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THE ISOMETRIES OF $L^2(\Omega, X)$

BY

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1. Introduction

Let X be a complex Banach space. A semi-inner-product (s.i.p.) compatible with the norm is a function $[\cdot, \cdot]$: $X \times X \to \mathbf{C}$ such that

$$(1) \qquad [\alpha x + y, z] = \alpha[x, z] + [y, z] \quad \text{for } x, y, z \in X \text{ and } \alpha \in \mathbf{C},$$

$$(2) \qquad [x, x] = \|x\|^2 \quad \text{for } x \in X,$$

$$(3) \qquad |[x, y]| \leq \|y\| \cdot \|x\| \quad \text{for any } x, y \in X.$$

It is known that for any Banach space X, there is a homogeneous semi-inner-product compatible with the norm, i.e.,

$$[x, \alpha y] = \bar{\alpha}[x, y] \quad \text{for all } x, y \in X \text{ and } \alpha \in \mathbf{C}.$$

An operator H: $X \to X$ is *hermitian* if

$$[Hx, x] \in \mathbf{R} \quad \text{for all } x \in X.$$

Let (Ω, Σ, μ) be a σ-finite measure space and let X be a separable Banach space. A.R. Sourour has shown [5] that if H is a hermitian operator on $L^p(\Omega, X)$, $1 \leq p < \infty$, $p \neq 2$, then $(Hf)(\cdot) = A(\cdot)f(\cdot)$ for some hermitian valued strongly measurable map A of Ω into $\mathscr{B}(X)$ (the set of all bounded operators on X). Using this result, A.R. Sourour [5] proved that if X is a separable Banach space with trivial L^p-structure (see [3]) for $1 \leq p < \infty$, $p \neq 2$, and if T is a surjective isometry on $L^p(\Omega, X)$, then

$$(Tf)(\cdot) = S(\cdot)h(\cdot)(\Phi(f))(\cdot) \quad \text{for } f \in L^p(\Omega, X),$$

where Φ is a set isomorphism of the measure space onto itself (for definition see [5]), S is a strongly measurable map of Ω into $\mathscr{B}(X)$ with $S(t)$ a surjective isometry of X for almost all $t \in \Omega$, and $h = (dv/d\mu)^{1/p}$ where $v(\cdot) = \mu(\Phi^{-1}(\cdot))$. On the other hand, the hermitian operators and isometries on l^2

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are not necessarily of the above forms. But A. Berkson and A.R. Sourour [1] have shown that if

$$T = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$$

is hermitian on $(X \oplus X)_2$, where I is the identity on X, then X is isometrically isomorphic to a Hilbert space. It is natural to ask under what conditions on X, the hermitian operators and isometries on $L^2(\Omega, X)$ have the above forms. In this article, we show that if

$$T = \begin{pmatrix} 0 & T_1 \\ T_2 & 0 \end{pmatrix}$$

is a hermitian operator on $(X \oplus X)_2$, where T_1 and T_2 are operators on X, then $Y_1 = \overline{T_1(X)}$ and $Y_2 = \overline{T_2(X)}$ are isometrically isomorphic to Hilbert spaces, and there exist two subspaces Z_1 and Z_2 of X such that

$$(Z_1 \oplus Y_1)_2 = X = (Z_2 \oplus Y_2)_2.$$

Using this result, we prove that if X is not 1-dimensional and if X is separable with trivial L^2-structure and (Ω, Σ, μ) is σ-finite, then the hermitian operators and isometries on $L^2(\Omega, X)$ have forms like the hermitian operators and isometries on $L^p(\Omega, X)$.

For more results about isometries on $L^p(\Omega, X)$, see [3] and its references.

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2. Hermitian operators on $(X_1 \oplus X_2)_2$

