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HILBERT-SCHMIDT INTERPOLATION IN CSL-ALGEBRAS

BY

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One form of interpolation problem in operator algebras is the following: given vectors x and y in a Hilbert space $\mathscr{H}$ and an operator algebra $\mathscr{A}$ acting on $\mathscr{H}$, does there exist an operator T in $\mathscr{A}$ such that $Tx = y$? Furthermore, if interpolation is possible for a pair of vectors x and y, what is the minimum norm possible for an operator T which maps x to y? A variation of this problem asks for the interpolation of a linearly independent set of vectors $\{x_1, \dots, x_n\}$ onto a second set of vectors $\{y_1, \dots, y_n\}$.

An early and particularly deep example of this type of interpolation theorem is Kadison's Transitivity theorem [7]: if $\mathscr{A}$ is a C^*-algebra which acts irreducibly on $\mathscr{H}$, if $\{x_1, \dots, x_n\}$ is a linearly independent set of vectors in $\mathscr{H}$ and if $\{y_1, \dots, y_n\}$ is any set of vectors in $\mathscr{H}$, then there is an operator T in $\mathscr{A}$ such that $Tx_i = y_i$, for all i. Another example is the following theorem, which was first proven for nest algebras by Lance [9] and then extended to all CSL algebras in [5]: let Alg $\mathscr{L}$ be a reflexive operator algebra with a commutative subspace lattice $\mathscr{L}$. Let x and y be vectors in $\mathscr{H}$. Then there is an operator $T \in \operatorname{Alg} \mathscr{L}$ such that $Tx = y$ if, and only if,

$$\sup_{E \in \mathscr{L}} \frac{\|E^{\perp} y\|}{\|E^{\perp} x\|} < \infty.$$

If this supremum is finite, then it is the minimum norm for an interpolating operator for x and y. (A fraction with both numerator and denominator equal to 0 is taken to be 0.)

A recent paper by N.J. Munch [12] solves the interpolation problem in the setting of nest algebras subject to the additional restriction that the interpolating operator must be a Hilbert-Schmidt operator. As it happens, nest algebras are of interest in linear system theory; indeed, Munch interprets some of his results in terms of signal reconstruction. Some authors in system theory have found it convenient to go beyond Hilbert resolution space and the correspond-

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ing causal operators (i.e., nest algebras) to consider partially ordered Hilbert resolution spaces (see, for example, [2], [3], [4], [13]). This setting is actually equivalent to commutative subspace lattices and their corresponding reflexive algebras, first introduced by Arveson in [1]. Thus it is natural both from a system theory point of view and an operator algebraic point of view to consider interpolation problems in the context of CSL algebras. The primary purpose of this note is to extend most of Munch's results to CSL algebras. We do so with proofs which are more general and, at the same time, somewhat simpler.

A commutative subspace lattice, $\mathscr{L}$, is a strongly closed lattice of projections acting on a separable Hilbert space $\mathscr{H}$. We always assume that the trivial projections 0 and I are contained in $\mathscr{L}$. The associated CSL algebra, Alg $\mathscr{L}$, is the algebra of all bounded operators acting on $\mathscr{H}$ which leave invariant each projection of $\mathscr{L}$. The main tool which we use is Arveson's representation theorem [1, Theorem 1.3.1]: let X be a compact metric space, let $\leq$ be a reflexive and transitive order on X whose graph is closed, and let μ be a finite Borel measure on X. Define a Borel subset A of X to be increasing if $x \in A$ and $x \leq y$ imply $y \in A$. For each Borel subset A of X let $E(A)$ denote the orthogonal projection acting on the Hilbert space $\mathscr{H} = L^2(X, \mu)$ which arises from multiplication by the characteristic function of A. The family of all projections $E(A)$ corresponding to increasing Borel sets A forms a commutative subspace lattice, which we denote by $\mathscr{L}(X, \leq, \mu)$. Every commutative subspace lattice acting on a separable Hilbert space is unitarily equivalent to the lattice of the form $\mathscr{L}(X, \leq, \mu)$.

This representation will replace the vector valued and operator valued measures used in [12]. For a specific nest algebra, it is much the same to determine the ingredients of the Arveson representation as the measures used in [12]. The Arveson representation, however, is particularly well suited to problems involving Hilbert-Schmidt operators, for these may be identified with L^2-kernel functions on $X \times X$. Furthermore, the Arveson representation permits a natural extension of many of the results in nest algebras to CSL algebras.

It will be notationally convenient in the sequel to let the same symbol denote both an orthogonal projection and its range subspace. Thus, for a projection $E \in \mathscr{L}$ and a vector $x \in \mathscr{H}$ we may write interchangeably $Ex = x$ and $x \in E$.

