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ALGEBRAIC ASPECTS OF CHEN'S TWISTING COCHAIN

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In Memoriam Kuo-Tsai Chen

1. Introduction

The present volume is not only a memorial to K. T. Chen, but provides the opportunity, quoting [HT], for "a somewhat revisionist view of his approach to deRham homotopy theory". Since our own work has been heavily influenced by Chen's, we offer the following further but somewhat different insights into his work. In particular, one of Chen's major contributions was a method for computing the homology of the loop space on a manifold in terms of the homology of the manifold. He effected this via a differential on the tensor algebra T^a on the desuspension of the reduced homology of X. The differential satisfied a certain condition (detailed in §1.2 below) with respect to an element ω of the completed tensor product of the forms on X with $T^a s^{-1} H_+(X)$. Chen called ω a *formal power series connection*. By identifying this tensor product with an appropriate Hom, Chen's condition becomes that of a *twisting cochain*, as he implies in [C3] after Theorem 7.1. To provide a multiplicative chain equivalence Θ: $C_*(\Omega X) \to (T^a s^{-1} H_+(X), \partial)$, Chen makes use of his iterated integrals.

Over the years, we have come to realize more fully the depth and significance of Chen's constructions. Initially, many of us focused excessively on two aspects of Chen's work: the analysis symbolized in his phrase "iterated integrals" and the homotopy theory of his alternative to Adams' cobar construction. From our current point of view, it is the algebraic aspects of Chen's work which have broadest significance and, within that context, some emphasis could be placed on his constructing a much smaller model than did Adams. The analysis used forced the homotopy theory into characteristic zero, whereas, with hindsight, we can see large portions of Chen's

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methods applying in general characteristic. The beginning of this broader appreciation of Chen occurs in [G2].

In the early 1960's, Gugenheim became interested in the idea of a twisting cochain and noticed that E. H. Brown's theorem [EB] was by no means the only occurrence. About 10 years later, he realized that twisting cochains and homotopies of twisting cochains are at the heart of Chen's work. This interest culminated in [G2] where Gugenheim gave a purely algebraic version of Chen's theorem on the homology of the loop space [C1], [G2]. Gugenheim's map is a purely algebraic analog of the map given by Chen's iterated integrals. This theorem then came to Stasheff's attention and the connection with A_∞ structures was made; from this source much of the recent work on perturbation theory grew. Much "western" work on this subject was thus inspired by Chen's ideas. There were independent developments in the USSR by Berikashvili, Kadeishvilli, Sanablidze and others. Contact between the western and USSR groups is quite recent and presumably due to the lifting of restrictions in the USSR. Unfortunately this came too late for Chen whose response to the "Georgian school" we would so much like to have seen.

One of the characteristic features of Chen's and Gugenheim's twisting cochains is that they yield twisted tensor products which are *acyclic*. This has already been mentioned in [GL] in connection with some of Gugenheim's work with J. P. May and also provides a link to earlier work of H. Cartan and J. C. Moore which we will exploit elsewhere.

In addition to the extended insights into Chen already mentioned, the present paper provides new perspectives on what has come to be called *homological perturbation theory*. That theory usually involves one of two methods—one being an obstruction theoretic approach and the other involving direct transfer of structure *up to homotopy*. Both methods parallel techniques developed originally in the homotopy theory of CW-complexes, especially those carrying the structure of an H-space. Originally, the obstruction method for constructing a twisting cochain, though more cumbersome, seemed to encode the total information efficiently and to have the additional advantage of producing strict derivations, multiplicative maps, etc. The relationship between various perturbation methods was given in some classical cases in [GL]; the general situation (in [GLS], [HK]) was much more subtle than originally envisioned, but resulted in an appreciation of previously hidden strengths of the Basic Perturbation Lemma. We make full use of the algebraic consequences of the direct method [GLS] in this paper.

In §1.1, we review the cobar construction and Adams' theorem. In §1.2, we review Chen's approach to loop space homology. In §1.3, we review twisting cochains and establish the equivalence of twisting cochains to Chen's power series connections. §2 is a review of basic perturbation theory: SDR-data, the obstruction method (especially as inspired by Chen) and the (co)algebra form of the basic perturbation lemma. With this background, in §2.3, we reexamine Chen's results and in particular his passage from connections with values in associative algebras (valid in any characteristic) to those with values in a Lie algebra (valid for commutative (co)chains in characteristic zero).

The final section addresses issues still more algebraic in significance. First is the relation to (algebraic) deformation theory, with emphasis on its relevance to a particular theorem of Chen. Then we discuss "formality" for DGA-algebras in general characteristic. We conclude with an attempt to provide a diagram showing how Chen, Adams, Gugenheim and even de Rham at the form level fit together.

We are grateful to the referee for insistence on and helpful suggestions for making our exposition more readable and, hopefully, more informative.

Throughout the paper we will use R for a commutative ring with 1 and K for a field of characteristic 0.

