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BOUNDEDNESS OF THE LITTLEWOOD-PALEY g-FUNCTION ON $Lip_\alpha(R^n)$ $(0 < \alpha < 1)$

BY

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I. Introduction

Let R^n be n-dimensional Euclidean space. We say that a scalar valued function ψ on R^n is a Littlewood-Paley function if the following three conditions are satisfied:

$$\psi \in L(R^n), \quad \int_{R^n} \psi(x)\, dx = 0, \tag{1.1}$$

$$|\psi(x)| \leq c(1 + |x|)^{-(n+1)}, \tag{1.2}$$

$$|\psi(x+y) - \psi(x)| \leq \frac{c|y|^\varepsilon}{(1+|x|)^{n+1+\varepsilon}}, \quad |y| \leq \frac{|x|}{2}, \text{ some } \varepsilon > 0. \tag{1.3}$$

For instance, these conditions are satisfied by the well-known functions

$$\psi(x) = \frac{\partial}{\partial t} \frac{c_n t}{(t^2 + |x|^2)^{(n+1)/2}}\bigg|_{t=1}$$

and

$$\psi_j(x) = \frac{\partial}{\partial x_j} \frac{1}{(1+|x|^2)^{(n+1)/2}} \quad (j = 1, 2, \ldots, n)$$

with $\varepsilon = 1$, and higher order derivatives of these functions of order k give examples with $\varepsilon = k$.

For a fixed Littlewood-Paley function ψ, let

$$g_\psi(f)(x) = g(f)(x) = \left\{ \int_0^\infty |f * \psi_t(x)|^2 \frac{dt}{t} \right\}^{1/2} \tag{1.4}$$

denote the Littlewood-Paley g-function of f. The Littlewood-Paley g-function

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of f is well defined under some assumptions on f, although it may be infinite on a set of positive measure.

As in [1], page 312, we have

$$\|g(f)\|_p \leq C\|f\|_p \quad (1 < p < \infty), \tag{1.5}$$

where C is independent of f.

Recently, Wang Silei [2] discussed the end-point cases of this result when $f \in L^\infty(R^n)$ or $f \in \mathrm{BMO}(R^n)$.

The aim of this paper is to study the behaviour of $g(f)$ when $f \in Lip_\alpha(R^n)$ $(0 < \alpha < 1)$.

THEOREM 1. *If $f \in Lip_\alpha(R^n)$, $0 < \alpha < \min\{\varepsilon, 1\}$, and $g(f)(x_0) < \infty$ for a single point x_0, then $g(f) \in Lip_\alpha(R^n)$ and*

$$\|g(f)\|_{\Lambda_\alpha} \leq C\|f\|_{\Lambda_\alpha},$$

where $\|f\|_{\Lambda_\alpha}$ denotes the Lip_α norm of f, and C is a constant depending only on n and α.

The assumption concerning the finiteness of $g(f)(x_0)$ is essential. In fact, consider the classical Littlewood-Paley g-function

$$g(f)(x) = \left\{\int_0^\infty t|\nabla u(x, t)|^2\, dt\right\}^{1/2},$$

where

$$p_t(y) = \frac{c_n t}{\left(t^2 + |y|^2\right)^{(n+1)/2}}, \quad c_n = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\pi^{(n+1)/2}},$$

$$u(x, t) = \int_{R^u} p_t(y) f(x - y)\, dy \quad (t > 0),$$

$$|\nabla u(x, t)|^2 = \left|\frac{\partial u}{\partial t}\right|^2 + \sum_{j=1}^n \left|\frac{\partial u}{\partial x_j}\right|^2.$$

THEOREM 2. *There exists a function $f \in Lip_\alpha(R^n)$ for all α $(0 < \alpha < 1)$, such that $g(f)(x) = \infty$ everywhere where g is the classical Littlewood-Paley g-function.*

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We say that a locally integrable function f on R^n is Lipschitz α $(0 < \alpha < 1)$, and denote this by $f \in Lip_\alpha$; if there is a constant c so that $|f(x) - f(y)| \leq c|x - y|^\alpha$ for every x, y in R^n, the smallest such constant c is called the Lip_α norm of f and is denoted by $\|f\|_{\Lambda_\alpha}$.