Before generalizing Munch's results on interpolation by Hilbert-Schmidt operators, we treat the (much simpler) case of interpolation by rank-one operators. Interpolation by trace class operators and by compact operators present two other interesting and challenging problems.

The solution to the rank-one interpolation problem requires the following definition: if $E \in \mathscr{L}$ let $E_- = \bigvee\{F \in \mathscr{L} | E \nleq F\}$. When $\mathscr{L}$ is a nest, E_- is either the immediate predecessor of E, when the immediate predecessor exists, or else $E_- = E$. If u and w are vectors in $\mathscr{H}$, let $u \otimes w^*$ denote the rank-one operator given by $u \otimes w^*(x) = \langle x, w \rangle u$. A well-known lemma of Longstaff [11] (Ringrose in the nest algebra case) asserts that $u \otimes w^* \in \operatorname{Alg} \mathscr{L}$ if, and only if, there is a projection $E \in \mathscr{L}$ such that $w \in (E_-)^\perp$ and $u \in E$.

Proposition 1. *Let x and y be non-zero vectors on $\mathscr{H}$ and let $\mathscr{L}$ be a commutative subspace lattice. Let $E = \bigwedge\{F \in \mathscr{L} | Fy = y\}$. There is a rank-one operator $u \otimes w^*$ in $\operatorname{Alg} \mathscr{L}$ such that $u \otimes w^*(x) = y$ if, and only if, $E_-^\perp x \neq 0$. If $E_-^\perp x \neq 0$, then the minimal norm for a rank-one interpolation operator is $\|y\| / \|E_-^\perp x\|$.*

Proof. Assume that $E_-^\perp x \neq 0$. Let $w = \|E_-^\perp x\|^{-2} E_-^\perp x$ and $u = y$. Then $u \otimes w^*$ is a rank-one operator in $\operatorname{Alg} \mathscr{L}$ which maps x to y; furthermore,

$$\|u \otimes w^*\| = \frac{\|y\|}{\|E_-^\perp x\|}.$$

Conversely, suppose that $u \otimes w^* \in \operatorname{Alg} \mathscr{L}$ and maps x to y. Since $y = u \otimes w^*(x) = \langle x, w \rangle u$, we see that u is a scalar multiple of y. In particular, $u \in E$ and E is the smallest projection in $\mathscr{L}$ containing u. Consequently, $w \in E_-^\perp$. But then, $0 \neq \langle x, w \rangle = \langle x, E_-^\perp w \rangle = \langle E_-^\perp x, w \rangle$, whence $E_-^\perp x \neq 0$. It is now clear that $u \otimes w^*$ also maps $E_-^\perp x$ to y, from which it follows that $\|u \otimes w^*\| \geq \|y\| / \|E_-^\perp x\|$.

The condition in Proposition 1 can be modified to obtain a necessary and sufficient condition for the existence of a finite-rank operator in $\operatorname{Alg} \mathscr{L}$ which maps x to y, provided that $\mathscr{L}$ is generated by finitely many nests. ($\mathscr{L}$ is said to have *finite width* in this case.)

Proposition 2. *Let x and y be non-zero vectors in $\mathscr{H}$ and let $\mathscr{L}$ be a finite width commutative subspace lattice. There exists a finite rank operator T such that $Tx = y$ if, and only if, there exist finitely many projections $E_1, \ldots, E_n$ in $\mathscr{L}$ such that $(E_i)_-^\perp x \neq 0$ for all i, and $y \in E_1 \vee \cdots \vee E_n$.*

Proof. Let $E = \bigwedge\{F \in \mathscr{L} | Fy = y\}$. Suppose T is a finite rank operator in $\operatorname{Alg} \mathscr{L}$ such that $Tx = y$. Since $\mathscr{L}$ has finite width, we may apply a theorem in [6] to write T as a sum of rank-one operators in $\operatorname{Alg} \mathscr{L}$, say

$$T = \sum_{i=1}^{n} u_i \otimes w_i^*,$$

where each $u_i \otimes w_i^* \in \operatorname{Alg} \mathscr{L}$.

Since the projection E lies in Alg $\mathscr{L}$, ET is another operator in Alg $\mathscr{L}$ which maps x to y. Therefore, we may replace T by ET and assume that $u_i \in E$ for all i. We may also assume without loss of generality that $\langle x, w_i\rangle \neq 0$ for all i. (Simply delete from the sum any terms for which $x \perp w_i$.)

For each $i = 1, \dots, n$, let E_i be the smallest projection in $\mathscr{L}$ which contains u_i. Then

$$y = \sum_{i=1}^{n} \langle x, w_i\rangle u_i \in E_1 \vee \cdots \vee E_n.$$

(In fact, $E = E_1 \vee \cdots \vee E_n$.) Since $u_i \otimes w_i^* \in$ Alg $\mathscr{L}$ and E_i is the smallest projection containing u_i, we have $w_i \in (E_i)_-^\perp$ for each i. Thus $(E_i)_-^\perp x \neq 0$ for all i and the condition in the proposition is satisfied.

