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A SIMULTANEOUS ALMOST EVERYWHERE CENTRAL LIMIT THEOREM FOR DIFFUSIONS AND ITS APPLICATION TO PATH ENERGY AND EIGENVALUES OF THE LAPLACIAN

BY

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1. Introduction and summary

Let M be a compact C^∞ manifold of dimension d. A C^∞ metric $\mathscr{g}$ on M gives rise to the notions of α-potential and α-energy, $\alpha > 0$. These were discussed in [2] and [4] in the context of the asymptotic behaviour of the diffusion on M with generator $L = \frac{1}{2}\Delta + V$, where Δ is the Laplace operator associated with $\mathscr{g}$, and V is a vector field. In this paper we shall continue that discussion, assuming for simplicity $V = 0$ and $d \geq 2$. In particular, we shall prove an almost everywhere central limit theorem (a.e. CLT) for the occupation measures of the diffusion, a theorem similar to the one we proved in [5] for IID random variables. The occupation measures assume their values in a nuclear space. We shall exploit "exponential mixing" of the diffusion. As an application of our a.e. CLT, we shall recover the spectrum of the operator Δ from the development of the α-energy on a typical diffusion path. A classical CLT for the occupation measures can be found in [2].

For background material on α-potentials and α-energy in $\mathbf{R}^d$ we refer the reader to [8].

In the case of a compact Riemannian manifold, the α-potential kernels $\{g_\alpha,\ \alpha > 0\}$ were defined in [3] in terms of the fundamental solution of the heat equation. To be precise, if p is the solution of

$$\frac{\partial p}{\partial t}(t, x, y) = \tfrac{1}{2}\Delta_y p(t, x, y), \quad p(0^+, x, y) = \gamma\delta_x(y),$$

with γ the total Riemann measure of $(M, \mathscr{g})$, for $\alpha > 0$ we let

$$(1.1) \qquad g_\alpha(x, y) = \frac{1}{\Gamma(\alpha)}\int_0^\infty t^{\alpha-1}\{p(t, x, y) - 1\}\, dt, \quad x, y \in M.$$

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The kernels $\{g_\alpha,\ \alpha > 0\}$ form a semigroup with respect to the normalized Riemann measure m on M. We recall that the Newtonian kernel $g = g_1$ is also the solution of the differential equation

$$\tfrac{1}{2}\Delta_y g(x, y) = -\gamma\delta_x(y) + 1 \tag{1.2)(a}$$

with the normalizing condition

$$\int_M g(x, y)\, dm(y) = 0. \tag{1.2)(b}$$

The potential kernels g_α, by the way, are symmetric, bounded below, and if $\alpha \le d/2$, have singularities on the diagonal which are bounded in both directions by

$$c[r(x, y)]^{-d+2\alpha}$$

if $\alpha < d/2$ and by

$$c\{1 + \log^-[r(x, y)]\}$$

if $\alpha = \frac{1}{2}d$. Here $r(x, y)$ is the geodesic distance between x and y.

We denote by G_α the operators

$$G_\alpha\colon L^2(M) \to L^2(M), \quad (G_\alpha f)(x) = \int_M g_\alpha(x, y) f(y)\, dm(y)$$

and let $G = G_1$. The inverse of G is $-\frac{1}{2}\Delta$. The operators G_α also act on finite measures on M, if we let

$$(G_\alpha\mu)(x) = \int_M g_\alpha(x, y)\, d\mu(y),$$

which is the α-potential of μ. For any finite measure μ on M, we define the α-energy of μ by

$$e_\alpha^2(\mu) = \int_M\int_M g_\alpha(x, y)\, d\mu(x)\, d\mu(y) = \int_M (G_\alpha\mu)(x)\, d\mu(x). \tag{1.3}$$

This expression may assume the value $+\infty$. For $\alpha = 1$, we have the classical Newtonian energy of μ. The semigroup property of $\{g_\alpha,\ \alpha > 0\}$ implies that

$$e_\alpha^2(\mu) = \int_M |G_{\alpha/2}\mu|^2\, dm. \tag{1.4}$$

It follows that $e_\alpha^2(\mu) \geq 0$ and $e_\alpha^2(\mu) = 0$ iff $\mu = cm$. If μ_1, μ_2 are finite measures with finite α-energy, one may define $e_\alpha^2(\mu)$ for $\mu = \mu_1 - \mu_2$ by either (1.3) or (1.4).

For a smooth measure μ, e.g., $d\mu = \varphi\, dm$, $\varphi \in L^2(M)$, $\varphi \geq 0$, we have $G_{\alpha/2}\mu \in L^2(M)$, hence $e_\alpha^2(\mu) < \infty$ for all $\alpha > 0$. In general, finiteness of $e_\alpha^2(\mu)$ or equivalently, the property that $G_{\alpha/2}\mu \in L^2(M)$, may serve as a measure of the regularity of μ. Obviously, $e_{\alpha_1}^2(\mu) < \infty$ implies $e_{\alpha_2}^2(\mu) < \infty$ for $\alpha_1 < \alpha_2$. Moreover, finiteness of $e_\alpha^2(\mu)$ for a given measure μ is an invariant of the C^∞ manifold M, i.e., does not depend on the particular metric $\mathscr{g}$.

There is an equivalent way of describing the regularity of μ, expressed by finiteness of $e_\alpha^2(\mu)$. It involves the Sobolev spaces $H^{\pm\alpha}(M)$, $\alpha \in [0, \infty]$. For a discussion of these spaces see [9] for example. May it suffice here to recall from [2] that the metric $\mathscr{g}$ on M induces in a natural way admissible scalar products on the Hilbertian spaces $H^{\pm\alpha}(M)$, $\alpha \in [0, \infty)$. Indeed, in $H^{+0}(M) = L^2(M)$ the scalar product is defined with the normalized Riemann measure m. This scalar product is then carried over to $H^{+\alpha}(M)$, $\alpha \in (0, \infty)$ by the Hilbertian space isomorphism

$$K_\alpha = G_{\alpha/2} + m\colon L^2(M) \to H^\alpha(M)$$

(see Theorem (3.2) in [2]), and the scalar products in $H^{+\alpha}(M)$ define scalar products in the dual spaces $H^{-\alpha}(M)$, $\alpha \in [0, \infty)$.

Now any finite measure μ on M defines $\tilde{\mu} \in H^{-\infty}(M)$ if we let

$$\tilde{\mu}(\varphi) = \int_M \varphi\, d\mu \quad \text{for } \varphi \in C^\infty(M).$$

Actually $\tilde{\mu} \in H^{-\alpha}(M)$ for any $\alpha > \frac{1}{2}d$ by the Sobolev embedding theorem. Using (1.4) and the isomorphisms K_α: $L^2(M) \to H^\alpha(M)$, $\alpha > 0$, it is not difficult to see that for $\alpha > 0$, $e_\alpha^2(\mu) < \infty$ iff $\tilde{\mu} \in H^{-\alpha}(M)$. In this case,

$$e_\alpha^2(\mu) = \|G_{\alpha/2}\mu\|_{L^2(M)}^2 = \|\tilde{\mu}\|_{H^{-\alpha}(M)}^2 - [\mu(M)]^2.$$

In this case also, μ does not charge any set of α-capacity 0, and for $f \in H^\alpha(M)$, $\tilde{\mu}(f) = \int_M \tilde{f}\, d\mu$, where $\tilde{f}$ is "f made precise". We recall that for any $f \in H^\alpha(M)$, there exists a sequence $\varphi_n \in C^\infty(M)$ which is a Cauchy sequence in $H^\alpha(M)$ and which converges outside a set of α-capacity 0 to a finite limit, the latter coinciding with f m-a.e. Moreover, if $\psi_n \in C^\infty(M)$ is a second sequence with the same properties, then $\lim_{n\to\infty} \varphi_n = \lim_{n\to\infty} \psi_n$ outside a set of α-capacity 0, and either may be used as $\tilde{f}$. Regarding "f made precise", we actually used in [2] an α-potential $G_{\alpha/2}\bar{f}$, $\bar{f} \in L^2(M)$, and in [4] a regularized version of f, defined in terms of its Lebesgue points (independently of α).

In this paper we are concerned with certain random measures on M which are singular with respect to m, namely with the occupation measures of the diffusion $(\Omega, \mathfrak{A}; P^x, x \in M; X_t, \mathscr{F}_t, \theta_t, t \geq 0)$ on M, which has generator $\frac{1}{2}\Delta$. The occupation measures L_t, $t \geq 0$, of the diffusion path up to time t, are defined by

$$L_t(f) = \int_M f(\xi)\, dL_t(\xi) = \int_0^t f(X_s)\, ds, \quad f \in C(M).$$

It was shown in [2] and [4] that the critical index for their regularity type is $\frac{1}{2}d - 1$. To be precise, for all $x \in M$, P^x-a.e. $e_\alpha^2(L_t) < \infty$ for $t \geq 0$ if $\alpha > \frac{1}{2}d - 1$, and $e_{d/2-1}^2(L_t) = \infty$ for $t > 0$. Equivalently

$$\tilde{L}_t \in \bigcap \{H^{-\alpha}(M); \alpha > \tfrac{1}{2}d - 1\} \setminus \{H^{-(d/2-1)}(M)\} \quad \text{for } t > 0.$$

For the Newtonian energy $\alpha = 1$, and this index is $> \frac{1}{2}d - 1$ iff $d \leq 3$. In other words, the classical Newtonian energy of the diffusion path is finite iff $d \leq 3$.

It may be worth noting that the measures L_t, supported by the range of nowhere differentiable paths on M, are somewhat less singular than one-dimensional Hausdorff measure on M, restricted to the range of smooth, nondegenerate paths. It is not hard to see, that for the latter the critical index is $\frac{1}{2}d - \frac{1}{2}$.

In [2] and [4] the asymptotic behaviour of L_t as $t \to \infty$ was studied in terms of the $H^{-\alpha}(M)$-valued process $\tilde{L}_t$, and more simply in terms of the real-valued process $e_\alpha^2(L_t)$ $(= \|\tilde{L}_t - tm\|_{H^{-\alpha}(M)}^2)$ for $\alpha > \frac{1}{2}d - 1$. Both processes satisfy laws of the iterated logarithm. Regarding $e_\alpha^2(L_t)$, we showed that for all $x \in M$, P^x-a.e.

$$\text{(1.5)} \qquad \operatorname*{cluster\ set}_{t \to \infty} \frac{e_\alpha^2(L_t)}{2t \log_2 t} = \left[0, 2\lambda_1^{-(\alpha+1)}\right], \quad \alpha > \tfrac{1}{2}d - 1,$$

where λ_1 denotes the smallest positive eigenvalue of $-\frac{1}{2}\Delta$. (See [4].) This implies in particular, that for all $x \in M$, P_x-a.e.

$$\text{(1.6)} \qquad \lim_{t \to \infty} e_\alpha^2\left(t^{-1}L_t - m\right) = \lim_{t \to \infty} \frac{1}{t^2} e_\alpha^2(L_t) = 0, \quad \alpha > \tfrac{1}{2}d - 1,$$

i.e., convergence of $t^{-1}L_t$ to m in the α-energy norm.