1.1. The cobar construction

Let R be a commutative ring with 1. For a simply connected space X, Adams [FA] defined a chain map which induces a natural isomorphism in homology

$$\mu \colon \overline{\Omega} C_*(X;R) \to CU_*(\Omega X;R)$$

where $\overline{\Omega} C_*(X)$ is the cobar construction of the reduced singular complex $C_*(X)$, and $CU_*(\Omega X;R)$ is the cubical chain complex of the loop space of X.

We will assume that X is of finite type, i.e. has the homotopy type of a CW complex with finitely many cells in each dimension, so that we have the approximately dual result that there are homology isomorphisms

$$\overline{B} C^*(X;R) \to (\overline{\Omega} C_*(X;R))^* \leftarrow CU^*(\Omega X;R)$$

where $\overline{B} C^*(X;R)$ is the bar construction of the cochain algebra $C^*(X;R)$ dual to the coalgebra $C_*(X;R)$ and $CU^*(\Omega X;R)$ is the coalgebra of cubical cochains of ΩX. Recall that the bar construction of an augmented (cochain) algebra A is the tensor coalgebra $T^c s^{-1} I_A$ of the desuspension of the augmentation ideal I_A with a differential $\bar{\partial}$ that is essentially the sum of the tensor product differential and the extension of the multiplication to a coderivation. It is often possible and convenient to use the submodule A^+ of positive degree elements of A in place of I_A.

1.2. Loop space homology

In this section, **K** will stand for either the field of real numbers or the field of complex numbers and homology or cohomology will be taken over **K**.

Chen's method for computing the homology of the loop space of a simply connected finite dimensional differentiable space X is essentially the following. Let $A = T^a s^{-1} H_+(X)$. Let ∂ be an element of $\mathrm{Der}^1(A)$, and let $\mathscr{B} = \{z_1, \dots, z_n\}$ be a basis for $H_+(X)$. Let $\Lambda^*(X)$ denote the algebra of differential forms on X and let $\{1, \omega_1, \dots, \omega_n\}$ be a set of closed forms whose classes in cohomology form a basis for $H^*(X)$ dual to $\mathscr{B}$. Let $X_i = s^{-1} z_i$. For $\alpha \in \Lambda^r X$, write $J\alpha = (-1)^r \alpha$ and extend J to all of $\Lambda^* X$ linearly. Extend J to $\Lambda^* X \otimes A$ by $J(\alpha \otimes x) = J(\alpha) \otimes x$. Chen's theorem [C1, (3.1.1)] is:

THEOREM. *For a given derivation $\partial \in \mathrm{Der}^1(A)$, if there exist forms $\omega_{ij}, \omega_{ijk}, \dots$ such that*

$$\omega = \sum \omega_i X_i + \omega_{ij} X_i X_j + \cdots \qquad (\omega_i \text{ and } X_i \text{ as above})$$

satisfies

$$\partial\omega + d\omega - J\omega \wedge \omega = 0, \tag{1.2.1}$$

the (A, ∂) is a DGA-algebra which is homology equivalent to the chains on the loop space ΩX by the DGA-algebra map

$$\Theta \colon C_*(\Omega X; K) \to A$$

given by

$$\Theta(c) = \langle 1, c \rangle + \sum \left\langle \int \omega_i, c \right\rangle X_i + \sum \left\langle \int \omega_i \omega_j + \omega_{ij}, c \right\rangle X_i X_j + \cdots .$$

Such an ω is called a formal power series connection; the integrals are Chen's iterated integrals which are mentioned in the introduction and [HT]. The existence and indeed uniqueness of (∂, ω) are addressed in Chen's section (3.3.3) where he refers to [C2] for an inductive argument giving an algorithm for such pairs (∂, ω). In fact, this argument was the inspiration for the homological perturbation argument given in [G2] and is briefly reviewed below. We call it "the basic obstruction argument". A similar argument was discovered independently by Kadeishvili in [TK] and is discussed further in [GLS2]. Another algorithm for producing such maps called "the basic perturbation lemma" in [LS] will have a great deal more to say about Chen's construction.

1.3. Twisting cochains and some identifications

If C is a supplemented differential graded coalgebra and A is an augmented differential graded algebra, a *twisting cochain* $\tau \colon C \to A$ is an

R-linear map of degree -1 satisfying the condition [G1]:

$$d\tau + \tau d = \tau \cup \tau. \tag{1.3.1}$$

The "cup-product" $\cup$ is defined in the R-module of R-module maps $C \to A$ by using the coproduct ψ in C and the product m in A. Given two maps $f, g\colon C \to A$,

$$f \cup g = m(f \otimes g)\psi.$$

Any R-module map $\tau\colon C \to A$ of degree -1 determines a graded algebra map $\overline{\Omega} C \to A$ and, if C is connected, a graded coalgebra map $C \to \overline{B}A$ and τ is a twisting cochain if and only if the corresponding maps are maps of differential algebras, respectively coalgebras.