For the converse, assume that $E_1, \dots, E_n$ in $\mathscr{L}$ are such that

$$y \in E_1 \vee \cdots \vee E_n$$

and $(E_i)_-^\perp x \neq 0$ for all i. Let $F_1 = E_1$ and

$$F_i = E_i \wedge (E_1 \wedge \cdots \wedge E_{i-1})^\perp, \quad i = 2, \dots, n.$$

Then the F_i are pairwise orthogonal and $F_1 \vee \cdots \vee F_n = E_1 \vee \cdots \vee E_n$. Let

$$u_i = F_i y \quad \text{and} \quad w_i = \|(E_i)_-^\perp x\|^{-2} (E_i)_-^\perp x$$

for each i. Since $F_i \leq E_i$, each rank-one operator $u_i \otimes w_i^*$ belongs to Alg $\mathscr{L}$. Furthermore, $(u_i \otimes w_i^*)x = u_i$ for each i. Finally, since $y = \sum_{i=1}^n u_i$ by the construction, the operator $T = \sum_{i=1}^n u_i \otimes w_i^*$ is a finite rank operator in Alg $\mathscr{L}$ which maps x onto y.

Remark. The finite width of the lattice $\mathscr{L}$ was not used in the proof of the sufficiency of the condition. Thus, in any CSL algebra, if there are finitely many projections $E_1, \dots, E_n$ in $\mathscr{L}$ satisfying $(E_i)_-^\perp x \neq 0$ for all i, and $y \in E_1 \vee \cdots \vee E_n$ then there is a finite rank operator T in the algebra such that $Tx = y$. If a lattice is not finite width, then it is possible that there exists a finite rank operator in Alg $\mathscr{L}$ which cannot be written as a sum of rank one operators in Alg $\mathscr{L}$. An example of such an operator with rank 2 is given in [6]. If T denotes this operator and if x and y are vectors such that $Tx = y$, then it is possible to find projections E_1, E_2 in $\mathscr{L}$ such that $(E_i)_-^\perp x \neq 0$, $i = 1, 2$ and $y \in E_1 \vee E_2$. Thus we are left with the following questions:

If T is a finite rank operator in a CSL algebra, Alg $\mathscr{L}$, such that $Tx = y$, then do there exist projections $E_1, \ldots, E_n$ in $\mathscr{L}$ such that $(E_i)_-^\perp x \neq 0$ for all i, and $y \in E_1 \vee \cdots \vee E_n$? Can n be chosen so that $n \leq \operatorname{rank} T$?

We now turn to the main questions to be addressed: given a CSL $\mathscr{L}$ and vectors $x, y \in \mathscr{H}$, when does there exist a Hilbert-Schmidt operator T in Alg $\mathscr{L}$ such that $Tx = y$? How should T be chosen so as to minimize the Hilbert-Schmidt norm $\|T\|_2$?

We assume hereafter that $\mathscr{L}$ has the form $\mathscr{L} = \mathscr{L}(X, \leq, \mu)$ as described above. Hilbert-Schmidt operators in $\mathscr{B}(L^2(X, \mu))$ are associated with L^2-kernel functions on $X \times X$ with respect to the measure $\mu \times \mu$. By a result in [1], a Hilbert-Schmidt operator lies in Alg $\mathscr{L}$ if, and only if, the support of the associated kernel function lies in the graph of the relation $\leq$.

Let G denote the characteristic function of the graph of $\leq$, i.e.,

$$G(s, t) = \begin{cases} 1 & \text{if } s \leq t \\ 0 & \text{otherwise.} \end{cases}$$

If x and y are vectors in $L^2(X, \mu)$ then the function $\overline{x(s)}\, y(t)$ on $X \times X$ is the kernel function for the rank-one operator $y \otimes x^*$. While this operator carries x to a scalar multiple of y, it generally fails to belong to Alg $\mathscr{L}$. This can be remedied by multiplying the kernel function $\overline{x(s)}\, y(t)$ by G. A further adjustment is then needed so that the resulting operator once again carries x to y: divide $\overline{x(s)}\, y(t) G(s, t)$ by the function $N_x(t)$ defined by

$$N_x(t) = \int |x(s)|^2 G(s, t)\, d\mu(s).$$

We then have the following generalization of Theorem 1.1 in [12].

