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THE L^p-CONJECTURE AND YOUNG'S INEQUALITY

BY

SADAHIRO SAEKI

Dedicated to Professor Edwin Hewitt for his great contributions to Harmonic Analysis

Let G be an arbitrary locally compact (Hausdorff) group. The conjecture in the title asserts that if $L^p(G)$ is closed under convolution for some $p \in (1, \infty)$, then G is compact. In the present paper, we shall confirm this conjecture.

In his 1961 paper [17], W. Zelazko solved the problem for all abelian groups. The truth of the conjecture has been established for $p > 2$ and arbitrary G by him [18] and M. Rajagopalan [11] independently; in the case where either (a) $p \geq 2$ and G is discrete, (b) $p = 2$ and G is totally disconnected, or (c) $p > 1$ and G is either a nilpotent group or a semi-direct product of two LCA groups by Rajagopalan [10]–[12]; for $p > 1$ and solvable groups by the above-mentioned two authors [13]; for $p = 2$ and arbitrary groups by N. Rickert [15]; and for $p > 1$ and amenable groups by F.P. Greenleaf [4]. Volume II of E. Hewitt–K.A. Ross [5] gives accounts of some of the above-mentioned cases. For related results and simplifications of known proofs, we refer to G. Crombez [1], [2], R.J. Gaudet–J.L.B. Gamlen [3], D.L. Johnson [6], P. Milnes [8], K. Urbanik [16], and W. Zelazko [19].

Let λ_G denote a left Haar measure on the locally compact group G. All the Lebesgue spaces $L^p = L^p(G)$ are taken with respect to λ_G. Now let f and g be two Haar measurable functions on G. Then the convolution product

$$(f * g)(x) = \int f(y) g(y^{-1}x)\, dy$$

is defined at each point x of G for which the function $y \to f(y)g(y^{-1}x)$ is λ_G-integrable. For $p \in [1, \infty]$, we write $f * g \in L^p(G)$ to mean that $|f| * |g| < \infty$ λ_G-almost everywhere, $f * g$ is Haar measurable on the set $\{|f| * |g| < \infty\}$, and $\|f * g\|_p < \infty$. It is easy to show that if either $\{f \neq 0\}$ or $\{g \neq 0\}$ has σ-compact closure, then $\{|f| * |g| < \infty\}$ is a Borel set and $f * g$ is Borel

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measurable on it. (Unfortunately, the treatment of measurability in [5; vol. I, pp. 288–290] has deficiencies; some of which are addressed in the addendum to vol. II of that treatise.) Finally f is said to be symmetric if $f^{\#} = f$, where $f^{\#}(x) = f(x^{-1})$ for all $x \in G$.

THEOREM 1. *Suppose that there exists $p \in (1, \infty)$ such that $f * g \in L^p(G)$ for all symmetric functions $f, g \in L^p(G)$. Then G is compact.*

To prove this, we need three lemmas. Our proof is *ab ovo* and completely self-contained. Given $p \in [1, \infty]$, let $p' = p/(p-1)$ if $p > 1$ and $p' = 1$ if $p = \infty$. Let $L_s^p = \{f \in L^p : f^{\#} = f\}$. We write $|A|$ for $\lambda_G(A)$ whenever A is a Haar measurable subset of G. For any set A, ξ_A denotes the characteristic function of A.

LEMMA 1.1. *Let A be a compact symmetric subset of the general locally compact group G. Then we have*

$$|A|^2|A^{m+n}| \leq |A^4| \cdot |A^m| \cdot |A^n| \quad \text{for } m, n \geq 1.$$

Proof. Let $m \in \mathbf{N}$ be given. If $k, l \in \mathbf{Z}^+$ and $k \leq m$, then

$$\xi_{A^m} * \xi_{A^{m+l}} \geq |A^{m-k}| \quad \text{on } A^{l+2k}. \tag{1}$$

In fact, pick any such k, l and any $x \in A^{l+2k}$. Then $x = abc$ for some $a, b \in A^k$ and $c \in A^l$. Since $A^{-1} = A$ by hypothesis, it follows that

$$\begin{aligned}(\xi_{A^m} * \xi_{A^{m+l}})(x) &= |A^m \cap (xA^{l+m})| \\ &\geq |A^m \cap (abA^m)| \\ &= |(a^{-1}A^m) \cap (bA^m)| \geq |A^{m-k}|,\end{aligned}$$

which confirms (1).