One of the essential properties of the bar construction is the full construction $A \otimes \overline{B}A$ with a twisted differential ∂ such that $(A \otimes \overline{B}A, \partial)$ is acyclic. Moreover, ∂ is given by $d \otimes 1 + 1 \otimes \bar{\partial} + \pi \cap$ where $\bar{\partial}$ is the usual bar construction differential and $\pi\colon \overline{B}A \to A$ is a twisting cochain, the universal twisting cochain given by projection:

$$\begin{aligned} \pi([\ ]) &= 0, \\ \pi([a_1]) &= a_1, \\ \pi([a_1|\cdots|a_n]) &= 0 \quad \text{if } n > 1. \end{aligned}$$

Similarly the cobar construction $\overline{\Omega} C$ fits into a twisted tensor product $\overline{\Omega} C \otimes C$ which is acyclic with respect to a total differential δ which is given by a twisting cochain $\iota\colon C \to \overline{\Omega} C$.

Strictly speaking, Chen's formal power series connections are elements of the completed tensor product

$$\Lambda^* X \mathbin{\hat{\otimes}} T^a s^{-1} H_+(X).$$

Keeping in mind our assumption that spaces have finite type and completing as necessary, we can compare the modules $T^a s^{-1} H_+(X)$ and $[T^c s^{-1} H^+(X), \mathbf{K}]_{\mathbf{K}}$, where $[\ \ ,\ \ ]_{\mathbf{K}}$ denotes the hom set in the category of $\mathbf{K}$-modules (not differential modules). We have a classical isomorphism

$$\Lambda^* X \mathbin{\hat{\otimes}} T^a s^{-1} H_+(X) \cong \left[T^c s^{-1} H^+(X), \Lambda^* X\right]_{\mathbf{K}}.$$

With these identifications, Chen's derivation ∂ in §1.2 corresponds to a *coderivation* of

$$T^c s^{-1} H^+(X)$$

and his formal power series connection ω above corresponds to an R-linear map

$$\bar{\omega}\colon T^c s^{-1} H^+(X) \to \Lambda^* X.$$

His condition (1.2.1) of 'local flatness' is then equivalent to the condition (1.3.1) for $\bar{\omega}$ to be a twisting cochain (as he remarks in [C3]). Since the algebra is that of differential forms, the abstract cup product in (1.3.1) corresponds to the 'wedge product' in (1.2.1) which in turn means that the form 'coefficients' are multiplied by the exterior product of forms and the monomials from $T^a s^{-1} H_+(X)$ are multiplied by juxtaposition.

2. Homological perturbation theory

In this section we will assume that we are given modules M and N and module maps

$$f\colon M \to N, \quad \nabla\colon N \to M, \quad \phi\colon M \to M$$

such that ∇ and f are chain maps and ϕ is a chain homotopy satisfying

$$f\nabla = \mathrm{id}_N, \quad \nabla f = \mathrm{id}_M + D(\phi)$$

where $D(\phi) = d_M \phi + \phi d_M$. This information will be summarized succinctly by the diagram

$$\left(N \underset{f}{\overset{\nabla}{\rightleftarrows}} M, \phi \right).$$

The chain homotopy ϕ can be chosen to satisfy the additional hypotheses called "side conditions" [LS], [GLS]:

$$f\phi = 0, \quad \phi\nabla = 0, \quad \phi\phi = 0.$$

Such objects and maps are called *SDR-data* [LS], [GS], [GL], [GLS], but are also known as E(ilenberg)-Z(ilber) data [GMu] or as contractions [EM1, 2], [JH1]. Homological perturbation theory is concerned with the transfer of structure from one module to the other through such SDR-data or with the comparison of structures on the two modules.

One of the motivating ideas is this: there are R-module maps $f\colon A \to M$ of differential graded augmented algebras which are not multiplicative, but which nonetheless produce induced maps in homology which are. Such is the case for the "integration map" from de Rham theory. In fact, as shown in

[G3], the integration map

$$\kappa\colon \Lambda^*X \to C^*(X) \tag{2.1.1}$$

is the initial map in a sequence of "higher homotopies", i.e., there is a sequence of maps

$$\kappa_i\colon \otimes^i(\Lambda^*X) \to C^*(X)$$

such that $\kappa_1 = \kappa$ and $d\kappa_2 + \kappa_2 d$ is the deviation of κ_1 from being multiplicative [GMu], and more generally the sequence fits together to induce a *twisting cochain* (cf. §1.3)

$$\tilde{\kappa}\colon \overline{B}\Lambda^*X \to C^*(X), \tag{2.1.2}$$

which, by the general theory of twisting cochains, induces a map of differential graded coalgebras

$$\bar{\kappa}\colon \overline{B}\Lambda^*X \to \overline{B}C^*(X). \tag{2.1.3}$$

There is an analogous situation in the case of H-spaces where there are continuous maps $X \to Y$ which are not maps of H-spaces, but nonetheless induce maps of classifying spaces $BX \to BY$ [S1].