THEOREM 1. *Let $\mathscr{L} = \mathscr{L}(X, \leq, \mu)$ be a commutative subspace lattice on the Hilbert space $L^2(X, \mu)$. Let $x, y \in L^2(X, \mu)$. Define*

$$N_x(t) = \int |x(s)|^2 G(s, t)\, d\mu(s) \quad \textit{for each } t \in X.$$

Let

$$C_{x,y} = \int \frac{|y(t)|^2}{N_x(t)}\, d\mu(t).$$

(Interpret $0/0$ as 0 in the integrand.) Then there is a Hilbert-Schmidt operator, T, in Alg $\mathscr{L}$ *such that $Tx = y$ if, and only if, $C_{x,y}$ is finite. If $C_{x,y} < \infty$, then*

this number is the minimum value for $\|T\|_2^2$; *the operator* T_h *associated with the kernel function*

$$h(s,t) = \frac{\overline{x(s)}y(t)G(s,t)}{N_x(t)}$$

is Hilbert-Schmidt, maps x *to* y, *and has Hilbert-Schmidt norm equal to* $C_{x,y}^{1/2}$.

Proof. Suppose first that there is a Hilbert-Schmidt operator T in Alg $\mathscr{L}$ such that $Tx = y$. Then there is a function $k \in L^2(X \times X, \mu \times \mu)$ which is supported on the graph of $\leq$ such that T is equal to the integral operator T_k. We claim that $C_{x,y} \leq \|k\|_2^2 = \|T\|_2^2$. Since $Tx = y$, we have for almost all t,

$$y(t) = \int k(s,t)x(s)\,d\mu(s) = \int k(s,t)x(s)G(s,t)\,d\mu(s).$$

Using the Cauchy-Schwartz inequality, we have

$$\begin{aligned}|y(t)|^2 &\leq \int |k(s,t)|^2\,d\mu(s) \cdot \int |x(s)|^2 G(s,t)\,d\mu(s)\\ &= N_x(t) \cdot \int |k(s,t)|^2\,d\mu(s).\end{aligned}$$

Therefore

$$\begin{aligned}C_{x,y} &= \int \frac{|y(t)|^2}{N_x(t)}\,d\mu(t) \leq \int\int |k(s,t)|^2\,d\mu(s)\,d\mu(t)\\ &= \|k\|_2^2 = \|T\|_2^2.\end{aligned}$$

Now suppose that $C_{x,y} < \infty$ and let

$$h(s,t) = \frac{\overline{x(s)}\,y(t)G(s,t)}{N_x(t)}.$$

We first observe that $h \in L^2(X \times X, \mu \times \mu)$. Indeed, using Fubini's theorem,

$$\begin{aligned}\int |h(s,t)|^2\,d\mu \times \mu(s,t) &= \int\int \frac{|x(s)|^2|y(t)|^2}{N_x(t)^2} G(s,t)\,d\mu(s)\,d\mu(t)\\ &= \int \frac{N_x(t)|y(t)|^2}{N_x(t)^2}\,d\mu(t)\\ &= \int \frac{|y(t)|^2}{N_x(t)}\,d\mu(t)\\ &= C_{x,y}.\end{aligned}$$

Thus, the Hilbert-Schmidt operator T_h associated with h satisfies $\|T_h\|_2^2 = C_{x,y}$. Also, $T_h \in \operatorname{Alg}\mathscr{L}$ since h is supported on the graph of $\leq$. All that remains is to show $T_h x = y$:

$$\begin{aligned}(T_h x)(t) &= \int h(s,t)x(s)\,d\mu(s)\\ &= \int \frac{\overline{x(s)}\,y(t)G(s,t)x(s)\,d\mu(s)}{N_x(t)}\\ &= \frac{y(t)}{N_x(t)}\int |x(s)|^2 G(s,t)\,d\mu(s)\\ &= \frac{y(t)}{N_x(t)}\cdot N_x(t)\\ &= y(t) \quad \text{for almost all } t.\end{aligned}$$

(It should be noted that the assumption that $C_{x,y} < \infty$ implies that $N_x(t) > 0$ for almost all t for which $y(t) \neq 0$.)

Remarks. If $T = T_h$ is the operator constructed in Theorem 1 and if S is any other operator in $C_2 \cap \operatorname{Alg}\mathscr{L}$ such that $Sx = y$ then $\operatorname{Tr}(S^*S) \geq \operatorname{Tr}(T^*T)$. A somewhat stronger minimality property for T is valid. To state this, we use the natural projection valued measure, E, defined on X: if Ω is a Borel subset of X, $E(\Omega)$ is the orthogonal projection on $L^2(X,\mu)$ given by multiplication by the characteristic function, χ_Ω, of Ω. With S and T as above, we have

$$\operatorname{Tr}(S^*E(\Omega)S) \geq \operatorname{Tr}(T^*E(\Omega)T)$$

for all Borel subsets Ω of X. (This relation can be denoted $S >_E T$.)