Let X_1 and X_2 be two Banach spaces, and let $[\cdot, \cdot]_1$ (resp. $[\cdot, \cdot]_2$) be a homogeneous s.i.p. compatible with the norm of X_1 (resp. X_2). Then

$$[(x_1, x_2), (x_1', x_2')] = [x_1, x_1']_1 + [x_2, x_2']_2$$

is a s.i.p. compatible with the norm $(X_1 \oplus X_2)_2$. If

$$T = \begin{pmatrix} 0 & T_1 \\ T_2 & 0 \end{pmatrix}$$

is a hermitian operator on $(X_1 \oplus X_2)_2$, where T_1 (resp. T_2) is an operator from X_2 (resp. X_1) into X_1 (resp. X_2), then

$$[T_1 x_2, x_1]_1 + [T_2 x_1, x_2]_2 \in \mathbf{R}$$

for any $x_1 \in X_1$, and any $x_2 \in X_2$. Replacing x_2 by ix_2 in this expression gives the conclusion that

$$i\{[T_1x_2, x_1]_1 - [T_2x_1, x_2]_2\} \in \mathbf{R}.$$

Therefore, for any x_1 in X_1 and x_2 in X_2,

$$[T_1x_2, x_1]_1 = \overline{[T_2x_1, x_2]_2}.$$

(This implies that $T_1 = 0$ if and only if $T_2 = 0$.) So for any x_1, x_1' in X_1, and x_2 in X_2,

$$(4) \qquad \begin{aligned}[T_1x_2, x_1 + x_1']_1 &= \overline{[T_2(x_1 + x_1'), x_2]_2} \\ &= \overline{[T_2x_1, x_2]_2} + \overline{[\ T_2x_1', x_2]_2} \\ &= [T_1x_2, x_1]_1 + [T_1x_2, x_1']_1.\end{aligned}$$

Similarly, for any $x_1 \in X_1$, and any $x_2, x_2' \in X_2$,

$$(5) \qquad [T_2x_1, (x_2 + x_2')]_2 = [T_2x_1, x_2]_2 + [T_2x_1, x_2']_2.$$

The restriction of $[\cdot, \cdot]_1$ to T_1X_2 is a homogeneous s.i.p. compatible with the norm such that

$$[x, y + z]_1 = [x, y]_1 + [x, z]_1 \quad \text{for any } x, y, z \in T_1X_2.$$

It is known that any homogeneous s.i.p. satisfying the above property is an inner product. So $Y_1 = \overline{T_1X_2}$ is a Hilbert space. Similarly, $Y_2 = \overline{T_2X_1}$ is a Hilbert space. But in order to show that there is a subspace Z_1 (resp. Z_2) of X_1 (resp. X_2) such that $X_1 = (Z_1 \oplus Y_1)_2$ (resp. $(X_2 = Z_2 \oplus Y_2)_2$), we need to prove the following strong property.

Let x and y be two linearly independent elements. If any s.i.p. $[\cdot, \cdot]$ compatible with the norm satisfies

$$(5') \qquad [y, \alpha x + \beta y] = [y, \alpha x] + [y, \beta y], \quad \alpha, \beta \in \mathbf{C},$$

then span(x, y) is isometrically isomorphic to l_2^2. So if $0 \neq y = T_1z \in T_1X_2$ and $x \in X_1$ are linearly independent, then by (5) x and y satisfy (5′); hence, span(x, y) is isometrically isomorphic to l_2^2.

(i) Without loss of generality, we may assume that $\|y\| = 1 = \|x\|$ and $\|x + \alpha y\| \geq 1$ for any $\alpha \in \mathbf{C}$. So there exists a linear function f such that $\|f\| = 1 = f(x)$ and $f(y) = 0$. We can find a homogeneous s.i.p. compatible with the norm so that $[y, x] = 0$. If $\|\alpha x + \beta y\| = 1$, then

$$|\beta| = |[y, \beta y]| = |[y, \alpha x] + [y, \beta y]| = |[y, \alpha x + \beta y]| \leq 1.$$

So we may choose the homogeneous s.i.p. compatible with the norm which satisfies $[x, y] = 0$.

(ii) Let Y denote the subspace span(x, y). We claim the norm of Y is smooth on

$$Y \setminus (\{\alpha y\colon \alpha \in \mathbf{C}\} \cup \{\alpha x\colon \alpha \in \mathbf{C}\}).$$

Suppose $\|\alpha x + \beta y\| = 1$, and $|\alpha| \neq 1 \neq |\beta|$. If the norm is not smooth at $\alpha x + \beta y$, then there exist two homogeneous s.i.p., $[\cdot, \cdot]_1$ and $[\cdot, \cdot]_{1'}$, compatible with the norm which satisfy

$$0 = [x, y]_1 = [x, y]_{1'} = [y, x]_1 = [y, x]_{1'}, \tag{6}$$

$$[\cdot, \alpha x + \beta y]_1|_Y \neq [\cdot, \alpha x + \beta y]_{1'}|_Y. \tag{7}$$

But by (4) and (2),

$$[y, \alpha x + \beta y]_1 = [y, \alpha x]_1 + [y, \beta y]_1 = \bar{\beta} = [y, \alpha x + \beta y]_{1'},$$

and

$$[\alpha x + \beta y, \alpha x + \beta y]_1 = 1 = [\alpha x + \beta y, \alpha x + \beta y]_{1'}.$$

We get a contradiction.