The algebraic situation where there is SDR-data

$$\left(M \underset{f}{\overset{\nabla}{\rightleftarrows}} A, \phi\right),$$

M and A are algebras (in any characteristic), and the projection f is *multiplicative* was studied in [GMu] (actually the dual, coalgebra case is done there). It was shown that the following algorithm constructs a twisting cochain (when A is complete):

$$\tau\colon \overline{B}M \to A, \quad \tau = \tau_1 + \tau_2 + \cdots,$$

where

$$\tau_0 = 0, \quad \tau_1 = \alpha\pi_1,$$
$$\tau_n = \sum_{i+j=n} \phi(\tau_i \cup \tau_j).$$

Furthermore, it was shown that the induced map of coalgebras

$$\bar{\tau}\colon \overline{B}M \to \overline{B}A$$

is a homology equivalence. We related this algorithm to the methods discussed below in [GL].

With less structured initial data, we have recourse to one of two main techniques in homological perturbation theory.

2.1. The obstruction method

In this section, we review the obstruction method which Gugenheim arrived at as an algebraic distillation [G2] of Chen's inductive perturbation argument [C2, 1.3.1] and which was generalized in [GS].

When M is *not* assumed to be an algebra, it was shown that there is still a differential ∂ on the tensor coalgebra $T^c s^{-1} M^+$ and a twisting cochain τ from the differential coalgebra $(T^c s^{-1} M^+, \partial)$ to the algebra A such that $(T^c s^{-1} M^+, \partial) \to \bar{B}A$ is a homology isomorphism. We summarize that argument briefly here.

Following Stasheff's terminology and notation, we consider an *A_∞-structure* on M, a differential ∂ which is a coderivation of the tensor coalgebra $T^c s^{-1} M^+$ such that for $m \in M^+$ we have $\partial sm = -sdm$. The differential coalgebra $(T^c s^{-1} M^+, \partial)$ is then denoted $\tilde{B}M$ and called the "tilde construction".

Now assume that we have SDR-data

$$\left(M \underset{f}{\overset{\nabla}{\rightleftarrows}} A, \phi \right)$$

in which A is an algebra and there are no assumptions on ∇ and f other than that they are chain maps. We seek an A_∞-structure on M and a twisting cochain τ from the differential coalgebra $T^c s^{-1} M^+$ to the algebra A. This is done by defining an appropriate filtration on $T^c s^{-1} M^+$ and with respect to this filtration defining $\partial = \lim \partial_i$, and $\tau = \lim \tau_i$ where

$$\partial_{n+1} = \partial_n + y_{n+1}, \quad \tau_{n+1} = \tau_n + x_{n+1}$$

and x_{n+1} is essentially given by $-\phi\Gamma_n$ and y_{n+1} is essentially given by extending $-\beta\Gamma_n$ as a coderivation where Γ_n is the "twisting cochain obstruction"

$$\Gamma_n = (d_A \tau_n + \tau_n \partial_n) - \tau_n \cup \tau_n .$$

The twisting cochain τ induces a coalgebra map $\bar{\tau}$ which is an isomorphism in homology:

$$\bar{\tau}_* : H\tilde{B}M \underset{\cong}{\to} H\bar{B}A .$$

A special case arises in which M is not assumed to be an algebra a priori but the differential in M is assumed to be zero, so that M has an algebra structure by virtue of being isomorphic to $H(A)$. The resulting structure on M is more subtle than one might expect. Although the induced multiplication on M is associative, neither the "inclusion" ∇ nor the projection f need be multiplicative, in general. In fact, in cases which arise naturally, it is not possible to find a map of coalgebras $\bar{B}M \to \bar{B}A$ which is a homology equivalence [GS]. The tilde construction above, however, does give a perturbation of the bar construction differential and a twisting cochain $\tilde{B}M \to A$ such that $\tilde{B}M \to \bar{B}A$ is a homology equivalence, as in Gugenheim's algebraization of Chen's algorithm. We quote from [G2, p. 197]: "Chen uses his iterated integrals; but the basic perturbation argument in 2.12 below is—apart from the context—exactly that of [7, 1.3.1]". (Gugenheim's reference [7] is our [C1].) We have called this form of perturbation argument the "basic obstruction method" because it inductively make use of any failure to have a "correct answer"' at each particular stage of the construction. Such an argument is at the heart of E. H. Brown's twisting cochain construction [EB]. An obstruction argument in the case we are currently focusing on was found independently by Kadeishvili [TK].