Integrating both sides of (1) over A^{l+2k}, we obtain

$$|A^{m-k}| \cdot |A^{l+2k}| \leq |A^m| \cdot |A^{m+l}|$$

whenever $k \leq m$. For $k = m - 1$, this reduces to

$$|A| \cdot |A^{l+2m-2}| \leq |A^m| \cdot |A^{m+l}|. \tag{2}$$

Taking $m = 4$ in (2), we get $|A| \cdot |A^{l+6}| \leq |A^4| \cdot |A^{l+4}|$; hence

$$|A| \cdot |A^j| \leq |A^4| \cdot |A^{j-2}| \tag{3}$$

holds for all $j \geq 6$. But (3) is obvious for $j = 3$ and 4. Moreover, one checks that (2) with $m = 3$ and $l = 1$ is nothing but (3) with $j = 5$. In other words, (3) holds for all $j \geq 3$.

To complete the proof, we may and do suppose that $m \leq n$ and $m + n \geq 3$. Letting $l = n - m$, we then have

$$\begin{aligned} |A|^2|A^{m+n}| &\leq |A| \cdot |A^4| \cdot |A^{m+n-2}| \quad \text{by (3)} \\ &= |A| \cdot |A^4| \cdot |A^{2m+l-2}| \\ &\leq |A^4| \cdot |A^m| \cdot |A^n| \quad \text{by (2)}, \end{aligned}$$

as desired.

The following two lemmas are easy generalizations of the corresponding results in Zelazko [19].

LEMMA 1.2. *Let $p, q, r \in [1, \infty]$ be such that $p^{-1} + q^{-1} - r^{-1} \neq 1$. Suppose that $L_s^p * L_s^q \subset L^r$, i.e., $f * g \in L^r$ whenever $f \in L_s^p$ and $g \in L_s^q$. Then G is unimodular, $L^p * L^q \subset L^r$, and there exists a positive finite constant C_0 such that*

$$\|f * g\|_r \leq C_0 \|f\|_p \cdot \|g\|_1 \quad \textit{for } f \in L^p \quad \textit{and } g \in L^q.$$

Proof. Notice that $(f, g) \to f * g$ is bilinear and that $f * g \geq 0$ whenever $f, g \geq 0$. So it is easy to see that there exists a finite positive constant C such that

$$\|f * g\|_r \leq C \|f\|_p \cdot \|g\|_q \quad \text{for } f \in L_s^p \quad \text{and } g \in L_s^q. \tag{4}$$

Now let Δ be the modular function of G. Pick any nonzero symmetric $f, g \in C_c^+(G)$ and any $a \in G$. Letting $b = a^{-1}$, we then have

$$\begin{aligned} \Delta(a)^{1/r'} \|f * g\|_r &= \|f * g * \delta_b\|_r \\ &= \|(\delta_a * f * \delta_b) * (\delta_a * g * \delta_b)\|_r \\ &\leq C \|\delta_a * f * \delta_b\|_p \|\delta_a * g * \delta_b\|_q \quad \text{by (4)} \\ &= C\Delta(a)^{1/p'} \Delta(a)^{1/q'} \|f\|_p \|g\|_q. \end{aligned}$$

Plainly $f * g$ is a nonzero continuous function on G, and

$$\frac{1}{r'} \neq \frac{1}{p'} + \frac{1}{q'}$$

by the hypotheses. Thus the last inequality implies $\inf \Delta(G) > 0$, which is equivalent to the unimodularity of G.

So G is unimodular. Therefore, $f \in L^p$ and $g \in L^q$ implies $\|f^\#\|_p = \|f\|_p$ and $\|g^\#\|_q = \|g\|_q$. Hence

$$\begin{aligned} \| |f| * |g| \|_r &\le \left\| (|f| + |f|^\#) * (|g| + |g|^\#) \right\|_r \\ &\le C \| |f| + |f|^\# \|_p \cdot \| |g| + |g|^\# \|_q \quad \text{by (4)} \\ &\le 4C \|f\|_p \|g\|_q. \end{aligned}$$

Thus the desired inequality obtains with $C_0 = 4C$.