LEMMA 1 [1, page 213]. *Suppose that f is a locally integrable function on R^n, $0 < \alpha < 1$. Then the following four statements are equivalent and the constants appearing on the right-hand side of each are also equivalent.*

(i) $|f(x) - f(y)| \leq c_1|x - y|^\alpha$, *all* x, y *in* R^n.

(ii)

$$\sup \frac{1}{|Q|^{1+\alpha/n}} \int_Q |f(x) - f_Q|\, dx = c_2 < \infty,$$

where the supremum ranges over all finite cubes Q in R^n whose sides are parallel to the axes, $|Q|$ is the Lebesgue measure of Q, and f_Q denotes the mean value of f over Q, namely,

$$f_Q = \frac{1}{|Q|} \int_Q f(y)\, dy.$$

(*We will consider only these cubes in what follows.*)

(iii) $|f(x) - f_Q| \leq c_3|Q|^{\alpha/n}$, *all* $x \in R^n$, $Q \subset R^n$.

(iv)

$$\sup \left\{ \frac{1}{|Q|^{1+\alpha p/n}} \int_Q |f(x) - f_Q|^p\, dx \right\}^{1/p} = c_4 < \infty, \quad 1 \leq p < \infty,$$

where the supremum is taken over the same range as in (ii). *The equivalence is to be understood to mean that each f which satisfies* (ii), (iii) *or* (iv), *can be modified in a set of measure zero so as to coincide with a continuous function which satisfies* (i) *as well.*

We will use freely c_1, c_2, c_3, c_4 as the Lip_α norm $\|f\|_{\Lambda_\alpha}$ of f.

Remark. In Lemma 1, if (ii) holds for some constant in place of f_Q then it also holds for f_Q.

LEMMA 2. *Suppose that* $f \in Lip_\alpha(R^n)$, $\beta > 0$, $0 < \alpha < \min\{\beta, 1\}$, Q *is a cube with center* x_0, *edge-length* h. *Then*

$$\int_{R^n \setminus 4\sqrt{n}Q} \frac{|f(x) - f_{4\sqrt{n}Q}|}{h^{n+\beta} + |x - x_0|^{n+\beta}} dx \le Ch^{\alpha-\beta} \|f\|_{\Lambda_\alpha}, \tag{1.6}$$

C a constant depending only on n, α *and* β.

Using Lemma 1, the proof of Lemma 2 is similar to a result of Fefferman-Stein's [3], and we omit it.

LEMMA 3. *Suppose that* $f \in Lip_\alpha(R^n)$ $(0 < \alpha < 1)$, Q *is a cube and* x_0 *is in its interior, then* $g(f\chi_Q)(x_0) < \infty$.

Proof. To see this we use different estimates on $\psi_t^*(f\chi_Q)(x_0)$ depending on the size of t. If $t \ge 1$ we observe that

$$|\psi_t * (f\chi_Q)(x_0)| \le C\int_{|y| \le C} \frac{t}{(t + |y|)^{n+1}} dy \le Ct^{-n}.$$

This uses only the fact that $f\chi_Q$ is bounded and has compact support. As for $t \le 1$, we write

$$\begin{aligned}\psi_t * (f\chi_Q)(x_0) &= \int_{|y| \le d} (f(x_0 - y) - f(x_0))\psi_t(y)\, dy \\ &\quad + \int_{|y| \ge d} \big(f(x_0 - y)\chi_Q(x_0 - y) - f(x_0)\big)\psi_t(y)\, dy\end{aligned}$$

where we choose $d > 0$ but $d \le$ distance from x_0 to the complement of Q. Then

$$\left|\int_{|y| \le d} (f(x_0 - y) - f(x_0))\psi_t(y)\, dy\right| \le C\int |y|^\alpha |\psi_t(y)|\, dy \le Ct^\alpha$$

since $f \in Lip_\alpha(R^n)$ and

$$\begin{aligned}&\left|\int_{|y| \ge d} \big(f(x_0 - y)\chi_Q(x_0 - y) - f(x_0)\big)\psi_t(y)\, dy\right| \\ &\quad \le C\int_{|y| \ge d} \frac{t}{(t + |y|)^{n+1}} dy \\ &\quad \le Ct,\end{aligned}$$

since fx_Q is bounded.