2.2. The basic perturbation method for algebras

In this section we discuss the basic perturbation lemma which first occurred in [RB] and independently in [G1]. Both authors mention [WS] as the inspiration. The setup is this: we are given SDR-data

$$\left(M \underset{f}{\overset{\nabla}{\rightleftarrows}} N, \phi\right)$$

and a new differential $\mathscr{D}$ on N. Letting $t = \mathscr{D} - d$, we have sequences

$$\{\nabla_n\}, \{f_n\}, \{\partial_n\}, \{\phi_n\}$$

for $n \geq 0$ where inductively,

$$\nabla_{n+1} = r_n\nabla, \quad f_{n+1} = fs_n,$$
$$\partial_{n+1} = d_M + f\Sigma_n\nabla, \quad \phi_n = \phi s_n$$

and

$$t_{n+1} = (t\phi)^n t, \quad \Sigma_n = t_1 + t_2 + \cdots t_n,$$
$$s_n = 1 + \Sigma_n\phi, \quad r_n = 1 + \phi\Sigma_n.$$

Assuming that there are complete filtrations on M and N so that ∇, f, ∂, and ϕ are filtration preserving and t lowers filtration, the sequences converge to new SDR-data which we denote by

$$\left((M,\partial_\infty) \underset{f_\infty}{\overset{\nabla_\infty}{\rightleftarrows}} (N,\mathscr{D}), \phi_\infty\right).$$

We call t the *initiator*.

It turns out that the results of the basic obstruction method in §2.1 can be obtained by the basic perturbation method. This possibility had arisen in discussions with Huebschmann [JH] and had been observed to work in specific cases involving torsion-free nilpotent groups by Lambe [LL]. A more general case including the known examples was given in [GL] and the general case, which was a bit more subtle than anyone had originally imagined, is given in [GLS] and the result was independently given in [HK]. The proof hinges on the rather surprising fact that the basic perturbation lemma actually preserves algebra or coalgebra structure.

Remark. We use the words "algebra" and "coalgebra" here in a general sense such as an "algebra over a category". There is then the requirement that there is an appropriate tensor product and that is the tensor product taken below. The details are in [GLS]. When the (co)algebras are (co)associative, the ordinary tensor product is taken.

We introduce a decreasing filtration on $\mathrm{Hom}(X,Y)$ for certain modules X and Y. First, consider $\mathscr{A}$, the free associative (tensor) algebra on the generators f, ∇, ϕ, d, and t, modulo the relations

$$dt + td + tt = 0, \quad d\phi + \phi d = \nabla f - \mathrm{Id},$$
$$f\nabla = \mathrm{Id}, \quad f\phi = 0, \quad \phi\nabla = 0, \quad \phi\phi = 0.$$

Now filter $\mathscr{A}$ by $\mathscr{A} = I^0 \supset I^1 \supset \cdots \supset I^n \supset \cdots$ where a monomial is in I^n if t occurs at least n times in the monomial. For X, Y equal to M or N, consider $\mathrm{Hom}(X,Y)$ as filtered by $I^0 \supset I^1 \supset \cdots \supset I^n \supset \ldots$ where I^n is interpreted as the intersection of $\mathrm{Hom}(X,Y)$ with the I^n in $\mathscr{A}$. If M, N are algebras with multiplication denoted m, then for X, Y equal to $M \otimes M, N$ or $N \otimes N, M$, construct analogous filtrations using additional generators $m(f \otimes f)$, $m(\nabla \otimes \nabla)$, $m(1 \otimes \phi)$, $m(\phi \otimes \nabla f)$, $d \otimes 1$, $1 \otimes d$ and $t \otimes t$, where $t \otimes t$ counts as two occurrences.

Algebra Perturbation Lemma [GLS]. *Suppose that*

$$\left(M \underset{f}{\overset{\nabla}{\rightleftarrows}} N, \phi\right)$$

is SDR-data with M and N algebras. Suppose that N has a new derivation $\mathscr{D}$ such that the initiator t is a **derivation**:

$$tm = m(t \otimes 1 + 1 \otimes t).$$

Suppose also that ϕ is an algebra homotopy:

$$\phi m = m(1 \otimes \phi + \phi \otimes \nabla f).$$

We now have sequences

$$\{\nabla_n\}, \{f_n\}, \{\partial_m\}, \{\phi_n\}$$

for $n \geq 0$ as above.

(i) *If ∇ is an algebra map then ∇_{n+1} is an algebra map* mod I_n.
(ii) *If f is an algebra map then f_{n+1} is an algebra map* mod I_n.
(iii) *If ∇ and f are algebra maps and d_M is a derivation then ∂_{n+1} is a derivation.*
(iv) *ϕ_{n+1} is an algebra homotopy* mod I_n.

COROLLARY. *Assuming that there are complete filtrations on M and N so that ∇, f, ∂, and ϕ are filtration preserving and t lowers filtration, the sequences converge to new SDR-data*

$$\left((M, \partial_\infty) \underset{f_\infty}{\overset{\nabla_\infty}{\rightleftarrows}} (N, \mathscr{D}), \phi_\infty\right),$$

which are respectively a derivation, algebra maps and an algebra homotopy.

The dual coalgebra, coderivation statements also hold.

We note that a somewhat different proof of the Algebra Perturbation Lemma was also given in Huebschmann-Kadeishvili [HK].