LEMMA 1.3. *Let $p, q, r \in [1, \infty]$ and C_0 be as in Lemma 1.2. Then we have*

$$(|A| \cdot |B|)^{1/p' + 1/q'} \le C_0^2 |AB|^{2/r'}$$

for all compact subsets A, B of G.

Proof. (Cf. [19]). Since $\xi_A * \xi_B = 0$ off AB, we have

$$\begin{aligned} |A| \cdot |B| &= \int \xi_A * \xi_B \, dx \\ &\le |AB|^{1/r'} \|\xi_A * \xi_B\|_r \quad \text{by Hölder's Inequality} \\ &\le C_0 |AB|^{1/r'} \|\xi_A\|_p \cdot \|\xi_B\|_q \quad \text{by Lemma 1.2} \\ &= C_0 |AB|^{1/r'} |A|^{1/p} |B|^{1/q}. \end{aligned}$$

Hence

$$|A|^{1/p'} \cdot |B|^{1/q'} \le C_0 |AB|^{1/r'}. \tag{5}$$

Moreover G is unimodular by Lemma 1.2. So $f \in L^q_+$ and $g \in L^p_+$ implies

$$\begin{aligned} \|f * g\|_r &= \left\| (f * g)^\# \right\|_r = \|g^\# * f^\#\|_r \\ &\le C_0 \|g^\#\|_p \|f^\#\|_q \quad \text{by Lemma 1.2} \\ &= C_0 \|f\|_q \|g\|_p. \end{aligned}$$

Therefore we may exchange p, q in (5):

$$|A|^{1/q'} |B|^{1/p'} \le C_0 |AB|^{1/r'}. \tag{6}$$

Multiplying (5) and (6), we arrive at the desired inequality.

Proof of the L^p-conjecture. Suppose that $1 < p < \infty$ and $L^p_s * L^p_s \subset L^p$. Then, by Lemma 1.2 with $p = q = r, G$ is unimodular, $L^p * L^p \subset L^p$, and there exists a finite positive constant C_0 such that

$$\|f * g\|_p \leq C_0 \|f\|_p \cdot \|g\|_p \quad \text{for } f, g \in L^p. \tag{7}$$

Letting $C_1 = C_0^{p'}$, we also have

$$|A| \cdot |B| \leq C_1 |AB| \tag{8}$$

for all compact subsets A, B of G (Lemma 1.3). In particular,

$$|A^n| / |A^{n+1}| \leq C_1 / |A| \quad \text{for } n \geq 1 \tag{9}$$

whenever A is a compact set having positive Haar measure.

Now let $q = p'$. Suppose, with a view toward reaching a contradiction, that G is not compact. Then G contains a compact symmetric set A, with $e \in A$, such that

$$|A| > 1 \quad \text{and} \quad C_1 / |A| < 2^{-(p+q)}. \tag{10}$$

For each integer $n \geq 2$, let

$$a_n = \{n(\log n)^2 |A^n|\}^{-1/p}, \tag{11}$$

$$b_n = \{n(\log n)^2 |A^n|\}^{-1/q}. \tag{12}$$

Writing $\xi_n = \xi_{A^n}$ for $n \geq 0$, we define

$$f = \sum_{n=2}^{\infty} a_n \xi_n \quad \text{and} \quad g = \sum_{n=2}^{\infty} b_n \xi_n \tag{13}$$

both pointwise.