(iii) We claim that for any $0 \leq \alpha \leq 1$, there is a unique $\beta \geq 0$ such that $\|\alpha x + \beta y\| = 1$. Suppose this is not true. Then we must have $\alpha = 1$, and we may choose a homogeneous s.i.p. compatible with the norm such that $[y, x + \beta y] = 0$. But this contradicts the fact $\beta = [y, x + \beta y]$. Similarly for any $0 \geq \alpha \geq -1$, there is a unique $\beta \geq 0$ such that $\|\alpha x + \beta y\| = 1$.

(iv) For $0 \leq \alpha \leq 1$ (resp. $0 \geq \alpha \geq -1$), let $f(\alpha)$ be the unique non-negative real number such that $\|\alpha x + f(\alpha) y\| = 1$. Since the norm is smooth on

$$Y \setminus (\{\beta y\colon \beta \in \mathbf{C}\} \cup \{\beta x\colon \beta \in \mathbf{C}\}),$$

$f(\alpha)$ is differentiable on $0 < \alpha < 1$ (resp. $-1 < \alpha < 0$) and there exists c such that

$$[\cdot, \alpha x + f(\alpha) y] = c\{[\cdot, -f'(\alpha) x] + [\cdot, y]\}.$$

But $[\alpha x + f(\alpha) y, \alpha x + f(\alpha) y] = 1$ and $[y, \alpha x + f(\alpha) y] = f(\alpha)$. We have

$$c = \frac{1}{-\alpha f'(\alpha) + f(\alpha)},$$

and f satisfies

$$\frac{1}{-\alpha f'(\alpha) + f(\alpha)} = f(\alpha) \quad \text{and} \quad -\frac{f'(\alpha) f(\alpha)}{1 - f^2(\alpha)} = \frac{1}{\alpha}.$$

So $1 - f^2(\alpha) = c\alpha^2$. Since $f(1) = 0 = f(-1)$,

$$f(\alpha) = \sqrt{1 - \alpha^2},$$

and span(x, y) is a Hilbert space. (Note: if $[x, y] = 0$ then $[x, e^{i\theta}y] = e^{-i\theta}[x, y] = 0$ for any $\theta \in \mathbf{R}$.)

Since Y_1 is reflexive, Y_1 is a proximinal subspace, i.e. for every $x \in X_1$, there is $y \in Y_1$ such that

$$\|x - y\| = \inf_{y' \in Y_1} \|x - y'\|.$$

Let

$$Z_1 = \left\{z \in X_1 \colon \|z\| = \inf_{y \in Y_1} \|z - y\|\right\}.$$

We claim that Z_1 is a vector space. Let $0 \neq z \in Z_1$ and $0 \neq y \in Y_1$. Since span$(y, z) = l_2^2$, $[y, z] = 0$, $\{y, z\}$ is an orthogonal basis of span(y, z), and

$$[y, z] = 0 = [z, y].$$

So if z' is another element in Z_1, then $[y, z'] = 0 = [z', y]$, and $[z + z', y] = 0$. But span$(z + z', y)$ is a Hilbert space. So $[y, z + z'] = 0$ and $z + z' \in Z_1$. The verification that $X_1 = (Z_1 \oplus Y_1)_2$ is left to the reader. Similarly, there exists a subspace Z_2 of X_2 such that $X_2 = (Y_2 \oplus Z_2)_2$.

Remark 1. It is known that if

$$H = \begin{pmatrix} T_1, T_2 \\ T_2, T_4 \end{pmatrix}$$

is a hermitian operator on $(X_1 \oplus X_2)_2$, then

$$\begin{pmatrix} T_1, 0 \\ 0, T_4 \end{pmatrix}$$

is hermitian and T_1 (resp. T_2) is a hermitian operator on X_1 (resp. X_2) (see [4]). So

$$\begin{pmatrix} 0, T_2 \\ T_3, 0 \end{pmatrix}$$

is hermitian.