Using the coalgebra version of the perturbation lemma, we can recapture the results of the obstruction method and obtain even more at the same time. Consider module SDR-data

$$\left(M \underset{f}{\overset{\nabla}{\rightleftarrows}} N, \phi\right)$$

where M and N are connected modules, i.e. $M = R \oplus M^+$ and $N = R \oplus N^+$. Form the tensor coalgebra version of this data [GL], [GLS]:

$$\left(T^c s^{-1} M^+ \underset{T^c s^{-1} f}{\overset{T^c s^{-1} \nabla}{\rightleftarrows}} T^c s^{-1} N^+, T^c s^{-1} \phi\right).$$

The maps $T^c s^{-1}\nabla$, $T^c s^{-1} f$ are the obvious tensor products and the tensor product homotopy $T^c s^{-1}\phi$ is given by

$$T^c s^{-1}\phi = \phi \oplus T_2\phi \oplus \cdots \oplus T_n\phi \oplus \cdots$$

where

$$T_n\phi = \phi \otimes \pi \otimes \cdots \otimes \pi + \cdots \quad \cdots + 1 \otimes \cdots \otimes 1 \otimes \phi \otimes \pi \otimes \cdots \otimes \pi$$
$$+ \cdots \quad \cdots + 1 \otimes \cdots \otimes 1 \otimes \phi$$

and $\pi = \nabla f$. The tensor coalgebra data is SDR-data with respect to the tensor product differentials. (As pointed out in [HK], this idea goes back to Eilenberg and MacLane [EM] who refer to a contraction of bar constructions.) We now assume that N is an algebra and give $T^c s^{-1} N^+$ a new differential, viz. the *bar construction* differential. The difference between this bar construction differential and the tensor product differential is our initiator t which, in this case, is just the "ordinary algebra" bar construction differential, i.e., the differential in the bar construction of the algebra obtained from N by forgetting the differential in N and thinking of N just as an ordinary algebra. As in [GLS], we note that the hypotheses of the coalgebra perturbation lemma are satisfied and so we obtain new SDR-data

$$\left(\tilde{B}M \underset{(T^c s^{-1} f)_\infty}{\overset{(T^c s^{-1}\nabla)_\infty}{\rightleftarrows}} \bar{B}N, (T^c s^{-1}\phi)_\infty \right),$$

where we are using the tilde notation mentioned earlier for $(T^c s^{-1} M^+, \partial_\infty)$. The maps

$$\alpha = (T^c s^{-1}\nabla)_\infty \quad \text{and} \quad \beta = (T^c s^{-1} f)_\infty$$

are differential coalgebra maps and the differential ∂_∞ is a coderivation by the coalgebra perturbation lemma. We thus obtain a coderivation and a twisting cochain as in the basic obstruction method; however we also have the coalgebra map β about which we will say just a few words here and come back to in another note. In the special case where the differential in M is zero, so that $M = H(N)$, we have already mentioned that Kadeishvili [TK] obtained such a coderivation and twisting cochain. He also obtained a "homotopy twisting cochain" in the other direction, i.e., a map $\bar{B}N \to M$ which induces a coalgebra map $\bar{B}N \to \tilde{B}M$ of differential coalgebras and is essentially our map β. We have therefore generalized this result to the case where the differential in M is non-zero. This is analyzed in detail in [GLS].

2.3. The basic perturbation lemma for Lie (co)algebras

One of Chen's major motivations and developments over the years was the restriction of his twisting cochains to have values in a Lie algebra, in particular, the free Lie algebra generated by the reduced homology of M. His work with what we have called the obstruction method was carried forward by Hain [RH], who also constructed twisting cochains starting from a differential graded Lie algebra. Here we need a variant of the algebra perturbation lemma in characteristic 0 (see the Remark in (2.2)). We will need to look at Lie (co)algebras and (co)commutative algebras.

Let us turn our attention to a differential graded algebra A which is commutative in the graded sense, henceforth denoted by labeling it a *CDGA*, e.g., $\Lambda^*(X)$. For any connected module M, $T^c s^{-1}M^+$ is a Hopf algebra with respect to the *shuffle product* [HC]. The module of indecomposables $QT^c s^{-1}M^+$ with respect to the shuffle product inherits the structure of a Lie coalgebra, in fact, it is the free Lie coalgebra if M is of finite type. (See [NJ, pp. 167–170] for the more familiar description of the free Lie algebra—Friederich's Theorem.) If further M is a differential module, QT^cM is a differential graded Lie coalgebra with respect to the inherited differential. Now consider module SDR-data

$$\left(M \underset{f}{\overset{\nabla}{\rightleftarrows}} N, \phi\right)$$

where M and N are connected modules. Form the Lie coalgebra of indecomposables of this data

$$\left(QT^c s^{-1}M^+ \underset{QT^c s^{-1}f}{\overset{QT^c s^{-1}\nabla}{\rightleftarrows}} QT^c S^{-1}N^+, \Phi\right).$$

The homotopy $T^c s^{-1}\phi$ does not respect shuffles and so does not induce a map on indecomposables. The map Φ above is constructed by fully symmetrizing $T^c s^{-1}\phi$, i.e., $\Phi = \Sigma\phi_n$ where ϕ_n is defined on n-fold tensor products as $(1/n!)\Sigma_\sigma \sigma T_n\phi$ and where the summation is over all permutations σ of $\{1, 2, \ldots, n\}$ acting on $T_n\phi$ with the appropriate signs. This homotopy does not satisfy the (co)algebra condition of the (co)algebra perturbation lemma, but does satisfy a corresponding symmetric relation. The proof of the algebra perturbation lemma in [GLS] is reworked carefully to accommodate this symmetry and provide a Lie coalgebra analog of the lemma.