We claim that $f \in L^p$ and $g \in L^q$. To confirm this, pick any $n \geq 2$. Then

$$\begin{aligned}(a_{n+1}/a_n)^p &= n(\log n)^2 |A^n| \Big/ \{(n+1)(\log(n+1))^2 |A^{n+1}|\} \quad \text{by (11)} \\ &\leq |A^n| / |A^{n+1}| \\ &\leq C_1 / |A| \leq 2^{-p} \quad \text{by (9) and (10)}.\end{aligned}$$

So $a_n - a_{n+1} = a_n(1 - a_{n+1}/a_n) \geq a_n/2$; hence

$$\sum_{n=k}^{\infty} a_n \leq 2 \sum_{n=k}^{\infty} (a_n - a_{n+1}) = 2a_k \quad \text{for } k \geq 2. \tag{14}$$

Thus

$$\begin{aligned}\|f\|_p^p &= \left\| \left(\sum_{n=2}^{\infty} a_n \right) \xi_2 + \sum_{k=3}^{\infty} \left(\sum_{n=k}^{\infty} a_n \right) (\xi_k - \xi_{k-1}) \right\|_p^p \quad \text{by (13)} \\ &= \left(\sum_{n=2}^{\infty} a_n \right)^p |A^2| + \sum_{k=3}^{\infty} \left(\sum_{n=k}^{\infty} a_n \right)^p (|A^k| - |A^{k-1}|) \\ &\leq 2^p \left\{ a_2^p |A^2| + \sum_{k=3}^{\infty} a_k^p |A^k| \right\} \quad \text{by (14)} \\ &= 2^p \sum_{k=2}^{\infty} \left\{ k (\log k)^2 \right\}^{-1} < \infty \quad \text{by (11)}.\end{aligned}$$

Therefore $f \in L^p$, and similarly $g \in L^q$.

Next we claim that $f * g \in L^q$. In fact, $h \in L_+^p$ implies

$$\begin{aligned}\int h(x)(f * g)(x^{-1})\, dx &= (h * f * g)(e) \\ &\leq \|h * f\|_p \|g^{\#}\|_q \quad \text{by Hölder's Inequality} \\ &\leq C_0 \|h\|_p \|f\|_p \|g\|_q \quad \text{by (7)},\end{aligned}$$

which is finite by the last claim. Since G is unimodular, this confirms that $f * g \in L^q$.

Now we are going to show that $\|f * g\|_q = \infty$, which will of course complete the proof. If m, $k \geq 1$ and $x \in A^k$, then

$$(\xi_m * \xi_{m+k})(x) = \left| A^m \cap (x A^{m+k}) \right| \geq |A^m|.$$

So

$$\begin{aligned}f * g &= \sum_{m=2}^{\infty} \sum_{n=2}^{\infty} a_m b_n (\xi_m * \xi_n) \quad \text{by (13)} \\ &\geq \sum_{k=2}^{\infty} \sum_{m=2}^{\infty} a_m b_{m+k} (\xi_m * \xi_{m+k}) \\ &\geq \sum_{k=2}^{\infty} \sum_{m=2}^{\infty} a_m b_{m+k} |A^m| \xi_k.\end{aligned}$$

Notice that $(\sum_{k=1}^{\infty} c_k)^q \geq \sum_{k=1}^{\infty} c_k^q$ for any sequence (c_k) of nonnegative numbers. Hence

$$\|f * g\|_q^q \geq \sum_{k=2}^{\infty} \left(\sum_{m=2}^{\infty} a_m b_{m+k} |A^m| \right)^q \int \xi_k \, dx$$
$$= \sum_{k=2}^{\infty} \left(\sum_{m=2}^{\infty} a_m b_{m+k} |A^m| \right)^q |A^k|. \tag{15}$$

To prove the divergence of the series in (15), note that $|A| > 1$ by (10); hence

$$|A^{m+k}| \leq |A^4| \cdot |A^m| \cdot |A^k| \quad \text{for } m, k \geq 1 \tag{16}$$

by Lemma 1.1. Let us only consider those pairs (m, k) of integers which satisfy $3 \leq k \leq m \leq 2k$. Then

$$(m + k)\{\log(m + k)\}^2 \leq 3k(\log 3k)^2 \leq 12k(\log k)^2.$$

This, combined with (16) and (12), shows that

$$b_{m+k} \geq \frac{1}{\{12|A^4| \cdot |A^m| \cdot |A^k| k(\log k)^2\}^{1/q}}. \tag{17}$$

Similarly

$$a_m \geq \frac{1}{\{8|A^m| k(\log k)^2\}^{1/p}} \tag{18}$$

by (11). Combine (17) and (18) to get

$$a_m b_{m+k} \geq \frac{1}{12|A^4| \cdot |A^m| k(\log k)^2 |A^k|^{1/q}}. \tag{19}$$