It is also possible to prove an analogue of a result of V. Strassen [10] for one-dimensional Brownian motion: We have for all $x \in M$, P^x-a.e.

$$(1.7)\quad \overline{\lim_{t\to\infty}} \frac{1}{t}\int_{e^2}^{t} ds\, \chi_{(c2s\log_2 s,\infty)}\left(e_\alpha^2(L_s)\right) = 1 - \exp\left\{-4\left(\frac{2}{c\lambda_1^{(\alpha+1)}} - 1\right)\right\},$$

for $\alpha > \frac{1}{2}d - 1$, $0 < c < 2\lambda_1^{-(\alpha+1)}$. (See [6].)

Notice that (1.5) allows us to recover λ_1, the smallest positive eigenvalue of $-\frac{1}{2}\Delta$, by observing the α-energy of a nonexceptional path for large values of t. In this paper we shall recover the whole spectrum of Δ, if for some nonexceptional ω and some $\alpha > \frac{1}{2}d - 1$ the function $t \mapsto e_\alpha^2(L_t(\omega))$, $t \geq t_0$, is given. (Theorem (1.11) below.) It appears that such recovery cannot be achieved by a $\log_2$-law, as in the case of λ_1.

In this context it may be of interest that for any $\alpha > 0$, the classical max min method permits the recovery of the spectrum of Δ from the function $\varphi \mapsto e_\alpha^2(\varphi\, dm) = \|\varphi\, dm\|_{H^{-\alpha}(M)}^2$, defined on the vector space $\{\varphi \in C^\infty(M);\ m(\varphi) = 0\}$, if the latter is endowed with the inner product $(\varphi_1, \varphi_2)_{L^2(M,m)}$. This recovery is of course also possible from the function $\varphi \mapsto \|\varphi\|_{H^\alpha(M)}^2$ for any $\alpha > 0$.

Let now $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \ldots$ be the eigenvalues of $-\frac{1}{2}\Delta$. By a theorem of H. Weyl,

$$(1.8)\qquad \lambda_n \sim c_d \gamma^{-2/d} n^{2/d},$$

which implies that

$$(1.9)\qquad \sum_{n\geq 1} \lambda_n^{-(\alpha+1)} < \infty \quad \text{iff } \alpha > \tfrac{1}{2}d - 1.$$

It follows that for $\alpha > \frac{1}{2}d - 1$, the function

$$(1.10)\qquad \Phi_\alpha(z) = \prod_{n=1}^{\infty}\left(1 + \lambda_n^{-(\alpha+1)} z\right)$$

is an entire function. Its zeros together with their multiplicities determine the spectrum of Δ. Notice that $\Phi_\alpha \geq 1$ on the positive real axis, and that Φ_α is determined by its restriction to the positive real axis. The latter can be observed on a nonexceptional path, by the following theorem.

(1.11) THEOREM. *For all $x \in M$, P^x-a.e. for all $\beta \geq 0$, $\alpha > \frac{1}{2}d - 1$,*

$$(1.12)\qquad \lim_{t\to\infty} \frac{1}{\log t}\int_1^t \frac{ds}{s}\exp\left\{-\beta\frac{e_\alpha^2(L_s)}{s}\right\} = [\Phi_\alpha(\beta)]^{-1/2}.$$

This theorem can be proved directly. However, we shall see that it is also an easy corollary of Theorem (1.15) below, which is an a.e. CLT for the $H^{-\alpha}(M)$-valued process $\{\tilde{L}_t,\ t \geq 0\}$, $\alpha > \frac{1}{2}d - 1$. In order to formulate this a.e. CLT, we note that the operator G determines for $\alpha > \frac{1}{2}d - 1$, in a natural way a normal distribution $N_G^{-\alpha}$ on $H^{-\alpha}(M)$. To be precise, for $\alpha > \frac{1}{2}d - 1$, there exists a unique probability measure $N_G^{-\alpha}$ on $H^{-\alpha}(M)$, such that

$$(1.13)\qquad \int_{H^{-\alpha}(M)} \exp\{il(f)\}\, dN_G^{-\alpha}(l) = \exp\{-(f, Gf)_{L^2(M,m)}\}$$
$$\text{for } f \in (H^{-\alpha}(M))^* = H^{\alpha}(M).$$

This follows from the general theory of normal distributions on Hilbert spaces (see [11] for example), since for $f_1, f_2 \in H^{\alpha}(M)$,

$$(f_1, Gf_2)_{L^2(M,m)} = (f_1, G_{\alpha+1} f_2)_{H^{\alpha}(M)},$$

and $G_{\alpha+1}$ is a trace class operator iff $\alpha > \frac{1}{2}d - 1$, by (1.9).

If $\alpha > \frac{1}{2}d - 1$, we have trace $G_{\alpha+1} = \sum_{n \geq 1} \lambda_n^{-(\alpha+1)} < \infty$, and the trace formula

$$(1.14)\qquad \int_{H^{-\alpha}(M)} \|l\|_{H^{-\alpha}(M)}^2\, dN_G^{-\alpha}(l) = \text{trace } G_{\alpha+1}.$$

In the following we denote by $\delta_l^{-\alpha}$, $l \in H^{-\alpha}(M)$, the probability measure on $H^{-\alpha}(M)$, which assigns its total mass to the point l.

(1.15) THEOREM. *For all $x \in M$, P^x-a.e. for all $\alpha > \frac{1}{2}d - 1$,*

$$(1.16)\qquad \lim_{t \to \infty} \frac{1}{\log t} \int_1^t \frac{ds}{s} \delta^{-\alpha}_{(\tilde{L}_s - sm)/\sqrt{s}} = N_G^{-\alpha},$$

where the convergence is weak convergence of probability measures on $H^{-\alpha}(M)$.

Notice that for $\alpha_2 > \alpha_1 > \frac{1}{2}d - 1$, $N_G^{-\alpha_2}$ is the image of $N_G^{-\alpha_1}$ under the embedding $H^{-\alpha_1}(M) \hookrightarrow H^{-\alpha_2}(M)$. Similarly, $\delta_l^{-\alpha_2}$ is the image of $\delta_l^{-\alpha_1}$. Therefore, the ω-set defined by (1.16) is increasing in $\alpha > \frac{1}{2}d - 1$. Hence, if for every $\alpha > \frac{1}{2}d - 1$, P^x-a.e. (1.16) holds, then P^x-a.e. (1.16) holds simultaneously for all $\alpha > \frac{1}{2}d - 1$.

We may rephrase Theorem (1.15) using the projective limit N_G of $N_G^{-\alpha}$, $\alpha > \frac{1}{2}d - 1$, on the linear space $\mathfrak{X} = \bigcap_{\alpha > \frac{1}{2}d - 1} H^{-\alpha}(M)$. The space $\mathfrak{X}$, endowed with the smallest topology which makes all embeddings

$$i_{-\alpha}: \mathfrak{X} \hookrightarrow H^{-\alpha}(M), \quad \alpha > \tfrac{1}{2}d - 1,$$

continuous, was introduced in [4]. It is a nuclear space in the sense of [7]. Notice that $\mathfrak{X}$ is separable, metrizable and complete. If we use on $\mathfrak{X}$ the Borel field $\mathfrak{B}_{\mathfrak{X}}$, then $\{\tilde{L}_s, s \geq 0\}$ is a continuous additive $\mathfrak{X}$-valued functional. Moreover, there exists a unique probability measure N_G on $(\mathfrak{X}, \mathfrak{B}_{\mathfrak{X}})$, such that for all $\alpha > \frac{1}{2}d - 1$, $N_G^{-\alpha}$ is the image of N_G under the embedding

$$i_{-\alpha}: \mathfrak{X} \to H^{-\alpha}(M).$$

Keeping in mind that $\mathfrak{X}^* = \bigcup_{\alpha > d/2-1}(H^{-\alpha}(M))^* = \bigcup_{\alpha > d/2-1} H^{\alpha}(M)$ (see p. 61 in [7] for example), we have that N_G is the unique normal probability measure on $(\mathfrak{X}, \mathfrak{B}_{\mathfrak{X}})$ such that

$$\int_{\mathfrak{X}} l(f)\, dN_G(l) = 0 \quad \text{for } f \in \mathfrak{X}^* \tag{1.17}$$

and

$$\int_{\mathfrak{X}} l(f_1) \cdot l(f_2)\, dN_G(l) = 2(f_1, Gf_2)_{L^2(M,m)} \quad \text{for } f_1, f_2 \in \mathfrak{X}^*. \tag{1.18}$$

Obviously, $N_G\{l \in \mathfrak{X}; l(1) = 0\} = 1$. For the existence of N_G we only remark that for $\alpha \geq 0$, $H^{-\alpha}(M)$ may be identified with the space of sequences of real numbers

$$\left\{l_n,\quad n = 0, 1, 2, \ldots;\ \sum_{n \geq 1} \lambda_n^{-\alpha} l_n^2 < \infty\right\}$$

and that the measure N_G on $(\mathfrak{X}, \mathfrak{B}_{\mathfrak{X}})$ may be obtained as the distribution of the random sequence $\sqrt{2}\{0, \lambda_1^{-1/2}\xi_1, \lambda_2^{-1/2}\xi_2, \ldots\}$ where the ξ_n, $n \in \mathbf{N}$, are independent $N(0,1)$-random variables. Notice that $N_G(H^{-(d/2-1)}(M)) = 0$.

Denoting by δ_l, $l \in \mathfrak{X}$, the probability measure on $\mathfrak{X}$ which assigns its total mass to l, we may rephrase Theorem (1.15) as follows.

(1.19) THEOREM. *For all $x \in M$, P^x-a.e.*

$$\lim_{t \to \infty} \frac{1}{\log t} \int_1^t \frac{ds}{s} \delta_{(\tilde{L}_s - sm)/\sqrt{s}} = N_G, \tag{1.20}$$

where the convergence is weak convergence of probability measures on $(\mathfrak{X}, \mathfrak{B}_{\mathfrak{X}})$.