To verify this, let k be the kernel function associated with S. A routine computation shows that the kernel function for $E(\Omega)S$ is $\chi_\Omega(t)k(s,t)$. It is also easy to verify that $C_{x,E(\Omega)y} \leq C_{x,y}$ and that the kernel function for the Hilbert-Schmidt operator in $\operatorname{Alg}\mathscr{L}$ of minimal Hilbert-Schmidt norm mapping x to $E(\Omega)y$ is

$$\frac{\overline{x(s)}\,\chi_\Omega(t)y(t)G(s,t)}{N_x(t)} = \chi_\Omega(t)h(s,t).$$

Thus $E(\Omega)T$ is the minimal operator carrying x to $E(\Omega)y$, whence

$$\operatorname{Tr}(S^*E(\Omega)S) = \|E(\Omega)S\|_2^2 \geq \|E(\Omega)T\|_2^2 = \operatorname{Tr}(T^*E(\Omega)T).$$

This observation is pointed out in [12] in the case in which $\mathscr{L}$ is a nest and Ω is an interval.

A less complete result than Theorem 1 is available for the problem of interpolation of n vectors. The author would like to thank Steve Power for suggesting the essential ingredients in this result.

THEOREM 2. *Let $\mathscr{L} = \mathscr{L}(X, \leq, \mu)$ be a commutative subspace lattice, let $x_1, \ldots, x_n$ be n linearly independent vectors in $L^2(x, \mu)$ and let $y_1, \ldots, y_n$ be n additional vectors in $L^2(X, \mu)$. For each pair i, j with $1 \leq i, j \leq n$, define*

$$N_{ij} = \int G(s, t)\overline{x_j(s)}\, x_i(x)\, d\mu(s).$$

Let $N(t) = [N_{ij}(t)]$ for all t. Assume that for almost all t, if

$$y(t) = (y_1(t), \ldots, y_n(t)) \neq 0$$

then $N(t)$ is invertible and that

$$\int \|y(t)\| \cdot \|N(t)^{-1}y(t)\|\, d\mu(t) < \infty.$$

Then there is an operator T in $C_2 \cap \operatorname{Alg} \mathscr{L}$ such that $Tx_i = y_i$ for $i = 1, \ldots, n$.

Proof. Let π_i: $\mathbf{C}^n \to \mathbf{C}$ be the ith coordinate projection, $i = 1, \ldots, n$. Let

$$h(s, t) = \sum_{i=1}^{n} G(s, t)\overline{x_i(s)}\, \pi_i\big(N(t)^{-1}y(t)\big).$$

(Where $y(t) = 0$, take $h(s, t) = 0$.) We claim that h is an L^2-kernel function and that the corresponding operator, which obviously lies in $C_2 \cap \operatorname{Alg} \mathscr{L}$, maps x_i to y_i for each i.

Let $L(t) = [L_{ij}(t)] = N(t)^{-1}$, for all values of t for which $N(t)$ is invertible. In the following, all integrals are over just this set of values of t.

$$\int |h(s,t)|^2 \, d\mu \times \mu(s,t)$$

$$= \int\int h(s,t)\overline{h(s,t)}\, d\mu(s) d\mu(t)$$

$$= \int\int \sum_{i,j=1}^{n} G(s,t)\overline{x_i(s)}\, x_j(s) \pi_i(L(t)y(t))\overline{\pi_j(L(t)y(t))}\, d\mu(s)\, d\mu(t)$$

$$= \int \sum_{i,j=1}^{n} N_{ji}(t)\pi_i(L(t)y(t))\overline{\pi_j(L(t)y(t))}\, d\mu(t)$$

$$= \int \sum_{i,j=1}^{n} N_{ji}(t)\left(\sum_{p=1}^{n} L_{ip}(t)y_p(t)\right)\overline{\left(\sum_{q=1}^{n} L_{jq}(t)y_q(t)\right)}\, d\mu(t)$$

$$= \int \sum_{j,p,q=1}^{n} \left(\sum_{i=1}^{n} N_{ji}(t)L_{ip}(t)\right)\overline{L_{jq}(t)}\, y_p(t)\overline{y_q(t)}\, d\mu(t)$$

$$= \int \sum_{j,p,q=1}^{n} \delta_{jp}\overline{L_{jq}(t)}\, y_p(t)\overline{y_q(t)}\, d\mu(t)$$

$$= \int \sum_{p,q=1}^{n} \overline{L_{pq}(t)}\, y_p(t)\overline{y_q(t)}\, d\mu(t)$$

$$= \int \sum_{p=1}^{n} y_p(t)\overline{\pi_p(L(t)y(t))}\, d\mu(t)$$

$$\leq \int \|y(t)\|\, \|L(t)y(t)\|\, d\mu(t)$$

$$= \int \|y(t)\|\, \|N(t)^{-1}y(t)\| d\mu(t).$$

The last quantity is finite by hypothesis; thus, h is an L^2-function.