Letting $C = C_A = 1/(12|A^4|)$, we infer from (15) and (19) that

$$\|f * g\|_q^q \geq \sum_{k=3}^{\infty} \left\{ \sum_{m=k}^{2k} \frac{C|A^m|}{|A^m| k(\log k)^2 |A^k|^{1/q}} \right\}^q |A^k|$$
$$\geq C^q \sum_{k=3}^{\infty} (\log k)^{-2q} = \infty.$$

Hence $\|f * g\|_q = \infty$, which completes the proof.

Now we are going to investigate the triples of indices for which Young's Inequality holds. Let $p, q, r \in [1, \infty]$. If $1/r = 1/p + 1/q - 1$, then we have

$$\|f * g\|_r \leq \|f\|_p \max\left(\|g\|_q, \|g^{\#}\|_q\right) \quad \text{for } f \in L_+^p \quad \text{and} \quad g \in L_+^q \tag{20}$$

by Young's Inequality (see Theorem (20.18) of [5, Vol. I]). On the other hand, $s \geq r \geq 1$ implies that $L^r \subset L^s$ for all discrete groups, and $L^s \subset L^r$ for all compact groups. Combining these facts, we have that if

$$\frac{1}{r} \leq \frac{1}{p} + \frac{1}{q} - 1,$$

then

$$L^p * L^q \subset L^r$$

for all discrete groups and if

$$\frac{1}{r} \geq \frac{1}{p} + \frac{1}{q} - 1,$$

then

$$L^p * L^q \subset L^r$$

for all compact groups. Thus we are naturally led to the following two problems:

Problem I. If

$$\frac{1}{r} < \frac{1}{p} + \frac{1}{q} - 1 \quad \text{and} \quad L^p * L^q \subset L^r,$$

does it follow that G is discrete?

Problem II. If

$$\frac{1}{r} > \frac{1}{p} + \frac{1}{q} - 1 \quad \text{and} \quad L^p * L^q \subset L^r,$$

does it follow that G is compact?

T.S. Quek and L.Y.H. Yap [9] give affirmative answers to these problems for abelian groups. On the other hand, Theorem 9 of R.A. Kunze and E.M.

Stein [7] states that if $G = SL(2,\mathbf{R})$ and $1 \le p < 2$, then $L^p * L^2 \subset L^2$ holds. In particular, Problem II is negative in general. However, we have:

THEOREM 2. *Suppose that the noncompact group G has the property that given $\varepsilon > 0$, there exists a compact subset $A = A_\varepsilon$ of G, with sufficiently large $|A|$, such that*

$$\liminf_{n\to\infty} n^{-1} \log\log |A^{2^n}| < \varepsilon. \qquad (*)$$

Let $1 < p < \infty$. Then there exists $f \in L^p_s \cap C_0^+(G)$ such that

$$f * L^q_s \not\subset L^r$$

for all $r, q \in [1,\infty]$ satisfying

$$\frac{1}{r} > \frac{1}{p} + \frac{1}{q} - 1.$$

To prove this, let $\|f\|_u$ denote the uniform norm of any function f on G. Define

$$\|f\|_{p,u} = \max\{\|f\|_p, \|f\|_u\}$$

for $f \in L^p \cap C_0(G)$, where $p \in [1,\infty]$. It is easy to check that $\|f\|_{p,u}$ is a complete norm on $L^p \cap C_0(G)$, that $C_c(G)$ is dense in $L_p \cap C_0(G)$, and that $L^p_s \cap C_0(G)$ is a closed subspace of $L^p \cap C_0(G)$.

LEMMA 2.1. *Suppose that $p, q, r \in [1,\infty]$, $p > 1$, and G satisfies $(L^p \cap C_0) * (L^q \cap C_0) \subset L^r$. Then G is unimodular, and there exists a finite positive constant C_1 such that*

$$\|f * g\|_r \le C_1 \|f\|_{p,u} \|g\|_{q,u} \quad \text{for } f \in L^p \cap C_0 \quad \text{and} \quad g \in L^q \cap C_0.$$

If, in addition, G is noncompact, then $r \ge \max(p, q)$.