(1.21) *Remark.* If ω satisfies (1.20), it obviously satisfies (1.16) for all $\alpha > \frac{1}{2}d - 1$. We shall see later that if ω satisfies (1.16) for $\alpha_n \downarrow \frac{1}{2}d - 1$, it satisfies (1.20).

We mention without proof that there are Donsker versions of Theorems (1.15) and (1.19).

An immediate corollary of Theorem (1.19) is the following simultaneous a.e. CLT.

(1.22) COROLLARY. *Outside one single ω-set which has P^x-measure 0 for all $x \in M$, we have simultaneously for all $\varphi \in C(\mathfrak{X})$ and all Borel sets $B \subseteq \mathbf{R}$ with* $\mathrm{Leb}(\partial B) = 0$,

$$\lim_{t\to\infty} \frac{1}{\log t} \int_1^t \frac{ds}{s} \chi_B\left(\varphi\left(\frac{\tilde{L}_s - sm}{\sqrt{s}}\right)\right) = \varphi(N_G)(B), \tag{1.23}$$

where $\varphi(N_G)$ is the image of N_G under φ.

Notice that for continuous linear functions

$$\varphi: l \mapsto l(f), \quad \text{with } f \in \bigcup_{\alpha > d/2-1} H^\alpha(M),$$

$\varphi(N_G)$ is the normal distribution with mean 0 and variance

$$\sigma_f^2 = 2(f, Gf)_{L^2(M,m)}$$

and, for nonconstant f, (1.23) reads

$$\lim_{t\to\infty} \frac{1}{\log t} \int_1^t \frac{ds}{s} \chi_B\left(\frac{L_s(\tilde{f}) - m(f)s}{\sqrt{s}}\right) = \frac{1}{\sqrt{2\pi\sigma_f^2}} \int_B e^{-u^2/2\sigma_f^2}\, du. \tag{1.23'}$$

Using $\varphi(l) = \|l\|^2_{H^{-\alpha}(M)}$, we have again Theorem (1.11) (apart from the identification of the limit).

Finally we shall prove a.e. convergence results for certain unbounded continuous functions on $\mathfrak{X}$, which do not follow immediately from the CLT.

(1.24) COROLLARY. *For all $x \in M$, P^x-a.e.*

$$\lim_{t\to\infty} \frac{1}{\log t} \int_1^t \frac{ds}{s} \frac{e_\alpha^2(L_s)}{s} = \operatorname{trace} G_{\alpha+1} \quad \textit{for } \alpha > \tfrac{1}{2}d - 1, \tag{1.25}$$

$$\lim_{t\to\infty} \frac{1}{\log t} \int_1^t \frac{ds}{s} \frac{L_s(\tilde{f})}{\sqrt{s}} = 0 \tag{1.26}$$

for all $f \in \bigcup_{\alpha > \frac{1}{2}d-1} H^\alpha(M)$ *such that* $m(f) = 0$,

$$(1.27) \qquad \lim_{t\to\infty} \frac{1}{\log t}\int_1^t \frac{ds}{s}\,\frac{L_s(\tilde{f}_1)L_s(\tilde{f}_2)}{s} = 2(Gf_1, f_2)_{L^2(M,m)}$$

for all $f_1, f_2 \in \bigcup_{\alpha > \frac{1}{2}d-1} H^\alpha(M)$ *such that* $m(f_1) = m(f_2) = 0$.

These latter results are a.e. analogues of (1.14), (1.17), (1.18). Moreover, (1.27) together with the ergodic theorem gives a new method for recovering the spectrum of G, hence the spectrum of Δ, from the random subspace $\{s \mapsto f(X_s), s \geq 0;\ f \in C^\infty(M)\}$ of $C(\mathbf{R}^+)$, observed on a nonexceptional path.

The key to the proofs is the Ornstein-Uhlenbeck process

$$\left\{\frac{W_{e^s}}{\sqrt{e^s}},\ s \in \mathbf{R}\right\},$$

where W is 1-dimensional Brownian motion. This process is stationary and allows an extensive use of ergodic theory. An immediate result is the following easy analogue of Theorem (1.11). Let $W^1, \dots, W^d$ be independent 1-dimensional Brownian motions, let $a_1, \dots, a_d > 0$ and let $Z_t = (W_{a_1 t}, \dots, W_{a_d t})$, $t > 0$. Then P-a.e. for all $\beta \geq 0$,

$$\begin{aligned}\lim_{t\to\infty} \frac{1}{\log t}\int_1^t \frac{ds}{s}\exp\left(-\tfrac{1}{2}\beta\frac{\|Z_s\|^2}{s}\right) &= E\exp\left(-\tfrac{1}{2}\beta\|Z_1\|^2\right) \\ &= \left\{\prod_{n=1}^d (1+\beta a_n)\right\}^{-1/2}.\end{aligned}$$

Thus P-a.e., d and $a_1, \dots, a_d$ may be recovered by observing $\{\|Z_t\|,\ t > 1\}$.

2. Maximal inequalities

In this section we shall present Lemma (2.7), a maximal inequality for certain stationary, mixing sequences of random variables in $L^{1+\delta}$ ($\delta > 0$ arbitrarily small). Its corollary, Lemma (2.16) seems indispensible for the proof of our key Lemma (4.11) below.

For $K > 0$, $\gamma > 0$, we say that the sequence $\{Y_i,\ i \in \mathbf{N}\}$ of random variables on a probability space $(\Omega, \mathfrak{A}, P)$ is (K, γ)-mixing, if for $r, t \in \mathbf{N}$, A_1 Borel $\subseteq$

$\mathbf{R}^t$, A_2 Borel $\subseteq \mathbf{R}^{\mathbf{N}}$,

$$\begin{aligned}(2.1)\qquad &|P\{(Y_1,\dots,Y_t)\in A_1,(Y_{t+r+1},Y_{t+r+2},\dots)\in A_2\}\\ &\quad -P\{(Y_1,\dots,Y_t)\in A_1\}\cdot P\{(Y_{t+r+1},Y_{t+r+2},\dots)\in A_2\}|\\ &\le Ke^{-\gamma r}P\{(Y_1,\dots,Y_t)\in A_1\}\cdot P\{(Y_{t+r+1},Y_{t+r+2},\dots)\in A_2\}.\end{aligned}$$

Notice that this is equivalent to saying, that for $r,t\in\mathbf{N}$, U: $\Omega\to\mathbf{R}$ integrable and $(Y_k,\ k\ge t+r+1)$-measurable,

$$(2.1')\qquad |E\{U|Y_1,\dots,Y_t\}-EU|\le Ke^{-\gamma r}E|U|,\quad P\text{-a.e.}$$

It is also equivalent to the following property. For all $r,t\in\mathbf{N}$, for all U_1: $\Omega\to\mathbf{R}$ integrable and $(Y_1,\dots,Y_t)$-measurable, for all U_2: $\Omega\to\mathbf{R}$ integrable and $(Y_k,\ k\ge t+r+1)$-measurable, such that U_1U_2 is integrable, we have

$$(2.1'')\qquad |E\{U_1U_2\}-EU_1\cdot EU_2|\le Ke^{-\gamma r}E|U_1|\cdot E|U_2|.$$

We shall call $\{Y_i,\ i\in\mathbf{N}\}$ an $SM(K,\gamma)$-sequence, if it is stationary and (K,γ)-mixing.

(2.2) Lemma. *Let K, $\gamma>0$. There exists $c>0$ with the following property. If $\{Y_i,\ i\in\mathbf{N}\}$ is an $SM(K,\gamma)$-sequence on a probability space $(\Omega,\mathfrak{A},P)$ such that $EY_1=0$, then for all $\delta\in[0,1]$ and all $n\in\mathbf{N}$,*

$$(2.3)\qquad E\left|\sum_{k=1}^{n}Y_k\right|^{1+\delta}\le cnE|Y_1|^{1+\delta},$$

$$(2.4)\qquad P\left\{\max_{k=1,\dots,n}\left|\sum_{i=1}^{k}Y_i\right|\ge\lambda\right\}\le c\frac{1}{\lambda^{1+\delta}}\cdot n\cdot E|Y_1|^{1+\delta}\quad \textit{for all } \lambda>0.$$

Proof. Estimate (2.3) for $n\in\mathbf{N}$ is trivial if $\delta=0$, and easy to prove for $\delta=1$. It then follows for $\delta\in(0,1)$, for example from the Riesz-Thorin interpolation theorem. It is also possible to prove (2.3) directly using (2.1″) and convexity of the function $x\mapsto|x|^{1+\delta}$, $x\in\mathbf{R}$.

As for (2.4), we may assume that $E|Y_1|^{1+\delta}<\infty$. Let $S_k=\sum_{i=1}^{k}Y_i$, $k\in\mathbf{N}$. Let

$$\tau=\begin{cases}\inf\{k\in\mathbf{N};\ |S_k|\ge\lambda\} & \text{if }\{\ \}\ne\varnothing\\ \infty & \text{if }\{\ \}=\varnothing.\end{cases}$$

Now,

$$P\Big\{\max_{k=1,\dots,n}|S_k|\geq\lambda\Big\}=P\{|S_{\tau\wedge n}|\geq\lambda\}\leq\frac{1}{\lambda^{1+\delta}}E|S_{\tau\wedge n}|^{1+\delta}.$$

Furthermore,

$$\begin{aligned}E|S_{\tau\wedge n}|^{1+\delta}&\leq 2^{\delta}E|S_n|^{1+\delta}+2^{\delta}E|S_n-S_{\tau\wedge n}|^{1+\delta}\\&\leq 2E|S_n|^{1+\delta}+2E|S_n-S_{\tau\wedge n}|^{1+\delta}.\end{aligned}$$

We may apply (2.3) to $E|S_n|^{1+\delta}$, and it remains to show that for all $n\in\mathbf{N}$, $\delta\in[0,1]$,

$$E|S_n-S_{\tau\wedge n}|^{1+\delta}\leq cnE|Y_1|^{1+\delta}. \tag{2.5}$$

Now, for $n=3,4,\dots,$

$$\begin{aligned}E|S_n-S_{\tau\wedge n}|^{1+\delta}&=\sum_{k=1}^{n}E\big\{\chi_{\{\tau=k\}}\cdot|S_n-S_k|^{1+\delta}\big\}\\&\leq 2\sum_{k=1}^{n-1}E\big\{\chi_{\{\tau=k\}}\cdot|Y_{k+1}|^{1+\delta}\big\}\\&\quad+2\sum_{k=1}^{n-2}E\big\{\chi_{\{\tau=k\}}\cdot|S_n-S_{k+1}|^{1+\delta}\big\}.\end{aligned} \tag{2.6}$$

The first sum on the right of (2.6) is majorized by

$$2\sum_{k=1}^{n-1}E|Y_{k+1}|^{1+\delta}\leq 2n\cdot E|Y_1|^{1+\delta}.$$

As for the second sum on the right of (2.6), we notice that $|S_n-S_{k+1}|^{1+\delta}$ is $(Y_{k+2},\dots)$-measurable, and we conclude from (2.1′) that for $n=3,4,\dots,$ and $k=1,2,\dots,n-1$,

$$E\big\{|S_n-S_{k+1}|^{1+\delta}|Y_1,\dots,Y_k\big\}\leq(1+Ke^{-\gamma})E|S_n-S_{k+1}|^{1+\delta},\quad P\text{-a.e.}$$

An application of (2.3) completes the proof of (2.5).