Now suppose that T is the integral operator associated with h. Clearly $T \in C_2 \cap \operatorname{Alg} \mathscr{L}$. For each i,

$$\begin{aligned}
(Tx_i)(t) &= \int h(s, t) x_i(s)\, d\mu(s) \\
&= \int \sum_{j=1}^{n} G(s, t)\overline{x_j(s)}\, \pi_j\big(N(t)^{-1} y(t)\big) x_i(s)\, d\mu(s) \\
&= \sum_{j=1}^{n} \pi_j\big(N(t)^{-1} y(t)\big) \cdot \int G(s, t)\overline{x_j(s)}\, x_i(s)\, d\mu(s) \\
&= \sum_{j=1}^{n} N_{ij}(t)\pi_j\big(N(t)^{-1} y(t)\big) \\
&= \pi_i\big(N(t)N(t)^{-1} y(t)\big) \\
&= \pi_i(y(t)) \\
&= y_i(t).
\end{aligned}$$

Thus T is an interpolating operator for the sets $\{x_1, \dots, x_n\}$ and $\{y_1, \dots, y_n\}$.

The interpolation problem for Hilbert-Schmidt operator suggests the analogous problem for the other Schatten C_p-classes and for the compact operators, $\mathscr{K}$.

Questions. Let $\mathscr{L} = \mathscr{L}(X, \leq, \mu)$ be a commutative subspace lattice and let x and y belong to $L^2(X, \mu)$. When does there exist an operator T in $C_p \cap \operatorname{Alg} \mathscr{L}$ (resp. in $\mathscr{K} \cap \operatorname{Alg} \mathscr{L}$) such that $Tx = y$? Can T in $C_p \cap \operatorname{Alg} \mathscr{L}$ be chosen so as to minimize the Schatten p-norm, $\|T\|_p$? Can compact T be chosen with minimal operator norm $\|T\|_\infty$?

Even in the case in which $\mathscr{L}$ is a nest, little is known about $\mathscr{C}_1$ interpolation. The most interesting case is, perhaps, the trace class operators, $\mathscr{C}_1$. Where $\mathscr{L}$ is a discrete nest with finite dimensional atoms, Proposition 1.3 in [12] gives a condition on x and y which guarantees that the minimal Hilbert-Schmidt interpolating operator is also trace class. This condition, together with an estimate for the trace class norm, can be extended to totally atomic CSL-algebras (without any restriction on the dimension of the atoms).

When $\mathscr{L}$ is totally atomic (on a separable Hilbert space), we may take X to be a countable set and μ to be counting measure: $\mu(\{s\}) = 1$, for all $s \in X$. The Hilbert space is thus $l^2(X)$. If Y is a subset of X, let $E(Y)$ be the orthogonal projection of $l^2(X)$ onto $l^2(Y)$, where the latter is viewed as a subspace of $l^2(X)$.

For each vector $x \in l^2(X)$, let

$$N_x(t) = \sum_{s \leq t} |x(s)|^2.$$

Also, let

$$P_x = \{t | N_x(t) \neq 0\}.$$

Then P_x is the smallest increasing set in X which contains the support of the vector x. Consequently, $E(P_x)$ is the smallest projection in $\mathscr{L}$ whose range contains the vector x.

PROPOSITION 3. *Let $\mathscr{L} = \mathscr{L}(X, \leq, \mu)$ be a totally atomic commutative subspace lattice with X countable (or finite) and μ equal to counting measure. Let $x, y \in l^2(X)$. Assume the following:*

$$y(t) = 0 \quad \text{for all } t \notin P_x. \tag{1}$$

$$\sum_{t \in P_x} \frac{|y(t)|^2}{N_x(t)} < \infty. \tag{2}$$

$$\sum_{s \in P_x} \left(\sum_{t \geq s} \frac{|y(t)|^2}{N_x(t)} \right)^{1/2} < \infty. \tag{3}$$

Then the Hilbert-Schmidt operator, T, corresponding to the kernel function

$$\frac{\overline{x(s)}\, y(t) G(s, t)}{N_x(t)}$$

is trace class and

$$\|T\|_1 \leq \sum_{s \in P_x} \left(\sum_{t \geq s} \frac{|y(t)|^2}{N_x(t)} \right)^{1/2}.$$

Proof. Assumptions (1) and (2) guarantee that the operator T with kernel

$$\frac{\overline{x(s)}\, y(t) G(s, t)}{N_x(t)}$$

is Hilbert-Schmidt, maps x to y, and has minimal Hilbert-Schmidt norm amongst interpolation operators. (Again, $0/0$ is taken as 0.) Conditions (1) and (2) are just the discrete version of the hypothesis in Theorem 1. We can, of course, view

$$\frac{\overline{x(s)}\, y(t) G(s,t)}{N_x(t)}$$

as the entry in row t, column s of the matrix for T. The rows and columns of matrices for operators in $\mathscr{B}(l^2(X))$ are indexed by the elements of X.