Proof. The existence of C_1 having the desired property is obvious.

To complete the proof, we may suppose that G is noncompact, Pick any nonzero $f, g \in C_c^+(G)$ and any $a \in G$. Letting $b = a^{-1}$, we then have

$$\begin{aligned}\|f * g\|_r &= \|f * \delta_b * \delta_a * g\|_r \\ &\le C_1 \|f * \delta_b\|_{p,u} \|\delta_a * g\|_{q,u} \\ &= C_1 \max\left\{\Delta(a)^{1/p'} \|f\|_p, \Delta(a)\|f\|_u\right\} \|g\|_{q,u}.\end{aligned}$$

Since $p' < \infty$, this ensures that G is unimodular. Also note that $f * g \in C_c^+(G)$. So, given $n \geq 1$, we can find $a_1, a_2, \ldots, a_n \in G$ so that the functions $\delta_{a_k} * f + \delta_{a_k} * f * g$, $1 \leq k \leq n$, have pairwise disjoint supports. It follows that

$$\begin{aligned} n^{1/r}\|f * g\|_r &= \left\| \sum_{k=1}^{n} \delta_{a_k} * f * g \right\|_r \\ &\leq C_1 \left\| \sum_{k=1}^{n} \delta_{a_k} * f \right\|_{p,u} \|g\|_{q,u} \\ &= C_1 \max\{n^{1/p}\|f\|_p, \|f\|_u\}\|g\|_{q,u}. \end{aligned}$$

Since n can be chosen as large as one wishes, we must necessarily have $r \geq p$. Also G is unimodular, so the set-inclusion in the hypotheses holds with p, q interchanged (see the proof of Lemma 1.3). Hence $r \geq q$, as desired.

LEMMA 2.2. *Let G, p, q, r and C_1 be as in Lemma 2.1. Then we have*

$$(|A| \cdot |B|)^{1/p' + 1/q'} \leq C_1^2 |AB|^{2/r'}$$

for all compact subsets A, B of G with $|A|, |B| \geq 1$.

Proof. The necessary arguments to prove this are quite similar to those in the proof of Lemma 1.3. So we omit the details.

Remark 2.3. In case $1/r > 1/p + 1/q - 1$, or equivalently $1/r' < 1/p' + 1/q'$, the proof of Lemma 1.2 shows that the apparently weaker inclusion $(L_s^p \cap C_0)*(L_s^q \cap C_0) \subset L^r$ already implies the hypothesis $(L^p \cap C_0)*(L^q \cap C_0) \subset L^r$ of Lemmas 2.1 and 2.2.

Proof of Theorem 2. Suppose that G satisfies the hypothesis of Theorem 2 and that $1 < p < \infty$.

Pick any $r, q \in [1, \infty]$ such that $1/r > 1/p + 1/q - 1$. To force a contradiction, suppose that

$$(L_s^p \cap C_0) * (L_s^q \cap C_0) \subset L^r.$$

Then $(L^p \cap C_0)*(L^q \cap C_0) \subset L^r$ by Remark 2.3. So Lemmas 2.1 and 2.2 provide a finite positive constant C_1 such that

$$(|A| \cdot |B|)^{1/p' + 1/q'} \leq C_1^2 |AB|^{2/r'} \tag{21}$$

for all compact subsets A, B of G with $|A|, |B| > 1$. Notice that $r' < \infty$ (since $p' < \infty$ and G is noncompact) and that our assumption on q, r is equivalent to the condition that $\beta > 1$, where

$$\beta = r'\left(\frac{1}{p'} + \frac{1}{q'}\right). \tag{22}$$

Letting $C_2 = C_1^{r'}$ and $A = B$ in (21), we obtain $|A|^\beta \le C_2|A^2|$ for all compact set A with $|A| > 1$. An inductive application of this inequality yields

$$(C_3|A|)^{\beta^n} \le C_3|A^{2^n}| \quad \text{for } n \ge 1 \tag{23}$$

for all such A, where $C_3 = C_2^{1/(1-\beta)}$ (recall $\beta > 1$).