Now assume that N is a CDGA A so that the bar construction differential $\bar{\partial}$ is also a derivation with respect to the shuffle product and hence passes to the module of indecomposables. Again we take the difference between this bar construction differential and the differential in the Lie coalgebra data above to be our initiator t and thus obtain new SDR-data

$$\left((QT^cM^+, \partial_\infty) \underset{(QT^cf)_\infty}{\overset{(QT^c\nabla)_\infty}{\rightleftarrows}} Q\bar{B}A, (\Phi)_\infty \right).$$

Applied to $A = \Lambda^*(X)$ and $M = H^*(X)$, this produces $(QT^c\nabla)_\infty$ which is determined by a twisting cochain $QT^cH^*(X) \to \Lambda^*(X)$. Using the classical isomorphism of page 4, we arrive at Chen's

$$\omega \in \Lambda(X) \hat{\otimes} \mathscr{P}H_*(X)$$

where $\mathscr{P}$ denotes the primitives of $T^aH_*(X)$ (i.e., the free Lie algebra on $H_*(X)$ with respect to "shuffle-coproduct"). [Compare [RH].]

3. Other applications

3.1. Deformation theory

We have emphasized that a major aspect of Chen's work is to provide a small and computable model for $C_*(\Omega X)$, but there is an alternate reading especially of [C1], although Chen himself did not pursue it. Chen's approach emphasizes a pair (ω, ∂) satisfying the twisting cochain condition, but he first considers a general $\partial \in \operatorname{Der} T^as^{-1}H_*(M)$. From the point of view of rational homotopy theory, it is reasonable to consider an arbitrary graded vector space H as being the homology of more than one space X, not even necessarily a manifold. Then assuming $\Lambda^*(X)$ is defined, e.g., via compatible forms on the singular simplicies of X, the question can be asked: Given $\partial \in \operatorname{Der} T^as^{-1}H$, does there exist a space X and a twisting cochain $\bar{\omega} \in [T^csH, \Lambda^*X]_{\mathbf{K}}$? If so, Chen's result is that $(T^as^{-1}H, \partial)$ has the homology of $H_*(\Omega X)$ as an algebra. There are two obvious necessary conditions. The twisting cochain condition (1.3.1) requires ∂ to be of degree 1 with respect to the total grading on $[T^csH, \Lambda^*X]_R$, and Chen shows that the twisting cochain equation implies that $\partial\partial = 0$.

Hain and Tondeur [HT] interpret Chen's construction of his connection form as the construction of the dual of the versal deformation of a trivial connection. Again from the point of view of rational homotopy theory, we have an alternate interpretation in terms of the classification of rational homotopy types of DGA's with $H^*(X)$ isomorphic to a fixed H, either as an algebra [SS] or more generally just as a vector space [YF]. According to [SS], generalized by dropping the commutativity restriction, the rational homotopy types with fixed cohomology algebra H are classified as follows: The multiplication of the algebra H determines a particular differential d on $T^cs^{-1}H$, namely the bar construction differential. The classification then proceeds in terms of $\theta \in \mathrm{Coder}(T^c s^{-1}H)$ such that $d + \theta$ is a differential, i.e., $(d + \theta)^2 = 0$. Expanded, this yields $d\theta + \theta d = \theta\theta$. The background for this approach is in deformation theory and this latter is known as the deformation equation.

The resemblance to the twisting cochain condition is striking. Of course, it is more than a resemblance. This can be seen in two ways. Given a twisting cochain $\omega \in [C, A]$, we form the twisted tensor product $C \otimes A$ with total differential $d_C \otimes 1 + 1 \otimes d_A + \omega \cap$. In particular, Chen's twisting cochain gives a twisted tensor product

$$T^a s^{-1} H_+(X) \hat{\otimes} \Lambda^*(X)$$

which is acyclic. Chen's differential ∂ on $T^a s^{-1} H_+(X)$ is "dual" to one on $T^c s^{-1} H^+(X)$ which is of Schlessinger and Stasheff form $d + \theta$ [SS].

On the other hand, given a θ of this type, Schlessinger and Stasheff show it can be regarded as a twisting cochain from a 'point' into $\Lambda\,\mathrm{Coder}(T^c s^{-1} H)$.

3.2. Formality and integer homology applications

First, recall that the integration map (2.1.1) from de Rham theory gives rise to a twisting cochain κ: $\bar{B}\Lambda^* X \to C^*(X)$. Notice that we are writing the *ordinary* bar construction here, i.e., there are no higher terms necessary in the differential. This is one of the central issues in [G3], [GMu], [BG], and [G4]. With rational, real or complex coefficients, we may work with the less complicated differential algebra $\Lambda^* X$ in place of $C^*(X)$ in the context of differential homological algebra. It is in descending to smaller complexes or coefficient rings that the A_∞ structure comes into play.