The action of T is given as follows: if $f \in l^2(X)$ then Tf is the element of $l^2(x)$ given by

$$Tf(t) = \sum_s \frac{f(s)\overline{x(s)}\, y(t) G(s,t)}{N_x(t)}.$$

If $t \notin P_x$, then $Tf(t) = 0$ (since $y(t) = 0$). Thus, range $T \subseteq E(P_x)$.

We now show that $\|T\|_1$ is finite and, in fact, is dominated by the quantity in (3).

If $t \notin P_x$ then $Tf(t) = 0$; if $t \in P_x$ then

$$\begin{aligned} Tf(t) &= \frac{y(t)}{N_x(t)} \sum_s f(s)\overline{x(s)}\, G(s,t) \\ &= \frac{y(t)}{N_x(t)} \sum_{s \in P_x} f(s)\overline{x(s)}\, G(s,t), \end{aligned}$$

since $x(s) = 0$ if $s \notin P_x$. Then

$$\begin{aligned} \|Tf\|^2 &= \sum_{t \in P_x} |Tf(t)|^2 \\ &= \sum_{t \in P_x} \frac{|y(t)|^2}{N_x(t)^2} \left| \sum_{s \in P_x} f(s)\overline{x(s)}\, G(s,t) \right|^2 = \sum_{t \in P_x} \frac{|y(t)|^2}{N_x(t)^2} \left| \sum_{\substack{s \in P_x \\ s \le t}} f(s)\overline{x(s)} \right|^2 \\ &\le \sum_{t \in P_x} \frac{|y(t)|^2}{N_x(t)^2} \left(\sum_{\substack{s \in P_x \\ s \le t}} |f(s)|^2 \right) \left(\sum_{\substack{s \in P_x \\ s \le t}} |x(s)|^2 \right) \end{aligned}$$

But

$$\sum_{\substack{s \in P_x \\ s \le t}} |x(s)|^2 = N_x(t),$$

so

$$\begin{aligned}
\|Tf\|^2 &\le \sum_{t\in P_x} \frac{|y(t)|^2}{N_x(t)} \left(\sum_{\substack{s\in P_x\\ s\le t}} |f(s)|^2 \right) \\
&= \sum_{t\in P_x} \sum_{\substack{s\in P_x\\ s\le t}} \frac{|y(t)|^2 |f(s)|^2}{N_x(t)} \\
&= \sum_{s\in P_x} \left(\sum_{\substack{t\in P_x\\ t\ge s}} \frac{|y(t)|^2}{N_x(t)} \right) |f(s)|^2 .
\end{aligned}$$

Now let

$$h(s) = \begin{cases} \left(\sum_{t\ge s} \frac{|y(t)|^2}{N_x(t)} \right)^{1/2} & \text{if } s \in P_x \\ 0 & \text{if } s \notin P_x . \end{cases}$$

Let H be the diagonal operator whose entries are $h(s)$. Thus $Hf(s) = h(s)f(s)$, all s, and

$$\begin{aligned}
\|Hf\|^2 &= \sum_s |h(s)f(s)|^2 \\
&= \sum_{s\in P_x} |h(s)|^2 |f(s)|^2 \\
&= \sum_{s\in P_x} \left(\sum_{t\ge s} \frac{|y(t)|^2}{N_x(t)} \right) |f(s)|^2 .
\end{aligned}$$

Thus $\|Tf\|^2 \le \|Hf\|^2$, for any $f \in l^2(x)$. This says that $\langle T^*Tf, f\rangle \le \langle H^2 f, f\rangle$ for all f, i.e., that $T^*T \le H^2$. By the monotonicity of square roots we have

$$(T^*T)^{1/2} \le H.$$

Consequently,

$$\begin{aligned}
\|T\|_1 = \operatorname{tr}(T^*T)^{1/2} \le \operatorname{tr} H &= \sum_{s\in P_x} h(s) \\
&= \sum_{s\in P_x} \left(\sum_{t\ge s} \frac{|y(t)|^2}{N_x(t)} \right)^{1/2} < \infty
\end{aligned}$$

by condition (3). This completes the proof.

Remark. The proof above can be formally written out in the general case. However, if μ has a continuous part, then the operator H is an L^∞-multiplication operator with respect to a measure which is not totally atomic. As such, it is not even compact, let alone trace class and consequently no estimate is obtained by this approach.