These estimates easily imply $g(f\chi_Q)(x_0) < \infty$.

II. Proof of Theorem 2

First we consider the case $n = 1$, and define the function

$$f(x) = \begin{cases} 1, & x \geq 1, \\ x, & 0 < x < 1, \\ 0, & x \leq 0. \end{cases}$$

To estimate $|f(x_1) - f(x_2)|$, without loss of generality, we assume that $x_1 < x_2$. If $x_1 < 0$ or $x_2 > 1$, observing that f is constant on $(-\infty, 0]$ and $[1, +\infty)$, by the triangle inequality, we may reduce these cases to $x_1 = 0$ or $x_2 = 1$. That is, we need only to deal with the special case $0 \leq x_1, x_2 \leq 1$. In this case, $|x_1 - x_2| \leq 1$, so

$$|f(x_1) - f(x_2)| = |x_1 - x_2| \leq |x_1 - x_2|^\alpha \quad \text{for } 0 < \alpha < 1.$$

We will prove that $g(f)(x) = \infty$ everywhere.

A simple calculation gives

$$\begin{aligned} \frac{\partial u}{\partial x}(x, t) &= -2c_1 \int_{R^1} f(x - y) \frac{ty}{(t^2 + y^2)^2} dy \\ &= -2c_1 \int_{-\infty}^{x-1} \frac{ty}{(t^2 + y^2)^2} dy - 2c_1 \int_{x-1}^{x} \frac{(x - y)ty}{(t^2 + y^2)^2} dy \\ &= c_1\left(\operatorname{arctg}\frac{x}{t} - \operatorname{arctg}\frac{x-1}{t}\right) \quad (t > 0). \end{aligned}$$

Now, by the mean value theorem, we have

$$\operatorname{arctg}\frac{x}{t} - \operatorname{arctg}\frac{x-1}{t} = \frac{1}{t} \cdot \frac{1}{1 + \xi^2} \quad \left(t > 0, \frac{x}{t} > \xi > \frac{x-1}{t}\right),$$

and consequently, if we let $\tilde{x}^2 = \max\{x^2, (x-1)^2\}$, it follows that

$$\begin{aligned} \frac{\partial u}{\partial x} &= \frac{c_1}{t} \cdot \frac{1}{1 + \xi^2} \geq c_1 \cdot \frac{1}{t} \cdot \frac{1}{1 + \max\left\{\frac{x^2}{t^2}, \frac{(x-1)^2}{t^2}\right\}} \\ &= \frac{c_1 t}{t^2 + \tilde{x}^2} \quad (t > 0). \end{aligned}$$

Thus,

$$g(f)(x) = \left\{\int_0^\infty t|\nabla u(x, t)|^2 \, dt\right\}^{1/2} \geq c_1 \left\{\int_0^\infty \frac{t^3}{(t^2 + \tilde{x}^2)^2} dt\right\}^{1/2} = \infty.$$

Next, for the case $n \geq 2$, for $x = (x_1, x_2, \dots, x_n) \in R^n$, define the function

$$f(x) = \begin{cases} 1, & x_n \geq 1, \\ x_n, & 0 < x_n < 1, \\ 0, & x_n \leq 0. \end{cases}$$

A similar proof can be used to show that $f \in Lip_\alpha(R^n)$ and $g(f)(x) = \infty$ everywhere, and the Theorem 2 follows.

III. Proof of Theorem 1

We first prove that $g(f)(x) < \infty$ a.e. on R^n. It suffices to prove that $g(f)(x) < \infty$ a.e. on each cube Q_0 containing x_0. Now, suppose that a cube Q_0 with edge-length h_0 is chosen such that x_0 is in its interior, and write

$$\begin{aligned} f(x) &= f_{4\sqrt{n}Q_0} + \left(f(x) - f_{4\sqrt{n}Q_0}\right)\chi_{4\sqrt{n}Q_0}(x) \\ &\quad + \left(f(x) - f_{4\sqrt{n}Q_0}\right)\chi_{R^n \setminus 4\sqrt{n}Q_0}(x) \\ &= f_1(x) + f_2(x) + f_3(x). \end{aligned}$$