(2.7) LEMMA. *For $K,\gamma>0$, $\delta\in[0,1]$, $\rho<\delta/(1+\delta)$, there exists c with the following property. If $\{Y_i,\ i\in\mathbf{N}\}$ is an $SM(K,\gamma)$-sequence with $EY_1=0$,* *then for all* $\lambda > 0$,

$$(2.8)\qquad P\left\{\sup_{k\geq 1}\frac{\left|\sum_{i=1}^{k} Y_i\right|}{k^{1-\rho}} \geq \lambda\right\} \leq \frac{c}{\lambda^{1+\delta}} E|Y_1|^{1+\delta}.$$

Proof. Letting $S_k = \sum_{i=1}^{k} Y_i$ again, $k \geq 1$, we have

$$\begin{aligned} P\left\{\sup_{k\geq 1}\frac{|S_k|}{k^{1-\rho}} \geq \lambda\right\} &\leq \sum_{n=0}^{\infty} P\left\{\sup_{2^n \leq k < 2^{n+1}} |S_k| \geq \lambda 2^{n(1-\rho)}\right\} \\ &\leq \sum_{n=0}^{\infty} P\left\{\sup_{k\leq 2^{n+1}} |S_k| \geq \lambda 2^{n(1-\rho)}\right\}, \end{aligned}$$

and the proof is completed by (2.4).

We shall now apply Lemma (2.7) to the diffusion

$$(\Omega, \mathfrak{A}; P^x, x \in M; X_t, \mathscr{F}_t, \theta_t, t \geq 0)$$

on M. We recall that the normalized Riemann measure m on M is the invariant measure of the diffusion, and we denote by P^m the law of the diffusion if its initial distribution is m. Under P^m the diffusion is stationary, i.e., P^m is preserved under θ_t, $t \geq 0$. It is well known that the diffusion is also mixing in the following strong sense. (See [1] for example.) There exist K, $\gamma > 0$ depending only on M and $\wp$ such that for $r \geq 1$, $t \geq 0$, Z: $\Omega \to \mathbf{R}$ integrable,

$$(2.9)\qquad |E^m\{Z \circ \theta_{r+t} | \mathscr{F}_t\} - E^m Z| \leq K e^{-\gamma r} E^m |Z|, \quad P^m\text{-a.e.}$$

Indeed, $E^m\{Z \circ \theta_{r+t} | \mathscr{F}_t\} = E^{X_t}(Z \circ \theta_r)$, P^m-a.e., and since for any $x \in M$,

$$E^x(Z \circ \theta_r) = E^x E^{X_r} Z = \int_M dm(y) p(r, x, y) E^y Z,$$

it follows that for $x \in M$, $r \geq 1$,

$$\begin{aligned} |E^x(Z \circ \theta_r) - E^m Z| &= \left|\int_M dm(y)\{p(r,x,y) - 1\} E^y Z\right| \\ &\leq \int_M dm(y) |p(r,x,y) - 1| \cdot E^y |Z| \leq K e^{-\gamma r} E^m |Z|, \end{aligned}$$

which proves (2.9). For the last estimate see [1] for example.

Now if $f \in L^1(M)$, then P^m-a.e. $f \in \bigcap_{t \geq 0} L^1(M, L_t)$, and if $f_1, f_2 \in L^1(M)$ coincide m-a.e., then P^m-a.e. $L_t(f_1) = L_t(f_2)$ for $t \geq 0$. Moreover, for $f \in L^1(M)$, the random variables

$$Y_i(f) = L_i(f) - L_{i-1}(f) - m(f), \quad i \in \mathbf{N},$$

form an $SM(K, \gamma)$-sequence on $(\Omega, \mathfrak{A}, P^m)$ with $K, \gamma > 0$ of (2.9), and $E^m Y_1(f) = 0$. We have

(2.10) LEMMA. *For $\delta \in [0, 1]$, $\rho < \delta/(1+\delta)$, there exists c such that for all $f \in L^{1+\delta}(M)$ and all $\lambda > m(|f|)$,*

$$P^m\left\{\sup_{t \geq 1} \frac{|L_t(f) - tm(f)|}{t^{1-\rho}} \geq \lambda\right\} \tag{2.11}$$
$$\leq \frac{c}{\{\lambda - m(|f|)\}^{1+\delta}} \int_M |f|^{1+\delta}\, dm.$$

Proof. Letting $Z_i(f) = L_i(|f|) - L_{i-1}(|f|)$, for $t \geq 1$ we have

$$\frac{|L_t(f) - tm(f)|}{t^{1-\rho}} \leq \frac{\left|\sum_{i=1}^{[t]} Y_i(f)\right|}{[t]^{1-\rho}} + 2^{1-\rho} \frac{Z_{[t]+1}}{([t]+1)^{1-\rho}} + m(|f|).$$

For the first term on the right, the tail probability of the maximal function is estimated by Lemma (2.7). As for the second term, we have obviously for any $\delta > 0$,

$$P^m\left\{\max_{1 \leq k \leq n} Z_k(f) \geq \lambda\right\} \leq \frac{n}{\lambda^{1+\delta}} E^m\{Z_1(f)\}^{1+\delta}$$
$$\leq \frac{n}{\lambda^{1+\delta}} \int_M |f|^{1+\delta}\, dm,$$

and the argument for Lemma (2.7) gives an estimate for the tail probability of its maximal function.

It is well known that weak type estimates such as Lemma (2.10) imply strong type estimates. For our purposes however the following lemma suffices.

(2.12) LEMMA. *For $\delta \in (0,1]$, $\rho < \delta/(1+\delta)$, $\delta' \in [0,\delta)$, there exist c_1, c_2 such that for all $f \in L^{1+\delta}(M)$,*

$$E^m\left\{\sup_{t\ge 1} \frac{|L_t(f) - tm(f)|}{t^{1-\rho}}\right\}^{1+\delta'} \le c_1 + c_2\int_M |f|^{1+\delta}\, dm. \tag{2.13}$$

Proof. Let

$$Z(f) = \sup_{t\ge 1} \frac{|L_t(f) - tm(f)|}{t^{1-\rho}}.$$

We have to show that

$$\int_0^\infty \lambda^{\delta'} P^m\{Z(f) \ge \lambda\}\, d\lambda \le c_1 + c_2\int_M |f|^{1+\delta}\, dm. \tag{2.13$'$}$$

If $m(|f|) \ge 1$, we conclude from (2.11), that

$$\begin{aligned}\int_{2m(|f|)}^\infty &\lambda^{\delta'} P^m\{Z(f) \ge \lambda\}\, d\lambda\\ &\le \frac{c}{[m(|f|)]^{\delta-\delta'}}\int_2^\infty \frac{u^{\delta'}}{\{u-1\}^{1+\delta}}\, du \cdot \int_M |f|^{1+\delta}\, dm\\ &\le c\int_M |f|^{1+\delta}\, dm.\end{aligned}$$

Since

$$\begin{aligned}\int_0^{2m(|f|)} \lambda^{\delta'} P^m\{Z(f) \ge \lambda\}\, d\lambda &\le c\{m(|f|)\}^{1+\delta}\\ &\le c\int_M |f|^{1+\delta}\, dm,\end{aligned}$$

(2.13′) follows.

If $m(|f|) < 1$, we conclude from (2.11), that

$$\begin{aligned}\int_2^\infty \lambda^{\delta'} P^m\{Z(f) \ge \lambda\}\, d\lambda &\le c\int_2^\infty \frac{\lambda^{\delta'}\, d\lambda}{\{\lambda-1\}^{1+\delta}} \cdot \int_M |f|^{1+\delta}\, dm\\ &\le c\int_M |f|^{1+\delta}\, dm,\end{aligned}$$

and (2.13′) follows.

(2.14) *Remark.* If $f \in L^\infty(M)$, we have $\int_M |f|^{1+\delta}\, dm < \infty$ for all $\delta \geq 0$, and in this case Lemma (2.12) implies that for $\rho < \frac{1}{2}$, $1 \leq p < 2$,

$$E^m\left\{\sup_{t \geq 1} \frac{|L_t(f) - tm(f)|}{t^{1-\rho}}\right\}^p < \infty. \tag{2.15}$$

In this case a stronger result is actually true. It is easy to conclude from Lemma (6.1) in [2], that for $f \in L^\infty(M)$, (2.15) holds for all $p \geq 1$.

We shall now apply Lemma (2.12) to the functions

$$f_\xi^\alpha(\cdot) = \left|\operatorname{grad} Gg_{\alpha/2}(\xi, \cdot)\right|^2, \quad \alpha > \tfrac{1}{2}d - 1,\ \xi \in M.$$

For $\alpha > \frac{1}{2}d - 1$ and $\xi \in M$, $f_\xi^\alpha \in L^1(M)$ (even $\in L^{1+\delta}(M)$ for $\delta > 0$ sufficiently small). Letting $A_t^\alpha(\xi) = L_t(f_\xi^\alpha)$, $t \geq 0$, and $\nu^\alpha(\xi) = m(f_\xi^\alpha)$, we have the following result.

