < 1 + |G|/12$ where g is the least genus of a surface on which G acts. Thus by Lemma 6.2, Γ is triangular as claimed. By Theorem A, G is normal in K unless $G \cong L_2(7)$. But the genus action for $L_2(7)$ is induced by the sequence

$$1 \to \Delta \to T(2,3,7) \to L_2(7) \to 1,$$

and one easily checks from [14] that this action admits no proper extension. Thus $G = K$ for G isomorphic to $L_2(7)$ and the lemma is proved.

We now proceed to the proof of:

THEOREM B. *Let G be a finite simple $(2, s, t)$-group with genus action on the Riemann surface S arising from the short exact sequence*

$$1 \to \Delta \to \Gamma \to G \to 1.$$

Then G is normal in Aut S. *Moreover, if Γ is a triangle group, then* Aut S *embeds faithfully in* Aut G.

Proof. Let K be any group which extends the action of G on S in accordance with the commutative row exact diagram.

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \Delta & \longrightarrow & \Lambda & \longrightarrow & K & \longrightarrow & 1 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 1 & \longrightarrow & \Delta & \longrightarrow & \Gamma & \longrightarrow & G & \longrightarrow & 1. \end{array}$$

We show G is a normal subgroup of K.

Recall from Section 3 that the genus g of S satisfies

$$g \leq 1 + \frac{|G|}{2}\left(\frac{1}{2} - \frac{1}{s} - \frac{1}{t}\right)$$

as G is $(2, s, t)$-generated. In terms of Euler characteristics this gives

$$\begin{aligned} \chi(S) &\geq |G|\left[2 - \left(1 - \frac{1}{2}\right) - \left(1 - \frac{1}{s}\right) - \left(1 - \frac{1}{t}\right)\right] \\ &= |G|\left(\frac{1}{s} + \frac{1}{t} - \frac{1}{2}\right) \\ &> -\frac{1}{2}|G|. \end{aligned}$$

As we have already shown that A_5 cannot be properly extended in its genus action (proof of Proposition 5.2), the theorem follows in this case. We also recall that A_5 is the only finite simple group which is not hyperbolic. We may therefore assume for the balance of the proof that G, so K as well, is hyperbolic.

The largest Euler characteristic that can be attained by any hyperbolic group H is given by

$$|H|\left[2 - \left(1 - \frac{1}{2}\right) - \left(1 - \frac{1}{3}\right) - \left(1 - \frac{1}{7}\right)\right] = -\frac{1}{42}|H|.$$

(This is the case where H is a Hurwitz group, i.e., $(2, 3, 7)$-generated.) Applying this to the case where H is the extended group K, we obtain

$$\chi(S) = |K|\left[2 - 2l - \sum\left(1 - \frac{1}{m_i}\right)\right] \leq -\frac{1}{42}|K|$$

so that

$$-\frac{1}{2}|G| < \chi(S) \leq -\frac{1}{42}|K|.$$

As easy calculation now yields $|K : G| < 21$; i.e., $|K : G| \leq 20$.

Consider next the action of K on its G-cosets. Clearly the kernel N of this action is a normal subgroup of K contained in G. As G is simple by assumption, we have either $N = 1$ or $N = G$. In the latter case G is normal in K as desired. In the former case K embeds in Σ_{20}, whence G embeds in A_{19} as it is a simple group which stabilizes the trivial coset $1G$. Application of Lemma 6.3 now yields the desired conclusion $G \triangleleft K$. We complete the proof by applying Proposition 5.2 to the case $K = \mathrm{Aut}\, S$, under the additional assumption that Γ is triangular.

Remark. It is of interest to determine which signatures $(m_1, \ldots, m_k)$ for Γ might actually arise from a genus action for a finite simple $(2, s, t)$-group G. The reader will have little trouble verifying them to be

1. (m_1, m_2, m_3) where $\dfrac{1}{m_1} + \dfrac{1}{m_2} + \dfrac{1}{m_3} > \dfrac{1}{2}$,
2. $(2, 2, 2, n)$, $n > 2$,
3. $(2, 2, 3, t)$, $3 \leq t \leq 5$.

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VILLANOVA UNIVERSITY
VILLANOVA, PENNSYLVANIA