Assuming that $|A|$ is large enough, we obtain

$$\log\log\left(C_3|A^{2^n}|\right) \ge n\log\beta + \log\log(C_3|A|)$$

for all $n \ge 1$, which clearly violates our hypothesis on G. Thus we have confirmed that

$$(L_s^p \cap C_0^+) * (L_s^q \cap C_0^+) \not\subset L^r \tag{24}$$

for all q, $r \ge 1$ with $1/r > 1/p + 1/q - 1$.

Now choose and fix any countable dense subset $\{(q_k, r_k)\}_{k=1}^\infty$ of

$$E = \left\{(q, r) \in [1, \infty)^2 : \frac{1}{r} > \frac{1}{p} + \frac{1}{q} - 1\right\}. \tag{25}$$

Given $k \ge 1$, (24) yields $f_k \in L_s^p \cap C_0^+$ such that

$$\|f_k\|_{p,u} < \frac{1}{k^2} \quad \text{and} \quad f_k * (L_s^{q_k} \cap C_0^+) \not\subset L^{r_k}.$$

Define $f = \sum_{k=1}^\infty f_k$ pointwise on G. Plainly $f \in L_s^p \cap C_0^+$ and

$$f * (L_s^{q_k} \cap C_0^+) \not\subset L^{r_k} \quad \text{for } k \ge 1. \tag{26}$$

To show that f has the desired property, pick any $q, r \in [1, \infty]$ such that $1/r > 1/p + 1/q - 1$. Suppose to the contrary that $f * L_s^q \subset L^r$. Notice that $f \in L^p$, so $f * L_s^1 \subset L^p$ by Young's Inequality, and $f * L_s^{p'} \subset L^\infty$ by Hölder's Inequality. Since $f \ge 0$, it follows that the mapping $g \to f * g$ is simultaneously of strong type (q, r), $(1, p)$ and (p', ∞) on $L_s^1 \cap L^\infty$. It follows from the Riesz-Thorin Convexity Theorem and its proof [20] that $f * L_s^a \subset L^b$ for all $a,b \in [1, \infty)$ such that the point $(1/a, 1/b)$ belongs to the triangle with vertices at $(1/q, 1/r)$, $(1, 1/p)$ and $(1/p', 0)$. Notice that the last two points lie on the line $v = u + 1/p - 1$ and that $(1/q, 1/r)$ is above this line. Thus the above triangle contains an interior point (u, v) with $v > u + 1/p - 1$. Consequently our choice of the (q_k, r_k) shows that $(1/q_k, 1/r_k)$ belongs to this triangle for at least one (in fact, for infinitely many) $k \geq 1$. This is of course absurd and therefore the proof is complete.

Remark 2.4. It is well known and easy to show that for each compact subset A of a LCA group G, there exists $k \in \mathbf{N}$ such that

$$|A^n| = 0(n^k) \quad \text{as } n \to \infty.$$

The following result is due to Quek and Yap [9]. Our proof is considerably simpler than theirs.

COROLLARY 2.5. *Let $p, q, r \in [1, \infty]$ and $p > 1$. Suppose that G is an infinite* LCA *group and that $L^p(G) * L^q(G) \subset L^r(G)$.*

(a) *If G is discrete, then $1/r \leq 1/p + 1/q - 1$.*

(b) *If G is compact, then $1/r \geq 1/p + 1/q - 1$.*

(c) *If G is neither discrete nor compact, then $1/r = 1/p + 1/q - 1$.*

Proof. (a) Suppose that G is discrete (and infinite). If G is a torsion group, then plainly G satisfies the condition in Theorem 2. So suppose that G is not torsion. Then G contains (a copy of) $\mathbf{Z}$. Given $m \in \mathbf{N}$, define $A = A_m = [0, m] \cap \mathbf{Z}$. Then $A^n = [0, mn] \cap \mathbf{Z}$ for all $n \geq 1$, so again G satisfies the condition in Theorem 2. Hence $1/r \leq 1/p + 1/q - 1$ in either case by Theorem 2, provided that $p < \infty$. But $p = \infty$ clearly implies $q = 1$ and $r = \infty$. Therefore $1/r \leq 1/p + 1/q - 1$ for all cases.