(2.16) Lemma. *For*

$$\alpha \in \left(\tfrac{1}{2}d - 1, \tfrac{3}{4}d - 1\right), \quad \delta \in \left[0, \frac{\alpha - \frac{1}{2}d + 1}{d - \alpha - 1}\right), \quad \rho < \frac{2}{d}\left(\alpha - \tfrac{1}{2}d + 1\right),$$

$$\rho < \frac{2}{d}\left(\alpha - \tfrac{1}{2}d + 1\right),$$

we have

$$\sup_{\xi \in M} E^m\left\{\sup_{t \geq 1} \frac{|A_t^\alpha(\xi) - t\nu^\alpha(\xi)|}{t^{1-\rho}}\right\}^{1+\delta} < \infty. \tag{2.17}$$

Proof. For $\alpha \in (0, d - 1)$, there exists c_α such that for $\xi \in M$,

$$\left|\operatorname{grad} Gg_{\alpha/2}(\xi, \cdot)\right|^2 \leq c_\alpha[r(\xi, \cdot)]^{-2(d-\alpha-1)}.$$

Now, for

$$\alpha \in \left(\tfrac{1}{2}d - 1, d - 1\right), \quad \delta \in \left[0, \frac{\alpha - \frac{1}{2}d + 1}{d - \alpha - 1}\right)$$

we have

$$2(d - \alpha - 1)(1 + \delta) < d,$$

and hence

$$\sup_{\xi \in M} \int_M |f_\xi^\alpha|^{1+\delta} \, dm < \infty. \tag{2.18}$$

Notice also that if $\alpha < \frac{3}{4}d - 1$ ($< d - 1$), then

$$\frac{\alpha - \frac{1}{2}d + 1}{d - \alpha - 1} < 1.$$

Moreover

$$\frac{\delta}{1+\delta} \uparrow \frac{2}{d}\left(\alpha - \tfrac{1}{2}d + 1\right) \quad \text{if } \delta \uparrow \frac{\alpha - \frac{1}{2}d + 1}{d - \alpha - 1}.$$

Now the assertion of the lemma follows at once from Lemma (2.12).

3. A lemma for time changed Brownian motions

(3.1) LEMMA. *For $\delta > 0$, $\rho \in \mathbf{R}$, $\delta' \in [0, \delta)$, $\eta \in (-\infty, \rho)$ there exist c_1, c_2, c_3 with the following property. If $\{W_t,\ t \geq 0\}$ is a one-dimensional Brownian motion on some probability space $(\Omega, \mathfrak{A}, P)$, and if $\tau_t\colon \Omega \to [0, \infty)$, $t \geq 1$, is a continuous process on $(\Omega, \mathfrak{A}, P)$, and if $\nu > 0$, then*

$$E\left\{\sup_{t \geq 1} \frac{|W_{\tau_t} - W_{\nu t}|^2}{t^{1-\eta}}\right\}^{1+\delta'} \leq c_1 + c_2 \nu + c_3 E\left\{\sup_{t \geq 1} \frac{|\tau_t - \nu t|}{t^{1-\rho}}\right\}^{1+\delta}. \tag{3.2}$$

Proof. In the following we fix $\sigma \in (0, (\delta - \delta')/(1 + \delta))$. For $\lambda > 0$,

$$P\left\{\sup_{t \geq 1} \frac{|W_{\tau_t} - W_{\nu t}|^2}{t^{1-\eta}} \geq \lambda\right\} \leq I_1(\lambda) + I_2(\lambda),$$

where

$$I_1(\lambda) = P\left\{\sup_{t \geq 1} \frac{|\tau_t - \nu t|}{t^{1-\rho}} \geq \lambda^{1-\sigma}\right\},$$

and

$$I_2(\lambda) = P\left\{\sup_{t\ge 1}\frac{|W_{\tau_t} - W_{\nu t}|^2}{t^{1-\eta}} \ge \lambda,\ \sup_{t\ge 1}\frac{|\tau_t - \nu t|}{t^{1-\rho}} < \lambda^{1-\sigma}\right\}.$$

Now for $\lambda > 0$,

$$I_1(\lambda) \le \frac{1}{\lambda^{(1-\sigma)(1+\delta)}} E\left\{\sup_{t\ge 1}\frac{|\tau_t - \nu t|}{t^{1-\rho}}\right\}^{1+\delta}.$$

This implies

$$\int_1^\infty \lambda^{\delta'} I_1(\lambda)\,d\lambda \le cE\left\{\sup_{t\ge 1}\frac{|\tau_t - \nu t|}{t^{1-\rho}}\right\}^{1+\delta},$$

hence

$$\int_0^\infty \lambda^{\delta'} I_1(\lambda)\,d\lambda \le 1 + cE\left\{\sup_{t\ge 1}\frac{|\tau_t - \nu t|}{t^{1-\rho}}\right\}^{1+\delta}. \tag{3.3a}$$

Turning now to $I_2(\lambda)$, we have

$$I_2(\lambda) \le \sum_{n=0}^{\infty} a_n(\lambda),$$

where

$$a_n(\lambda) = P\left\{\sup_{e^n \le t < e^{n+1}}\frac{|W_{\tau_t} - W_{\nu t}|^2}{t^{1-\eta}} \ge \lambda,\ \sup_{t\ge 1}\frac{|\tau_t - \nu t|}{t^{1-\rho}} < \lambda^{1-\sigma}\right\},$$

for $n \ge 0$.

Letting $\lambda_1 = \lambda$ if $\eta \le 1$, $\lambda_1 = \lambda e^{1-\eta}$ if $\eta > 1$, and $\lambda_2 = \lambda$ if $\rho \le 1$, $\lambda_2 = \lambda e^{-(1-\rho)/(1-\sigma)}$ if $\rho > 1$, we obtain

$$\begin{aligned} a_n(\lambda) &\le P\left\{\sup_{e^n \le t < e^{n+1}} |W_{\tau_t} - W_{\nu t}| \ge \lambda_1^{1/2} e^{(1-\eta)n/2},\ \sup_{t\ge 1}\frac{|\tau_t - \nu t|}{t^{1-\rho}} < \lambda^{1-\sigma}\right\} \\ &\le P\Big\{\sup\Big[|W_{\nu t'} - W_{\nu t}|;\ e^n \le t < e^{n+1}, t' > 0, \\ &\qquad |t' - t| < \frac{1}{\nu}\lambda_2^{1-\sigma} e^{(1-\rho)(n+1)}\Big] \ge \lambda_1^{1/2} e^{(1-\eta)n/2}\Big\}. \end{aligned}$$

In order to estimate this last expression we shall argue, that there exist

$c_1, c_2 > 0$ such that for all $\nu > 0$, $T_2 > T_1 > 0$, $\varepsilon > 0$, $\theta > 0$,

$$P\{\sup[|W_{\nu t'} - W_{\nu t}|; T_1 \le t \le T_2, t' > 0, |t' - t| < \varepsilon] \ge \theta\} \le c_1\left(\frac{T_2 - T_1}{\varepsilon} + 1\right)\frac{\sqrt{\nu\varepsilon}}{\theta}\exp\left(-c_2\frac{\theta^2}{\nu\varepsilon}\right). \tag{3.4}$$

Distinguishing the cases $t < t'$ and $t' < t$, one shows first that the left side in (3.4) is majorized by

$$2P\{\sup[|W_{\nu t'} - W_{\nu t}|; 0 \le t \le T_2 - T_1, t < t' < t + \varepsilon] \ge \theta\}. \tag{+}$$

To be more precise, in the first case we use the fact that

$$\{W_{\nu t}, t \ge 0\} \quad \text{and} \quad \{W_{\nu T_1 + \nu t} - W_{\nu T_1}, t \ge 0\}$$

have the same laws, and in the second case we use the fact that

$$\{W_{\nu t}, 0 \le t \le T_2\} \quad \text{and} \quad \{W_{\nu T_2 - \nu t} - W_{\nu T_2}, 0 \le t \le T_2\}$$

have the same laws.

Using similar transformation laws for Brownian motion, we see that the expression $(+)$ equals

$$\begin{aligned}
&2P\left\{\sup\left[|W_{\nu\varepsilon s'} - W_{\nu\varepsilon s}|; 0 \le s \le \frac{T_2 - T_1}{\varepsilon}, s < s' < s + 1\right] \ge \theta\right\} \\
&\quad = 2P\left\{\sup\left[|W_{s'} - W_s|; 0 \le s \le \frac{T_2 - T_1}{\varepsilon}, s < s' < s + 1\right] \ge \frac{\theta}{\sqrt{\nu\varepsilon}}\right\} \\
&\quad \le 2\sum_{l=0}^{[(T_2 - T_1)/\varepsilon]} P\left\{\sup[|W_{s'} - W_s|; l \le s \le l + 1, s < s' < s + 1] \ge \frac{\theta}{\sqrt{\nu\varepsilon}}\right\} \\
&\quad \le 2\left(\frac{T_2 - T_1}{\varepsilon} + 1\right)P\left\{\sup[|W_{s'} - W_s|; 0 \le s, s' \le 2] \ge \frac{\theta}{\sqrt{\nu\varepsilon}}\right\} \\
&\quad \le 2\left(\frac{T_2 - T_1}{\varepsilon} + 1\right)P\left\{2\sup[|W_s|; 0 \le s \le 2] \ge \frac{\theta}{\sqrt{\nu\varepsilon}}\right\} \\
&\quad \le 4\left(\frac{T_2 - T_1}{\varepsilon} + 1\right)P\left\{2\sqrt{2}\sup[W_s; 0 \le s \le 1] \ge \frac{\theta}{\sqrt{\nu\varepsilon}}\right\} \\
&\quad \le 8\left(\frac{T_2 - T_1}{\varepsilon} + 1\right)P\left\{W_1 \ge \frac{\theta}{\sqrt{8\nu\varepsilon}}\right\} \\
&\quad \le c_1\left(\frac{T_2 - T_1}{\varepsilon} + 1\right)\frac{\sqrt{\nu\varepsilon}}{\theta}\exp\left(-c_2\frac{\theta^2}{\nu\varepsilon}\right),
\end{aligned}$$

proving (3.4).