Note that the fact that $\int_{R^n} \psi(y)\, dy = 0$ implies that

$$g(const.) = 0, \text{ so } g(f_1) \equiv 0. \tag{3.2}$$

As for $f_2(x)$, using (1.5) and Lemma 2 (iv) with $p = 2$, we have

$$\begin{aligned} \int_{Q_0} |g(f_2)(x)|^2\, dx &\leq \int_{R^n} |g(f_2)(x)|^2\, dx \\ &\leq C \int_{R^n} |f_2(x)|^2\, dx \\ &= C \int_{4\sqrt{n}Q_0} |f(x) - f_{4\sqrt{n}Q_0}|^2\, dx \\ &\leq C|Q_0|^{1+(2/n)\alpha} \|f\|_{\Lambda_\alpha}^2, \end{aligned} \tag{3.3}$$

C a constant, not necessarily the same at each occurrence, depending only on n and α. Thus by the Cauchy-Schwarz inequality,

$$\begin{aligned} \int_{Q_0} |g(f_2)(x)|\, dx &\leq |Q_0|^{1/2} \left(\int_{Q_0} |g(f_2)(x)^2\, dx \right)^{1/2} \\ &\leq C|Q_0|^{1+\alpha/n} \|f\|_{\Lambda_\alpha}. \end{aligned} \tag{3.4}$$

It also follows from (3.4) that

$$g(f_2)(x) < \infty \quad \text{a.e. on } Q_0. \tag{3.5}$$

Now we consider $g(f_3)(x)$, $x \in Q_0$. We have $g(f_3)(x)$ bounded by the sum of

$$I_1(x) = \left\{ \int_0^{h_0} |\psi_t * f_3(x) - \psi_t * f_3(x_0)|^2 \frac{dt}{t} \right\}^{1/2}$$

and

$$I_2(x) = \left\{ \int_{h_0}^{\infty} |\psi_t * f_3(x) - \psi_t * f_3(x_0)|^2 \frac{dt}{t} \right\}^{1/2},$$

and

$$g(f_3)(x_0) = \left\{ \int_0^{\infty} |\psi_t * f_3(x_0)|^2 \frac{dt}{t} \right\}^{1/2}.$$

Applying Lemma 3 to the function $f - f_{4\sqrt{n}\,Q_0}$ and the cube $4\sqrt{n}\,Q_0$, $g(f_2)(x_0) < \infty$, and so

$$g(f_3)(x_0) \le g(f_2)(x_0) + g(f)(x_0) < \infty. \tag{3.6}$$

By (1.2),

$$\begin{aligned} I_1(x) &= \left\{ \int_0^{h_0} \left[\int_{R^n \setminus 4\sqrt{n}\,Q_0} [\psi_t(x-y) - \psi_t(x_0-y)] \right.\right. \\ &\qquad \left.\left. \times \left[f(y) - f_{4\sqrt{n}\,Q_0} \right] dy \right]^2 \frac{dt}{t} \right\}^{1/2} \\ &\le \left\{ \int_0^{h_0} \left[\int_{R^n \setminus 4\sqrt{n}\,Q_0} [|\psi_t(x-y)| + |\psi_t(x_0-y)|] \right.\right. \\ &\qquad \left.\left. \times |f(y) - f_{4\sqrt{n}\,Q_0}|\, dy \right]^2 \right\}^{1/2} \\ &\le C \left\{ \int_0^{h_0} t \left[\int_{R^n \setminus 4\sqrt{n}\,Q_0} \left[\frac{1}{(t + |x-y|)^{n+1}} + \frac{1}{(t + |x_0 - y|)^{n+1}} \right] \right.\right. \\ &\qquad \left.\left. \times |f(y) - f_{4\sqrt{n}\,Q_0}|\, dy \right]^2 dt \right\}^{1/2}. \end{aligned} \tag{3.7}$$