(b) Suppose that G is compact (and infinite). Then G is either totally disconnected (if the dual $\hat{G}$ is a torsion group) or contains a compact subgroup G_0 such that $G/G_0 \cong \mathbf{T}$ (otherwise). To obtain the desired inequality, we may suppose that $r > 1$ and $1/r \neq 1/p + 1/q - 1$.

Now let C_0 be the finite positive constant furnished by Lemma 1.2. Then we have

$$|A|^{1/p' + 1/q'} \leq C_0 |A^2|^{1/r'} \quad \text{for all compact } A \subset G \tag{27}$$

by Lemma 1.3 with $A = B$. Since G is nondiscrete and $r' < \infty$, (27) implies $1/p' + 1/q' > 0$. Letting $\beta = r'(1/p' + 1/q')$ and $C_2 = C_0^{r'}$, we have

$$|A|^{\beta} \leq C_2 |A^2| \quad \text{for all compact } A \subset G. \tag{28}$$

If G is totally disconnected, then every neighborhood of $e \in G$ contains a compact-open subgroup A. Therefore (28) is possible only when $\beta \geq 1$, or equivalently only when $1/r \geq 1/p + 1/q - 1$. (Notice that this part does not require the commutativity of G.)

In case G is not totally disconnected, G contains a compact subgroup G_0 such that $G/G_0 \cong \mathbf{T}$, as was observed above. Let λ_0 denote the normalized

Haar measure of G_0. Then we have

$$(\lambda_0 * L^p) * (\lambda_0 * L^q) \subset \lambda_0 * L^r$$

by the hypotheses. Therefore, by Fourier transform or by any other methods, we have that $L^p * L^q \subset L^r$ holds for $\mathbf{T}$. Taking $A = [0, t]$ in (28), we obtain $t^\beta \leq 2C_2 t$ for all $t \in [0, 2\pi]$. This is of course possible only when $1/r \geq 1/p + 1/q - 1$. Plainly this establishes (b).

(c) Finally suppose that G is neither discrete nor compact. If $p = \infty$, then it is clear that $q = 1$ and $r = \infty$. So we may suppose $p < \infty$.

Now consider the special case where G contains an open subgroup of the form $\mathbf{R} \times H$ for some locally compact group H. Since $L^p * L^q \subset L^r$ holds for G by hypothesis, it is clear that the same inclusion holds for $\mathbf{R} \times H$ and hence for $\mathbf{R}$. Let $C_0 < \infty$ be as in (27) with $G = \mathbf{R}$ (in case $1/r = 1/p + 1/q - 1$, take $C_0 = 1$). Then we have $r' < \infty$ since $\mathbf{R}$ is noncompact and $1/p' + 1/q' > 0$ (recall $p > 1$). Taking $A = [0, t]$ in (28), we obtain $t^\beta \leq 2C_2 t$ for all real $t > 0$. Plainly this is possible if and only if $\beta = 1$, i.e., $1/r = 1/p + 1/q - 1$.

In case G does not contain any open subgroup of the above form, G contains a compact-open subgroup H (see (9.8) of [5, Vol. I]). Since G is nondiscrete, part (b) applied to H ensures that $1/r \geq 1/p + 1/q - 1$. On the other hand, G is noncompact, so G/H is an infinite discrete abelian group. Hence, arguing as in the proof of part (a), we get $1/r \leq 1/p + 1/q - 1$. Consequently we obtain $1/r = 1/p + 1/q - 1$, as desired.

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Remarks 2.6 (Added on March 23, 1990).

(i) The conclusion of Theorem 2 may be strengthened as follows: there exists $f \in L_s^1 \cap C_0^+$ such that $f^{1/p} * L_s^q \not\subset L^r$ for all $p, q, r \geq 1$ satisfying $1/r > 1/p + 1/q - 1$. A similar result holds in each of the three cases considered in Corollary 2.5.

(ii) Professor N. Lohoué kindly pointed out to me that his paper *Estimations L^p des coefficients de représentation et opérateurs de convolution* (Advances in Math., vol. 38 (1980), pp. 178–221) resolved the L^p-conjecture for almost connected groups.

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Kansas State University
Manhattan, Kansas