We shall apply (3.4) to complete the estimate for $a_n(\lambda)$. We let

$$T_1 = e^n,\ T_2 = e^{n+1},\ \varepsilon = \frac{1}{\nu}\lambda_2^{1-\sigma}e^{(1-\rho)(n+1)},\ \theta = \lambda_1^{1/2}e^{(1-\eta)n/2}.$$

For $n \geq 0$, $\lambda > 0$ we obtain

$$\frac{T_2 - T_1}{\varepsilon} \leq c\frac{\nu e^{n\rho}}{\lambda^{1-\sigma}},\quad \frac{\theta^2}{\nu\varepsilon} \geq c\lambda^{\sigma}e^{(\rho-\eta)n},$$

and therefore

$$a_n(\lambda) \leq c'\left(\frac{\nu}{\lambda^{1-\sigma}}e^{\rho n} + 1\right)\lambda^{-\sigma/2}\exp\{-c''\lambda^{\sigma}e^{(\rho-\eta)n}\},$$

with $c', c'' > 0$. Choosing $\lambda_0 > 0$ such that for $n \geq 0$, $\lambda \geq \lambda_0$,

$$c''\lambda^{\sigma}e^{(\rho-\eta)n} - \rho n \geq \tfrac{1}{2}c''\lambda^{\sigma}e^{(\rho-\eta)n},$$

we obtain for $n \geq 0$, $\lambda \geq \lambda_0$,

$$a_n(\lambda) \leq c'''(\nu + 1)\exp\{-\tfrac{1}{2}c''\lambda^{\sigma}e^{(\rho-\eta)n}\},$$

with $c''' > 0$. We conclude that for $\lambda \geq \lambda_0$,

$$\begin{aligned}\sum_{n=1}^{\infty} a_n(\lambda) &\leq c'''(\nu + 1)\int_0^{\infty}\exp\{-\tfrac{1}{2}c''\lambda^{\sigma}e^{(\rho-\eta)t}\}\,dt\\ &\leq c(\nu+1)\exp\{-\tfrac{1}{2}c''\lambda^{\sigma}\}.\end{aligned}$$

It follows that for $\lambda \geq \lambda_0$,

$$I_2(\lambda) \leq \sum_{n=0}^{\infty} a_n(\lambda) \leq c(\nu + 1)\exp\{-\tfrac{1}{2}c''\lambda^{\sigma}\},$$

and hence for any $p \geq 1$,

$$\int_0^{\infty}\lambda^{p-1}I_2(\lambda)\,d\lambda \leq c_1 + c_2\nu. \tag{3.3b}$$

Estimate (3.2) now follows from (3.3a) and (3.3b).

4. The almost everywhere central limit theorem

(4.1) LEMMA. *For $\varphi \in C^{\infty}(M)$, $x \in M$, we have P^x-a.e.*

$$\lim_{t\to\infty}\frac{1}{\log t}\int_1^t\frac{ds}{s}\exp\left\{i\frac{\tilde{L}_s(\varphi) - m(\varphi)s}{\sqrt{s}}\right\} = \exp\{-(\varphi, G\varphi)_{L^2(M,m)}\}. \tag{4.2}$$

Proof. We may assume that $\varphi \not\equiv$ constant, or equivalently that $(\varphi, G\varphi)_{L^2(M,m)} > 0$. Furthermore, it suffices to prove that (4.2) holds P^m-a.e. Indeed, if $\Lambda \in \mathfrak{A}$ is the event defined by (4.2), then the function $x \mapsto P^x(\Lambda)$, $x \in M$, is harmonic and hence constant. Harmonicity follows from shift invariance of Λ. The latter means that for any $u \geq 0$, $\omega \in \Lambda$ iff $\theta_u\omega \in \Lambda$. For its verification, we observe that

$$\left|\exp\left\{i\frac{L_s(\omega)(\varphi) - m(\varphi)s}{\sqrt{s}}\right\} - \exp\left\{i\frac{L_s(\theta_u\omega)(\varphi) - m(\varphi)s}{\sqrt{s}}\right\}\right|$$

is majorized by

$$s^{-1/2}|L_s(\omega)(\varphi) - L_s(\theta_u\omega)(\varphi)| \leq 2us^{-1/2}\|\varphi\|_\infty$$

for $s \geq u$, and therefore converges to 0, as $s \uparrow \infty$.

In order to show that (4.2) holds P^m-a.e., notice first that with $\varphi \in C^\infty(M)$, we also have $G\varphi \in C^\infty(M)$ and $f_\varphi =_{\text{def}} |\text{grad } G\varphi|^2 \in C^\infty(M)$. We recall from (4.1) in [2] that

$$(4.3)\quad M_s =_{\text{def}} L_s(\varphi) - m(\varphi)s + (G\varphi)(X_s) - (G\varphi)(X_0), \quad s \geq 0,$$

is an obviously square integrable P^m-martingale with increasing process $s \mapsto L_s(f_\varphi)$, $s \geq 0$. There exists a probability space $(\Omega', \mathfrak{A}', P')$ supporting a one-dimensional Brownian motion $\{W_t, t \geq 0\}$ and a continuous time change $\{\tau_t, t \geq 0\}$ such that the P^m-law of M and the P'-law of $W \circ \tau$ coincide. As the increasing process of the square integrable P'-martingale $W \circ \tau$ is τ, it follows that the P^m-law of $L_t(f_\varphi)$, $t \geq 0$, and the P'-law of τ_t, $t \geq 0$, coincide.

Now in order to verify (4.2), P^m-a.e., it suffices to prove that

$$(4.4)\quad \lim_{t\to\infty} \frac{1}{\log t}\int_1^t \frac{ds}{s} \exp\left\{i\frac{M_s}{\sqrt{s}}\right\} = \exp\{-(\varphi, G\varphi)_{L^2(M,m)}\}, \; P^m\text{-a.e.},$$

since by (4.3),

$$\left|\exp\{is^{-1/2}[L_s(\varphi) - m(\varphi)s]\} - \exp\{is^{-1/2}M_s\}\right|$$

is majorized by

$$s^{-1/2}|M_s - L_s(\varphi) + m(\varphi)\cdot s| \leq 2s^{-1/2}\|G\varphi\|_\infty,$$

and therefore converges to 0 as $s \uparrow \infty$. Moreover, (4.4) is equivalent to

$$\lim_{t\to\infty} \frac{1}{\log t} \int_1^t \frac{ds}{s} \exp\left\{ i \frac{W_{\tau_s}}{\sqrt{s}} \right\} = \exp\{-(\varphi, G\varphi)_{L^2(M,m)}\}, \quad P'\text{-a.e.} \tag{4.5}$$

Letting

$$\nu = m(f_\varphi) = \int_M |\text{grad } G\varphi|^2 \, dm = 2(\varphi, G\varphi)_{L^2(M,m)},$$

we shall obtain (4.5) from

$$\lim_{t\to\infty} \frac{1}{\log t} \int_1^t \frac{ds}{s} \exp\left\{ i \frac{W_{\nu s}}{\sqrt{s}} \right\} = e^{-\nu/2}, \quad P'\text{-a.e.} \tag{4.6}$$

and

$$\lim_{s\to\infty} \frac{W_{\tau_s} - W_{\nu s}}{\sqrt{s}} = 0, \quad P'\text{-a.e.}, \tag{4.7}$$

as

$$\left| \exp\{iW_{\tau_s}/\sqrt{s}\} - \exp\{iW_{\nu s}/\sqrt{s}\} \right| \le s^{-1/2} |W_{\tau_s} - W_{\nu s}|.$$

Equation (4.6) follows from the ergodic theorem. Indeed, as the process $\{W_{\nu e^u}/\sqrt{e^u},\ u \in \mathbf{R}\}$ is stationary and has a trivial tail-field, we have P'-a.e.

$$\begin{aligned} \lim_{n\to\infty} \frac{1}{n} \int_0^n du \exp\{iW_{\nu e^u}/\sqrt{e^u}\} &= \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \int_k^{k+1} du \exp\{iW_{\nu e^u}/\sqrt{e^u}\} \\ &= E \int_0^1 du \exp\{iW_{\nu e^u}/\sqrt{e^u}\} \\ &= E \exp\{iW_\nu\} \\ &= e^{-\nu/2}. \end{aligned}$$

This implies, that P'-a.e.

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t du \exp\{iW_{\nu e^u}/\sqrt{e^u}\} = e^{-\nu/2},$$

which proves (4.6).

In order to prove (4.7), we recall that f_φ is bounded. We conclude by Remark (2.14), that for $\rho < \frac{1}{2}$, $0 \le \delta < 1$,

$$E^m\left\{\sup_{t\ge 1}\frac{|L_t(f_\varphi) - t\nu|}{t^{1-\rho}}\right\}^{1+\delta} < \infty,$$

or equivalently

$$E'\left\{\sup_{t\ge 1}\frac{|\tau_t - t\nu|}{t^{1-\rho}}\right\}^{1+\delta} < \infty.$$

It follows from Lemma (3.1), that for all $\eta < \frac{1}{2}$,

$$E'\left\{\sup_{t\ge 1}\frac{|W_{\tau_t} - W_{\nu t}|^2}{t^{1-\eta}}\right\} < \infty,$$

which implies (4.7) and completes the proof of the lemma.

Notice that for $f_\varphi \in L^\infty(M)$, finiteness a.e. of

$$\sup_{t\ge 1}|\tau_t - \nu t|/t^{1-\rho}, \quad \rho < \tfrac{1}{2},$$

follows also from a $\log_2$-law for $L_t(f_\varphi)$. Then (4.7) could be proved like Lemma (3.5) in [5].

In § 1 we fixed in the Hilbertian spaces $H^\alpha(M)$, $\alpha \ge 0$, the norms carried over from $L^2(M,m)$ by the isomorphisms K_α. In $H^{-\alpha}(M)$, $\alpha \ge 0$, we fixed the dual norms.