The conditions x_0, $x \in Q_0$ and $y \notin 4\sqrt{n}\,Q_0$, $0 < t < h_0$, imply that

$$|x - y| \geq C|x_0 - y| \geq C(|x_0 - y| + h_0)$$

and

$$\frac{1}{(|x_0 - y| + t)^{n+1}} \leq \frac{C}{|x_0 - y|^{n+1} + h_0^{n+1}},$$
$$\frac{1}{(|x - y| + t)^{n+1}} \leq \frac{C}{|x_0 - y|^{n+1} + h_0^{n+1}}.$$

Therefore, we obtain

$$I_1(x) \leq C\left\{\int_0^{h_0} t\left[\int_{R^n \setminus 4\sqrt{n}\,Q_0} \frac{|f(y) - f_{4\sqrt{n}\,Q_0}|}{|x_0 - y|^{n+1} + h_0^{n+1}}\,dy\right]^2 dt\right\}^{1/2}.$$

By Lemma 2 with $\varepsilon = 1$,

$$I_1(x) \leq C\left\{\int_0^{h_0} t \cdot h_0^{2\alpha-2}\|f\|_{\Lambda_\alpha}^2\,dt\right\}^{1/2} \leq Ch_0^{\alpha}\|f\|_{\Lambda_\alpha} \leq C|Q_0|^{\alpha/n}\|f\|_{\Lambda_\alpha}. \tag{3.8}$$

Now we estimate $I_2(x)$. The fact that x_0, $x_1 \in Q_0$ and $y \notin 4\sqrt{n}\,Q_0$ implies that

$$|(x - y) - (x_0 - y)| \leq \tfrac{1}{2}|x_0 - y|,$$

and by (1.3),

$$|\psi_t(x - y) - \psi_t(x_0 - y)| \leq ct^{-n}\left|\frac{x - x_0}{t}\right|^{\varepsilon} \cdot \frac{1}{\left(1 + \frac{|x_0 - y|}{t}\right)^{n+1+\varepsilon}}$$
$$\leq ct \cdot h_0^{\varepsilon} \cdot \frac{1}{(t + |x_0 - y|)^{n+1+\varepsilon}}. \tag{3.9}$$

Applying Minkowski's inequality we have

$$\begin{aligned} I_2(x) &= \left\{ \int_{h_0}^{\infty} \left| \int_{R^n \setminus 4\sqrt{n}\,Q_0} [\psi_t(x-y) - \psi_t(x_0-y)] \right.\right. \\ &\qquad\qquad \left.\left. \times \left[f(y) - f_{4\sqrt{n}\,Q_0} \right] dy \right|^2 \frac{dt}{t} \right\}^{1/2} \\ &\le \int_{R^n \setminus 4\sqrt{n}\,Q_0} \left[\int_{h_0}^{\infty} |\psi_t(x-y) - \psi_t(x_0-y)|^2 \right. \\ &\qquad\qquad \left. \times |f(y) - f_{4\sqrt{n}\,Q_0}|^2 \frac{dt}{t} \right]^{1/2} dy \\ &\le \int_{R^n \setminus 4\sqrt{n}\,Q_0} h_0^{\varepsilon} |f(y) - f_{4\sqrt{n}\,Q_0}| \\ &\qquad \times \left[\int_{h_0}^{\infty} \frac{t^2}{(t + |x_0 - y|)^{2(n+1+\varepsilon)}} \cdot \frac{dt}{t} \right]^{1/2} dy, \end{aligned} \tag{3.10}$$

and for $x_0 \in Q_0$, $y \in R^n \setminus 4\sqrt{n}\,Q_0$,

$$\begin{aligned} &\left[\int_{h_0}^{\infty} \frac{t^2}{(t + |x_0 - y|)^{2(n+1+\varepsilon)}} \cdot \frac{dt}{t} \right]^{1/2} \\ &\qquad \le C \left[\int_{h_0}^{\infty} \frac{t}{\left(t^2 + |x_0 - y|^2\right)^{(n+1+\varepsilon)}} \, dt \right]^{1/2} \\ &\qquad \le C \left[\frac{1}{\left(h_0^2 + |x_0 - y|^2\right)^{n+\varepsilon}} \right]^{1/2} \\ &\qquad \le \frac{C}{|x_0 - y|^{n+\varepsilon} + h_0^{n+\varepsilon}}. \end{aligned} \tag{3.11}$$