If X contains a nontrivial l^2 complemented Hilbert space, then it must contain a one-dimensional l^2 complement. So we have proved the following theorem.

THEOREM 1. *Suppose that X_1 and X_2 are two Banach spaces such that there is no subspace Z_1 (resp. Z_2) of X_1 (resp. X_2) which satisfies $X_1 = (Z_1 \oplus \mathbf{C})_2$ (resp. $X_2 = (Z_2 \oplus \mathbf{C})_2$). If*

$$H = \begin{pmatrix} T_1 & T_2 \\ T_3 & T_4 \end{pmatrix}$$

is a hermitian operator on $(X_1 \oplus X_2)_2$, then $T_2 = 0$ (resp. $T_3 = 0$), and T_1 (resp. T_4) is a hermitian operator on X_1 (resp. X_2).

3. The isometries on $L^2(\Omega, X)$

We say a complex Banach space has property $(*)$ if there is a subspace Y of X such that $X = (Y \oplus \mathbf{C})_2$. Since $\mathbf{C} = 0 \oplus \mathbf{C}$, $\dim(X) > 1$ if X does not have the property $(*)$. Before proving the main theorems, we need the following lemma.

LEMMA 2. *Let X be a complex Banach space without property $(*)$. Then $L^2(\Omega, X)$ does not have property $(*)$.*

Proof. Suppose this is not true. Then there is f in $L^2(\Omega, X)$ such that if f and g are linearly independent, then $\operatorname{span}(f, g)$ is isometrically isomorphic to l_2^2.

(i) Let $A = \operatorname{supp}(g)$. If $f|_A \neq 0$, then

$$\begin{aligned} &\int_A \|f(t) + g(t)\|^2 \, d\mu + \int_{\Omega \setminus A} \|f(t)\|^2 \, d\mu \\ &\quad = \|f + g\|^2 = \|f\|^2 + \|g\|^2 \\ &\quad = \int_A \|f(t)\|^2 \, d\mu + \int_A \|g(t)\|^2 \, d\mu + \int_{\Omega \setminus A} \|f(t)\|^2 \, d\mu. \end{aligned}$$

So $\operatorname{span}(f|_A, g)$ is isometrically isomorphic to l_2^2.

(ii) Since $0 \neq f \in L^2(\Omega, X)$, there is $x \neq 0$ in X such that for any $\varepsilon > 0$,

$$\mu\{t: \|f(t) - x\| < \varepsilon\} > 0.$$

Let $A_\varepsilon = \{t: \|f(t) - x\| < \varepsilon\}$, and let y be any element in X such that $[y, x] = 0$ and $\|y\| = \|x\|$. Let T be the mapping from $\operatorname{span}(x, y)$ onto $\operatorname{span}(f|_{A_\varepsilon}, y \cdot \chi_{A_\varepsilon})$ such that

$$T(x) = f|_{A_\varepsilon} \quad \text{and} \quad T(y) = y \cdot \chi_{A_\varepsilon}.$$

Then

$$\|T\| \cdot \|T^{-1}\| \leq \frac{1}{(1-\varepsilon)^2}.$$

This implies that $\operatorname{span}(x, y)$ is isometrically isomorphic to l_2^2 for any y such that $[y, x] = 0$. We get a contradiction. ■

By the technique in [5], we have the following theorems.

THEOREM 3. *Assume that for each $n \in \mathbf{N}$, X_n is a separable complex Banach space without property $(*)$ and $(\Omega_n, \Sigma_n, \mu_n)$ is σ-finite. An operator H on $(\Sigma \oplus L^2(\Omega_n, X_n))_2$ is hermitian if and only if*

$$H((f_n)(\cdot)) = (A_n(\cdot)f_n(\cdot))$$

for hermitian valued strongly measurable maps A_n of Ω_n into $\mathscr{B}(X_n)$.