If now μ is a finite measure on M, such that for an $\alpha > 0$, $e_\alpha^2(\mu) < \infty$ or equivalently $\tilde{\mu} \in H^{-\alpha}(M)$, we have for $f \in H^\alpha(M)$,

$$\begin{aligned}(4.8)\qquad \tilde{\mu}(f) &= \int_M \bar{f}\,d\mu = \left(\bar{f}, G_{\alpha/2}\mu + \mu(M)\right)_{L^2(M,m)}\\ &= \left(f, G_\alpha\mu + \mu(M)\right)_{H^\alpha(M)},\end{aligned}$$

where $f = K_\alpha\bar{f}$, m-a.e., $\bar{f} \in L^2(M)$. It follows that for finite measures μ_1, μ_2 on M, such that $e_\alpha^2(\mu_i) < \infty$, $i = 1, 2$,

$$\begin{aligned}(4.9)\quad \|\tilde{\mu}_2 - \tilde{\mu}_1\|_{H^{-\alpha}(M)} &= \left\|G_\alpha(\mu_2 - \mu_1) + \mu_2(M) - \mu_1(M)\right\|_{H^\alpha(M)}\\ &= \left\|G_{\alpha/2}(\mu_2 - \mu_1) + \mu_2(M) - \mu_1(M)\right\|_{L^2(M,m)}.\end{aligned}$$

If we apply (4.9) to the measures $\mu_1 = tm$ and $\mu_2 = L_t$, we obtain for $\alpha > \frac{1}{2}d - 1$

$$\|\tilde{L}_t - tm\|^2_{H^{-\alpha}(M)} = \int_M dm(\xi)\left[\int_0^t g_{\alpha/2}(\xi, X_s)\, ds\right]^2 \tag{4.10}$$
$$= e_\alpha^2(L_t - tm).$$

(4.11) LEMMA. *For $\alpha > \frac{1}{2}d - 1$, there exists $\delta' > 0$, such that*

$$E^m \sup_{t \geq e} \frac{1}{\log t} \int_1^t \frac{ds}{s} \left\{ \frac{\|\tilde{L}_s - sm\|_{H^{-\alpha}(M)}}{\sqrt{s}} \right\}^{2+2\delta'} < \infty. \tag{4.12}$$

Proof. It suffices to consider α sufficiently close to $\frac{1}{2}d - 1$. We shall assume that $\alpha \in (\frac{1}{2}d - 1, \frac{3}{4}d - 1)$, and we shall show that (4.12) holds for any

$$\delta' \in \left[0, \frac{\alpha - \frac{1}{2}d + 1}{d - \alpha - 1}\right).$$

For

$$\alpha \in \left(\tfrac{1}{2}d - 1, \tfrac{3}{4}d - 1\right), \quad \delta \in \left[0, \frac{\alpha - \frac{1}{2}d + 1}{d - \alpha - 1}\right)$$

we have by (2.18),

$$\sup_{\xi \in M} \int_M \left|\operatorname{grad}_2 g_{\alpha/2+1}(\xi, \eta)\right|^{2+2\delta} dm(\eta) < \infty. \tag{4.13}$$

Similarly, one gets

$$\sup_{\eta \in M} \int_M \left|\operatorname{grad}_2 g_{\alpha/2+1}(\xi, \eta)\right|^{2+2\delta} dm(\xi) < \infty, \tag{4.13'}$$

and certainly,

$$\sup_{\eta \in M} \int_M \left|g_{\alpha/2+1}(\xi, \eta)\right|^2 dm(\xi) < \infty. \tag{4.14}$$

The subscript 2 after grad in (4.13) indicates differentiation with respect to the second variable.

As we want to use formula (4.10) for $\|\tilde{L}_s - sm\|_{H^{-\alpha}(M)}$, we define by analogy with (4.3), for $\xi \in M$,

$$M_s^\alpha(\xi) = \int_0^s g_{\alpha/2}(\xi, X_u)\, du + g_{\alpha/2+1}(\xi, X_s) - g_{\alpha/2+1}(\xi, X_0), \quad s \geq 0. \tag{4.15}$$

For all $\xi \in M$, the process $\{M_s^\alpha(\xi),\ s \geq 0\}$ is a square integrable P^m-martingale (because $\alpha > \frac{1}{2}d - 1$) with the process $s \mapsto A_s^\alpha(\xi)$, $s \geq 0$, of Lemma (2.16) as its increasing process. (See also (6.4), (6.5) in [2]).

From (4.10), (4.15) and (4.14) we have

$$\|\tilde{L}_s - sm\|_{H^{-\alpha}(M)}^2 \leq c + c\int_M dm(\xi)|M_s^\alpha(\xi)|^2,$$

and hence

$$\|\tilde{L}_s - sm\|_{H^{-\alpha}(M)}^{2+2\delta'} \leq c + c\int_M dm(\xi)|M_s^\alpha(\xi)|^{2+2\delta'}. \tag{4.10$'$}$$

Thus, in order to verify (4.12), it suffices to prove

$$E^m \sup_{t \geq e} \frac{1}{\log t}\int_1^t \frac{ds}{s} \frac{\int_M dm(\xi)|M_s^\alpha(\xi)|^{2+2\delta'}}{s^{1+\delta'}} < \infty. \tag{4.16}$$

For the proof of (4.16) it is sufficient to prove

$$\int_M dm(\xi) E^m \sup_{t \geq e} \frac{1}{\log t}\int_1^t \frac{ds}{s} \frac{|M_s^\alpha(\xi)|^{2+2\delta'}}{s^{1+\delta'}} < \infty, \tag{4.17}$$

and for the proof of (4.17) it suffices to prove

$$\sup_{\xi \in M} E^m \sup_{t \geq e} \frac{1}{\log t}\int_1^t \frac{ds}{s} \frac{|M_s^\alpha(\xi)|^{2+2\delta'}}{s^{1+\delta'}} < \infty. \tag{4.18}$$

In the notation to be introduced now, we shall suppress the dependence on α, which is kept fixed anyway. For $\xi \in M$, there is again a probability space $(\Omega_\xi, \mathfrak{A}_\xi, P_\xi)$ supporting a one-dimensional Brownian motion $\{W_t^\xi,\ t \geq 0\}$ and a continuous time change $\{\tau_t(\xi),\ t \geq 0\}$ such that the P^m-law of $M^\alpha(\xi)$ and the P_ξ-law of $W^\xi \circ \tau(\xi)$ coincide. As $M^\alpha(\xi)$ is square integrable, so is $W^\xi \circ \tau(\xi)$. It follows that $\tau(\xi)$ is the increasing process for $W^\xi \circ \tau(\xi)$. Hence the P^m-law of $A^\alpha(\xi)$ and the P_ξ-law of $\tau(\xi)$ coincide.

Obviously, assertion (4.18) is equivalent to

$$\sup_{\xi\in M} E_\xi \sup_{t\geq e} \frac{1}{\log t}\int_1^t \frac{ds}{s}\frac{\left|W^\xi_{\tau_s(\xi)}\right|^{2+2\delta'}}{s^{1+\delta'}} < \infty \tag{4.19}$$

Here E_ξ denotes the expectation with respect to P_ξ on $(\Omega_\xi, \mathfrak{A}_\xi)$, not to be confused with P^ξ on the measurable space $(\Omega, \mathfrak{A})$, which supports the diffusion.

Let

$$\nu^\alpha(\xi) = \int_M \left|\operatorname{grad}_2 g_{\alpha/2+1}(\xi,\eta)\right|^2 dm(\eta),$$

as in Lemma (2.16). Obviously, $\nu^\alpha(\xi) > 0$ for all $\xi \in M$, and (4.13) with $\delta = 0$ reads

$$\sup_{\xi\in M} \nu^\alpha(\xi) < \infty. \tag{4.20}$$

Moreover, for the proof of (4.19) it suffices to prove

$$\sup_{\xi\in M} E_\xi \sup_{t\geq e} \frac{1}{\log t}\int_1^t \frac{ds}{s}\frac{\left|W^\xi_{\nu^\alpha(\xi)\cdot s}\right|^{2+2\delta'}}{s^{1+\delta'}} < \infty, \tag{4.21}$$

and

$$\sup_{\xi\in M} E_\xi \sup_{s\geq 1} \frac{\left|W^\xi_{\tau_s(\xi)} - W^\xi_{\nu^\alpha(\xi)\cdot s}\right|^{2+2\delta'}}{s^{1+\delta'}} < \infty. \tag{4.22}$$

Assertion (4.21) follows from the dominated ergodic theorem. Indeed, for fixed $\xi \in M$, the process

$$\left\{\frac{W^\xi_{\nu^\alpha(\xi)e^u}}{\sqrt{e^u}},\quad u\in\mathbf{R}\right\}$$

on $(\Omega_\xi, \mathfrak{A}_\xi, P_\xi)$ is stationary. If we fix $\delta > \delta'$, and for $\xi \in M$ let

$$Z^k_\xi = \int_k^{k+1} du\left\{\frac{\left|W^\xi_{\nu^\alpha(\xi)\cdot e^u}\right|}{\sqrt{e^u}}\right\}^{2+2\delta'}, \quad k = 0,1,2,\ldots,$$

$$Z^*_\xi = \sup_{n\in\mathbf{N}} \frac{1}{n}\sum_{k=0}^{n-1} Z^k_\xi,$$

then, by the dominated ergodic theorem, for all $\xi \in M$ we have

$$\begin{aligned}
E_\xi \sup_{n\in\mathbf{N}} \frac{1}{n}\int_0^n du \left\{ \frac{\left|W^{\xi}_{\nu^\alpha(\xi)\cdot e^u}\right|}{\sqrt{e^u}} \right\}^{2+2\delta'} \\
&= E_\xi Z^*_\xi \le \left[E_\xi \big(Z^*_\xi\big)^{(2+2\delta)/(2+2\delta')} \right]^{(2+2\delta')/(2+2\delta)} \\
&\le c\left[E_\xi \big(Z^0_\xi\big)^{(2+2\delta)/(2+2\delta')} \right]^{(2+2\delta')/(2+2\delta)} \\
&\le c + cE_\xi \big(Z^0_\xi\big)^{(2+2\delta)/(2+2\delta')} \le c + cE_\xi \int_0^1 du \left| \frac{W^{\xi}_{\nu^\alpha(\xi)\cdot e^u}}{\sqrt{e^u}} \right|^{2+2\delta} \\
&= c + cE_\xi \left| W^{\xi}_{\nu^\alpha(\xi)} \right|^{2+2\delta} = c + c[\nu^\alpha(\xi)]^{1+\delta}.
\end{aligned}$$

It follows that for all $\xi \in M$,

$$E_\xi \sup_{t\ge 1} \frac{1}{t}\int_0^t du \left| \frac{W^{\xi}_{\nu^\alpha(\xi)\cdot e^u}}{\sqrt{e^u}} \right|^{2+2\delta'} \le c + c[\nu^\alpha(\xi)]^{1+\delta},$$

which in view of (4.20) implies (4.21).

We shall now derive (4.22) from Lemma (2.16) and from Lemma (3.1). We fix

$$\delta \in \left(\delta', \frac{\alpha - \frac{1}{2}d + 1}{d - \alpha - 1}\right), \quad \rho \in \left(0, \frac{2}{d}\left(\alpha - \tfrac{1}{2}d + 1\right)\right), \quad \eta = 0$$

and conclude by Lemma (3.1) that the left side of (4.22) is majorized by

$$c_1 + c_2 \sup_{\xi\in M} [\nu^\alpha(\xi)] + c_3 \sup_{\xi\in M} E_\xi \left\{ \sup_{t\ge 1} \frac{|\tau_t(\xi) - \nu^\alpha(\xi)t|}{t^{1-\rho}} \right\}^{1+\delta}.$$

In view of (4.20), it remains to show that the last term in this sum is finite. Since for all $\xi \in M$,

$$E_\xi \left\{ \sup_{t\ge 1} \frac{|\tau_t(\xi) - \nu^\alpha(\xi)t|}{t^{1-\rho}} \right\}^{1+\delta} = E^m \left\{ \sup_{t\ge 1} \frac{|A^\alpha_t(\xi) - \nu^\alpha(\xi)t|}{t^{1-\rho}} \right\}^{1+\delta},$$

we just need to appeal to (2.17).