In rational homotopy theory [GM], [DS], [DGMS], [BG], [HS], [BL], considerable interest has been evoked by the concept of *formality*, both for spaces and CDGA's. The original definition of formality in rational homotopy theory refers only to commutative DGA-algebra's over $\mathbf{Q}$, though the ground ring being a field of characteristic 0 is all that is relevant. The vanishing of the terms of higher order was captured loosely in the expression of formality as the vanishing of all Massey products in $H(A)$, later refined to emphasize the simultaneous vanishing. Over a field of characteristic $p \neq 0$, an equivalent definition of formality appears in [G2]: A DGA-algebra is formal iff the higher order terms in the differential ∂_∞ vanish, i.e. if ∂_∞ is the usual bar construction differential for $H^*(A)$. For not-necessarily commutative algebras over any ring, there are several alternative definitions.

As the referee pointed out to us, the first definition over rings other than Q was introduced by Anick [A]. Merle showed in his Nice Ph.D. thesis that Anick's definition reduced to the CDGA definition for CDGA's over a field of characteristic zero. El Haouari [EH] showed that Anick's definition coincides with the following:

DEFINITION. A DGA-algebra A is *formal* if there exist DGA-algebras

$$A = A_0, A_1, \dots, A_{2n} = H(A)$$

and multiplicative chain maps that induce isomorphisms in homology

$$A = A_0 \leftarrow A_1 \rightarrow \cdots \leftarrow A_{2n-1} \rightarrow H(A).$$

It follows easily that a DGA-algebra A is formal iff there is a multiplicative map which induces an isomorphism in homology

$$\overline{\Omega}\overline{B}H(A) \rightarrow A.$$

For commutative DGA-algebras in characteristic 0, similar remarks apply with $\overline{B}$ replaced by $\mathscr{L}^c$ where $\mathscr{L}^c$ denotes Quillen free Lie coalgebra complex on $H(A)$. See, for example [DT].

These remarks help us to understand what Chen was driving at in his section "Integer homological applications" in [C1, pp. 862–864]. The objects he discusses are **Z**-formal in the sense of this paper. However, we may now see that also when the homology of X is torsion free, Chen's algebraic twisting cochain, obtained by either the basic obstruction method or the basic coalgebra perturbation lemma applied to tensor coalgebra data, yields a coderivation ∂ of T^cH^*X and a twisting cochain τ: $T^cH^*X \rightarrow C^*(X)$ which induces a differential algebra map $T^cH^*X \rightarrow \overline{C}^*(X)$ that is an isomorphism in homology:

$$H\tilde{B}H(X) \cong H\overline{B}C^*(X) \cong H(\Omega X).$$

(Indeed, what we have just said holds over any field, e.g., $\mathbf{Z}/p$.)

Moreover, when A and $H(A)$ are free R-modules (and R is a principal ideal domain), then there is SDR data

$$\left(H(A) \underset{f}{\overset{\nabla}{\rightleftarrows}} A, \phi \right)$$

which induces an equivalence $\overline{B}A \simeq \tilde{B}H(A)$, thus recovering a result of Baues and Lemaire [BL]. In the same way, we recover (with the same hypotheses) Proposition 2.2 of [BL]: Two DGA's A_1 and A_2 have the same homotopy type if and only if $\tilde{B}HA_1 \cong \tilde{B}HA_2$.

4. Some final remarks

In this section we will put many of the ideas discussed above into a roughly coherent framework. For a more precise view of the relationships between Chen's iterated integrals, the integration map, Chen's, Adams', and Eilenberg and Moore's theorem, see [G4], and [G5], [DT].

First, we have the comparison of Chen's connection ω with Gugenheim's twisting cochain τ:

$$\omega \in \Lambda^* \hat{\otimes}_R T^a s^{-1} H_* \xrightarrow{\text{Bourbaki}} \overline{\omega} \in [T^c s^{-1} H^*, \Lambda^*]_R$$
$$\xrightarrow[\text{integration}]{} [T^c s^{-1} H^*, C^*] \ni \tau.$$

Passing to the corresponding (completed) coalgebra maps $\hat{\omega}$ and $\hat{\tau}$, we have the following diagram, which, we conjecture, is commutative, at least up to homotopy:

$$\begin{tikzcd}
\hat{\omega} \in [\tilde{B}H^*, \bar{B}\Lambda^*]_R \arrow[r] \arrow[d, "\text{Chen}"'] & {[\tilde{B}H^*, \bar{B}C^*]_R \ni \hat{\tau}} \\
{[\tilde{B}H^*, \Lambda^*\Omega]_R} \arrow[r] & {[\tilde{B}H^*, C^*\Omega]_R.} \arrow[u, "\text{dual Adams}"']
\end{tikzcd}$$

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