Proof. Suppose that $A \in \Sigma_n$ with $\mu_n(A) \neq 0$. Then

$$\begin{aligned}&\left(\sum \oplus L^2(\Omega_m, X_m)\right)_2 \\ &\quad = \left(L^2(A, X_n) \oplus L^2(\Omega_n \setminus A, X_n) \oplus \left(\sum_{m \neq n} \oplus L^2(\Omega_m, X_m)\right)_2\right)_2.\end{aligned}$$

By Lemma 2, neither $(L^2(\Omega_n \setminus A, X_n) \oplus (\Sigma_{m\neq n} \oplus L^2(\Omega_m, X_m))_2)_2$ nor $L^2(A, X_n)$ has property $(*)$. So if H is a hermitian operator on $(\Sigma \oplus L^2(\Omega_m, X_m))_2$, then

$$\begin{aligned}&H\left(\left(L^2(\Omega_n \setminus A, X_n) \oplus \left(\sum_{m\neq n} \oplus L^2(\Omega_m, X_m)\right)_2\right)_2\right) \\ &\quad \subseteq \left(L^2(\Omega_n \setminus A, X_n) \oplus \left(\sum_{m\neq n} \oplus L^2(\Omega_m, X_m)\right)_2\right)_2\end{aligned}$$

and

$$H(L^2(A, X_n)) \subseteq L^2(A, X_n).$$

By Theorem 3.1 and Theorem 4.2 in [5], we have proved the theorem. ∎

THEOREM 4. *Assume that for each $n \in \mathbf{N}$, X_n (resp. Y_n) is a separable complex Banach space with trivial L^2-structure and* $\dim(X_n) > 1$ *(resp.* $\dim(Y_n) > 1$*), and $(\Omega_n, \Sigma_n, \mu_n)$ (resp. $(\Omega_n', \Sigma_n', \psi_n')$) is σ-finite. If for any $i \neq j$, X_i (resp. Y_i) and X_j (resp. Y_j) are not isometrically isomorphic, and if T is a surjective isometry from $(\Sigma \oplus L^2(\Omega_n, X_n))_2$ onto $(\Sigma \oplus L^2(\Omega_n', Y_n))_2$, then*

$$T\left(\sum \oplus f_n\right)(\cdot) = S(\cdot)h(\cdot)\left(\Phi\left(\sum \oplus f_n\right)\right)(\cdot)$$

where π is a permutation on $\mathbf{N}$, Φ is a set isomorphism from $\bigcup_{n=1}^{\infty}\Omega_n$ onto $\bigcup_{n=1}^{\infty}\Omega_n'$ such that $\Phi(\Omega_n) = \Omega_{\pi(n)}$, S is a strongly measurable map of $\bigcup_{n=1}^{\infty}\Omega_n$ into $\bigcup_{n=1}^{\infty}\mathscr{B}(X_n, Y_{\pi(n)})$ with $S(t)$ an isometry from X_n onto $Y_{\pi(n)}$ for almost all $t \in \Omega_n$, and

$$h = \sum \left(\frac{d\left(\mu_n \circ \Phi^{-1}\right)}{d\mu'_{\pi(n)}} \right)^{1/2}.$$

Proof. Let $A \in \Sigma_n$ such that $\mu_n(A) > 0$. If H is the hermitian projection from the space $(\Sigma \oplus L^2(\Omega_m, X_m))_2$ onto $L^2(A, X_n)$, then $H_1 = THT^{-1}$ is a hermitian projection. By Theorem 3,

$$THT^{-1}((f_m)(\cdot)) = (P_m(\cdot)f_m(\cdot))$$

where $P_m(t)$ is a hermitian projection on X_m for almost all $t \in \Omega_m$. By the proof of Theorem 5.2 in [5], $P_m(t) = I$ or 0 for almost all $t \in \Omega_m$. By Theorem 3.1, Corollary 3.2 and the proof of Theorem 5.2 in [5], we have

$$Tf(t) = A(t)(h(t)(\Phi f)(t))$$

where Φ is a Boolean isomorphism from $\cup\Sigma_n$ onto $\cup\Sigma_n'$, and $A(t)$ is an isometry from X_n onto Y_m if $t \in \Omega_n$ and $\Phi(t) \in \Omega_m'$. But if $n \neq n'$, then Y_n (resp. X_n) is not isometrically isomorphic to Y_n' (resp. $X_{n'}$). So $\Phi(\Sigma_n) = \Sigma_{\pi(n)}'$ where π is a permutation of $\mathbf{N}$. ∎

Let m be Lebesgue measure on $[0,1]$, and let X be any Banach space. It is known that $L^2([0,1], m, X)$ is isometrically isomorphic to $L^2([0,1], m, (\Sigma_{n=1}^{\infty} \oplus X)_2)$. So we have the following theorem.