Then we use Lemma 2 to obtain

$$\begin{aligned} I_2(x) &\le C \int_{R^n \setminus 4\sqrt{n}\,Q_0} h_0^{\varepsilon} \frac{|f(y) - f_{4\sqrt{n}\,Q_0}|}{|x_0 - y|^{n+\varepsilon} + h_0^{n+\varepsilon}} dy \\ &\le C h_0^{\varepsilon} \cdot h_0^{\alpha - \varepsilon} \|f\|_{\Lambda_\alpha} \le C |Q_0|^{\alpha/n} \|f\|_{\Lambda_\alpha}. \end{aligned} \tag{3.12}$$

Combining (3.6), (3.8) and (3.12), we have $g(f_3)(x) < \infty$, and so

$$g(f)(x) \le g(f_2)(x) + g(f_3)(x) < \infty \quad \text{a.e. on } Q_0.$$

Next let Q be any cube with edge-length h. Write

$$\begin{aligned} f(x) &= f_{4\sqrt{n}Q} + \big(f(x) - f_{4\sqrt{n}Q}\big)\chi_{4\sqrt{n}Q}(x) + \big(f(x) - f_{4\sqrt{n}Q}\big)\chi_{R^n \setminus 4\sqrt{n}Q}(x) \\ &= f_1(x) + f_2(x) + f_3(x). \end{aligned} \tag{3.13}$$

Then $g(f_1)(x) \equiv 0$, and repeating the process to prove (3.4) with Q_0 replaced by Q, we have

$$\int_Q |g(f_2)(x)|\, dx \le C|Q|^{1+\alpha/n}\|f\|_{\Lambda_\alpha}. \tag{3.14}$$

It has been shown that $g(f)(x) < \infty$ a.e., so that

$$g(f_3)(x) \le g(f_2)(x) + g(f)(x) < \infty \qquad a.e..$$

Therefore, there must be a point $\bar{x} \in Q$ for which $g(f_3)(\bar{x}) < \infty$. Repeating the proof of (3.8) and (3.12) with Q_0 replaced by Q, we have

$$\left\{\int_0^\infty |\psi_t * f_3(x) - \psi_t * f_3(\bar{x})|^2 \frac{dt}{t}\right\}^{1/2} \le C|Q|^{\alpha/n}\|f\|_{\Lambda_\alpha}, \quad x \in Q.$$

Thus for $x \in Q$,

$$\begin{aligned} |g(f_3)(x) - g(f_3)(\bar{x})| &\le \left[\int_0^\infty |\psi_t * f_3(x) - \psi_t * f_3(\bar{x})|^2 \frac{dt}{t}\right]^{1/2} \\ &\le C|Q|^{\alpha/n}\|f\|_{\Lambda_\alpha} \end{aligned}$$

and consequently

$$\int_Q |g(f_3)(x) - g(f_3)(\bar{x})|\, dx \le C|Q|^{1+\alpha/n}\|f\|_{\Lambda_\alpha}. \tag{3.15}$$

Therefore, we have

$$\begin{aligned}
&\int_Q |g(f)(x) - g(f_3)(\bar{x})| \, dx \\
&\leq \int_Q |g(f)(x) - g(f_3)(x)| \, dx + \int_Q |g(f_3)(x) - g(f_3)(\bar{x})| \, dx \\
&\leq \int_Q |g(f_2)(x)| \, dx + \int_Q |g(f_3)(x) - g(f_3)(\bar{x})| \, dx \\
&\leq C|Q|^{1+\alpha/n} \|f\|_{\Lambda_\alpha}. \qquad (3.16)
\end{aligned}$$

Finally, by the remark after Lemma 1, for each cube Q,

$$\frac{1}{|Q|^{1+\alpha/n}} \int_Q |g(f)(x) - (g(f))_Q| \, dx \leq C\|f\|_{\Lambda_\alpha},$$

namely,

$$\|g(f)\|_{\Lambda_\alpha} \leq C\|f\|_{\Lambda_\alpha},$$

C a constant depending only on n and α.

This completes the proof of Theorem 1.

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