We will close with a brief discussion of approximate interpolation. Both parts of Proposition 4 below are proven in [12] for nest algebras. The proof of part (1) given there works equally well for CSL-algebras; it is repeated here for the convenience of the reader. The second part of Proposition 4 is not valid for general CSL-algebras. It does hold for all commutative subspace lattices which are completely distributive, a class which includes all nests. The usual definition of complete distributivity is that distributive laws for the two lattice operations hold for families of arbitrary cardinality. We will use an equivalent technical condition; see [11] for the proof of the equivalence or [8] for a thorough discussion of complete distributivity. Two definitions are needed: for each $P \in \mathscr{L}$, define

$$P_+ = \wedge\{Q \in \mathscr{L} | Q \nleq P\}$$

and for each $L \in \mathscr{L}$, define

$$L_\# = \vee\{P_+ | P \in \mathscr{L} \text{ and } L \nleq P\}.$$

Then $\mathscr{L}$ is completely distributive if, and only if $L = L_\#$, for all $L \in \mathscr{L}$.

A notable property of a completely distributive commutative subspace lattice $\mathscr{L}$ is that the sums of rank one elements in Alg $\mathscr{L}$ are weakly dense in Alg $\mathscr{L}$; indeed, this property is actually equivalent to complete distributivity. (See [10].) If $\mathscr{L}$ is not completely distributive, then Alg $\mathscr{L}$ may contain no non-zero compact operators. This makes it clear that part (2) of Proposition 4 admits no extension to all CSL-algebras. If x is a vector in $\mathscr{H}$, then the smallest projection in $\mathscr{L}$ whose range contains x in $P_x = \wedge\{F \in \mathscr{L} | Fx = x\}$.

PROPOSITION 4. *Let $\mathscr{L}$ be a commutative subspace lattice on a Hilbert space $\mathscr{H}$. Let x and y be vectors in $\mathscr{H}$ such that $Tx \neq y$ for all $T \in C_2 \cap$ Alg $\mathscr{L}$.*

(1) *Suppose there is a net $T_\nu \in C_2 \cap$ Alg $\mathscr{L}$ such that $\lim T_\nu x = y$. Then* $\lim \|T_\nu\|_2 = \infty$,

(2) *Assume further that $\mathscr{L}$ is completely distributive. Then there is a net $T_\nu \in C_2 \cap$ Alg $\mathscr{L}$ such that $\lim T_\nu x = y$ if, and only if, $y \in P_x$.*

Proof. (1) If $\lim \|T_\nu\| \neq \infty$ then there is a bounded net, S_ν, in $C_2 \cap$ Alg $\mathscr{L}$ such that $\lim S_\nu x = y$. By passing to a subnet, if necessary, we may assume that S_ν is weakly convergent to some operator S in $C_2 \cap$ Alg $\mathscr{L}$. Thus $S_\nu x$ converges to y in norm and to Sx weakly. Therefore $Sx = y$, a contradiction.

(2) First assume that T_ν is a net in $C_2 \cap \operatorname{Alg} \mathscr{L}$ such that $T_\nu x$ converges to y. Since $P_x \in \mathscr{L}$, $T_\nu x \in P_x$ for all ν, whence $y \in P_x$.

For the converse, assume $y \in P_x$. Since $\mathscr{L}$ is completely distributive, $P_{x\sharp} = P_x$. Using the fact that $\mathscr{H}$ is separable and the definition of $\sharp$, we can conclude that there is a sequence P^n of projections in $\mathscr{L}$ such that $P_x \nleq P^n$ for all n and $P_x = \bigvee P_+^n$. Let $Q_1 = P_+^1$ and inductively define

$$Q_n = P_n^+ \setminus \left(\bigvee_{j=1}^{n-1} Q_j \right) \quad \text{for all } n.$$

Then the Q_i are mutually orthogonal and

$$P_x = \sum_{i=1}^{\infty} Q_i = \bigvee_{i=1}^{\infty} Q_i.$$

Since $y = P_x y = \sum_{i=1}^{\infty} Q_i y$, we have $\sum_{i=1}^{n} Q_i y \to y$ as $n \to \infty$. Now, for each n, $P_x - P^n P_x \neq 0$, since $P_x \nleq P^n$. Thus $P^n P_x$ is a proper subprojection of P_x; therefore its range cannot contain x. In other words, $(P_x - P^n P_x)x \neq 0$, for all n. Let $\lambda_n = \|(P_x - P^n P_x)x\|^{-2}$. Let T_n be the rank one operator

$$\lambda_n Q_n y \otimes \left((P_x - P^n P_x)x \right)^*.$$

By the dual of the Ringrose-Longstaff lemma [11], $T_n \in \operatorname{Alg} \mathscr{L}$.

We have $T_n x = Q_n y$, for each n. Let $S_n = \sum_{i=1}^{n} T_i$. Then $S_n \in C_2 \cap \operatorname{Alg} \mathscr{L}$ (in fact, rank $S_n \leq n$) and $S_n x = \sum_{i=1}^{n} Q_i y \to y$. This completes the proof.

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