THEOREM 5. *Assume that for each $n \in \mathbf{N}$, X_n (resp. Y_n) is a separable complex Banach space with trivial L^2-structure. Then $L^2([0,1], m, (\sum \oplus X_n)_2)$ and $L^2([0,1], m, (\sum \oplus Y_n)_2)$ are isometrically isomorphic, if and only if for each $n \in \mathbf{N}$, there exists m (resp. m') such that X_n (resp. Y_n) and Y_m (resp. $X_{m'}$) are isometrically isomorphic.*

Proof. We only need to show that it is a necessary condition. By Lemma 2, the space

$$\left(\sum \oplus L^2(\Omega_n, X_n)\right)_2$$

has property $(*)$ if and only if X_n has property $(*)$ for some $n \in \mathbf{N}$. This implies that if $\dim(X_n) = 1$ for some $n \in \mathbf{N}$, then $\dim(Y_m) = 1$ for some $m \in \mathbf{N}$.

Let T be a surjective isometry from

$$\left(\left(\sum \oplus L^2(\Omega_n, X_n)\right)_2 \oplus L^2\right)_2$$

onto

$$\left(\left(\sum \oplus L^2(\Omega_n', Y_n)\right)_2 \oplus L^2\right)_2.$$

We claim that $T(L^2) \subseteq L^2$ (so $T^{-1}(L^2) = L^2$ and $T(L^2) = L^2$). If this is not true, then there is an $f \in L^2$ such that $T(f) \notin L^2$.

(i) For any

$$g \oplus h \in \left(\left(\sum \oplus L^2(\Omega_n, X_n)\right)_2 \oplus L^2\right)_2,$$

if $g \oplus h$ and $0 \oplus f$ are linear independent, then

$$\operatorname{span}(g \oplus h, 0 \oplus f) \quad \text{and} \quad \operatorname{span}\big(T(g \oplus h), T(0 \oplus f)\big)$$

are isometrically isomorphic to l_2^2. So if

$$T(0 \oplus f) \quad \text{and} \quad \bar{g} \oplus \bar{h} \in \left(\left(\sum \oplus L^2(\Omega_n', X_n)\right)_2 \oplus L^2\right)_2$$

are linear independent, then

$$\operatorname{span}\big(T(0 \oplus f), \bar{g} \oplus \bar{h}\big)$$

is isometrically isomorphic to l_2^2.

(ii) By the assumption, there is an $n \in \mathbf{N}$ such that $A = \operatorname{supp}(T(0 \oplus f)) \cap \Omega_n'$ has measure greater than 0. By the proof of Lemma 2, if $\operatorname{supp}(\bar{g}) \subseteq A$, and if $\bar{g}$ and $T(0 \oplus f)|_A$ are linearly independent, then $\operatorname{span}(\bar{g}, T(0 \oplus f)|_A)$ is isometrically isomorphic to l_2^2. This implies that $L^2(A, Y_n)$ has property $(*)$. We get a contradiction.

If

$$T\Big(\big(\sum \oplus L^2(\Omega_n, X_n)\big)_2\Big) \nsubseteq \big(\sum \oplus L^2(\Omega_n', Y_n)\big)_2,$$

then there is

$$g \in \big(\sum \oplus L^2(\Omega, X_n)\big)_2$$

such that $T(g \oplus 0) = \bar{g} \oplus \bar{h}$ for some $\bar{h} \in L^2$. But $T(L^2) = L^2$, so there exists $h \in L^2$ such that $T(0 \oplus h) = 0 \oplus \bar{h}$. This implies

$$\|g \oplus -h\| = \|T(g \oplus -h)\| = \|\bar{g} \oplus 0\| < \|\bar{g} \oplus \bar{h}\| = \|g \oplus 0\|.$$

So we get a contradiction and we must have

$$T\Big(\big(\sum \oplus L^2(\Omega_n, X_n)\big)_2\Big) \subseteq \big(\sum \oplus L^2(\Omega_n, X_n)\big)_2.$$

By the proof of Theorem 4, for each n, there is an m such that X_n is isometrically isomorphic to Y_m. Similarly, for each n there is an m' such that Y_n is isometrically isomorphic to $X_{m'}$. ■

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