(4.23) *Remark.* It follows immediately from Lemma (4.11) that there is a function δ': $(\frac{1}{2}d - 1, \infty) \to (0,1)$ such that for all $x \in M$, P^x-a.e. for all

$\alpha > d/2 - 1$,

$$\sup_{t \geq e} \frac{1}{\log t} \int_1^t \frac{ds}{s} \left\{ \frac{\|\tilde{L}_s - sm\|_{H^{-\alpha}(M)}}{\sqrt{s}} \right\}^{2+2\delta'(\alpha)} < \infty, \tag{4.24}$$

as the events defined by (4.24) are shift-invariant. It can actually be shown that the assertion of Lemma (4.11) remains true if in (4.12) "E^m" is replaced by "$\sup_{x \in M} E^x$".

(4.25) LEMMA. *For all $x \in M$, P^x-a.e. (4.2) with φ replaced by f, holds for all $f \in \bigcup_{\alpha > \frac{1}{2}d-1} H^\alpha(M)$.*

Proof. Let $\alpha_k = \frac{1}{2}d - 1 + 1/k$, $k \in \mathbf{N}$. It suffices to show that for all $k \in \mathbf{N}$, $x \in M$, P^x-a.e., (4.2) holds for all $f \in H^{\alpha_k}(M)$. Now, there is a countable set of functions in $C^\infty(M)$, which is dense in $H^{\alpha_k}(M)$. By Lemma (4.1), (4.2) holds simultaneously for all functions in this set. Moreover, for any $f_1, f_2 \in H^{\alpha_k}(M)$,

$$\begin{aligned} &\left|\exp\{is^{-1/2}[\tilde{L}_s(f_1) - m(f_1)s]\} - \exp\{is^{-1/2}[\tilde{L}_s(f_2) - m(f_2)s]\}\right| \\ &\quad \leq s^{-1/2}\left|\tilde{L}_s(f_1 - f_2) - sm(f_1 - f_2)\right| \\ &\quad \leq s^{-1/2}\|\tilde{L}_s - sm\|_{H^{-\alpha_k}(M)} \cdot \|f_1 - f_2\|_{H^{\alpha_k}(M)}, \end{aligned}$$

and by Remark (4.23) for all $x \in M$, P^x-a.e.

$$\sup_{t \geq e} \frac{1}{\log t} \int_1^t \frac{ds}{s} \frac{1}{\sqrt{s}} \|\tilde{L}_s - ms\|_{H^{-\alpha_k}(M)} < \infty.$$

This completes the proof.

For $\alpha > \frac{1}{2}d - 1$, we define the random probability measures $\mu_t^{-\alpha}$, $t > 1$, on $H^{-\alpha}(M)$ by

$$\mu_t^{-\alpha} = \frac{1}{\log t} \int_1^t \frac{ds}{s} \delta^{-\alpha}_{(\tilde{L}_s - sm)/\sqrt{s}}.$$

(4.26) LEMMA. *Let $\alpha > \frac{1}{2}d - 1$. For all $x \in M$, P^x-a.e. the family*

$$\{\mu_t^{-\alpha}, t \geq e\}$$

is tight.

Proof. Fix $\varepsilon \in (0, \alpha - \frac{1}{2}d + 1)$. The operator K_ε on $L^2(M)$ also defines an isomorphism

$$K_\varepsilon \colon H^{\alpha-\varepsilon}(M) \to H^\alpha(M).$$

Therefore the adjoint is an isomorphism

$$K_\varepsilon^* \colon H^{-\alpha}(M) \to H^{-(\alpha-\varepsilon)}(M) \subseteq H^{-\alpha}(M).$$

As $\|f\|_{H^{\alpha-\varepsilon}(M)} = \|K_\varepsilon f\|_{H^\alpha(M)}$ for $f \in H^{\alpha-\varepsilon}(M)$, we get

$$\|l\|_{H^{-\alpha}(M)} = \|K_\varepsilon^* l\|_{H^{-(\alpha-\varepsilon)}(M)} \quad \text{for } l \in H^{-\alpha}(M). \tag{4.27}$$

Moreover as K_ε is a compact and symmetric operator on $H^\alpha(M)$, K_ε^* is a compact operator on $H^{-\alpha}(M)$. Indeed, if $l(\cdot) = (h_l, \cdot\,)_{H^\alpha(M)}$ with $h_l \in H^\alpha(M)$, then $(K_\varepsilon^* l)(\cdot) = l(K_\varepsilon\, \cdot\,) = (K_\varepsilon h_l, \cdot\,)_{H^\alpha(M)}$.

If we now let $B_\eta = \{l \in H^{-\alpha}(M);\ \|l\|_{H^{-\alpha}(M)} \le \eta\}$, then $K_\varepsilon^* B_\eta \subseteq H^{-(\alpha-\varepsilon)}(M)$ is a compact set in $H^{-\alpha}(M)$, and we shall show that $\mu_t^{-\alpha}\{(K_\varepsilon^* B_\eta)^c\}$ is small uniformly in t if η is large. From (4.27) we have

$$K_\varepsilon^* B_\eta = \left\{l \in H^{-(\alpha-\varepsilon)}(M);\ \|l\|_{H^{-(\alpha-\varepsilon)}(M)} \le \eta\right\}.$$

Therefore,

$$\begin{aligned}
\mu_t^{-\alpha}\left(K_\varepsilon^* B_\eta\right) &= \mu_t^{-(\alpha-\varepsilon)}\left(K_\varepsilon^* B_\eta\right) \\
&= 1 - \mu_t^{-(\alpha-\varepsilon)}\left\{l \in H^{-(\alpha-\varepsilon)}(M);\ \|l\|_{H^{-(\alpha-\varepsilon)}(M)} > \eta\right\} \\
&\ge 1 - \frac{1}{\eta^2}\int \|l\|^2_{H^{-(\alpha-\varepsilon)}(M)}\, d\mu_t^{-(\alpha-\varepsilon)}(l) \\
&= 1 - \frac{1}{\eta^2}\frac{1}{\log t}\int_1^t \frac{ds}{s}\frac{\|\tilde{L}_s - sm\|^2_{H^{-(\alpha-\varepsilon)}(M)}}{s}.
\end{aligned}$$

By Remark (4.23), for all $x \in M$, P^x-a.e.

$$\sup_{t \ge e} \frac{1}{\log t}\int_1^t \frac{ds}{s}\frac{\|\tilde{L}_s - sm\|^2_{H^{-(\alpha-\varepsilon)}(M)}}{s} < \infty,$$

which completes the proof.

Proof of Theorem (1.15). As we explained in § 1, it suffices to show that for all $\alpha > \frac{1}{2}d - 1$, $x \in M$, P^x-a.e. (1.16) holds. But this follows from Lemma (4.25) and Lemma (4.26).

Proof of Remark (1.21). We want to show that if ω satisfies (1.16) for $\alpha_n = \frac{1}{2}d - 1 + 1/n$, $n \in \mathbf{N}$, then it satisfies (1.20). We define the random probability measures μ_t, $t > 1$, on $(\mathfrak{X}, \mathfrak{B}_{\mathfrak{X}})$ by

$$\mu_t = \frac{1}{\log t}\int_1^t \frac{ds}{s}\delta_{(\tilde{L}_s - sm)/\sqrt{s}}.$$

Obviously, the $\mu_t^{-\alpha}$, $\alpha > \frac{1}{2}d - 1$, are the image measures of μ_t under the embeddings

$$i_{-\alpha}\colon \mathfrak{X} \hookrightarrow H^{-\alpha}(M).$$

We shall argue first that the family $\{\mu_t,\ t \geq e\}$ on $(\mathfrak{X}, \mathfrak{B}_{\mathfrak{X}})$ is tight. As for $n \in \mathbf{N}$, the families $\{\mu_t^{-\alpha_n},\ t \geq e\}$ on $H^{-\alpha_n}(M)$ are tight, we know that for all $n \in \mathbf{N}$, $\varepsilon > 0$, there exist compact sets K_n in $H^{-\alpha_n}(M)$ such that

$$\inf_{t \geq e} \mu_t^{-\alpha_n}(K_n) \geq 1 - \frac{\varepsilon}{2^n},$$

i.e.,

$$\inf_{t \geq e} \mu_t(K_n \mathfrak{X}) \geq 1 - \frac{\varepsilon}{2^n}.$$

Letting $K = \bigcap_{n \geq 1}(K_n \mathfrak{X}) = \bigcap_{n \geq 1} K_n$, we have $\inf_{t \geq e} \mu_t(K) \geq 1 - \varepsilon$. Moreover, a diagonal argument shows that K is compact in $\mathfrak{X}$; i.e., for any sequence $l_n \in K$, there exists a subsequence l_{n_k} converging to an $l_0 \in K$, in the topology of all $H^{-\alpha}(M)$, $\alpha > \frac{1}{2}d - 1$.

It remains to show, that if $t_k \uparrow \infty$, $\lim_{k\to\infty} \mu_{t_k} = \mu_0$, then $\mu_0 = N_G$. Now, $\lim_{k\to\infty} \mu_{t_k} = \mu_0$ implies that

$$\lim_{k\to\infty} \mu_{t_k}^{-\alpha_n} = i_{-\alpha_n}(\mu_0) \quad \text{for all } n \in \mathbf{N},$$

therefore $i_{-\alpha_n}(\mu_0) = N_G^{-\alpha_n} = i_{-\alpha_n}(N_G)$, for all $n \in \mathbf{N}$. It follows that μ_0 and N_G coincide on the traces on $\mathfrak{X}$, of the Borel fields on $H^{-\alpha}(M)$, $\alpha > \frac{1}{2}d - 1$, and therefore that $\mu = N_G$ on $\mathfrak{B}_{\mathfrak{X}}$.

Proof of Theorem (1.11). We apply Theorem (1.15). As

$$\frac{1}{s}e_\alpha^2(L_s) = \left\| \frac{\tilde{L}_s - sm}{\sqrt{s}} \right\|^2_{H^{-\alpha}(M)},$$

and $l \mapsto \exp\{-\beta \|l\|^2_{H^{-\alpha}(M)}\}$ is a bounded continuous function on $H^{-\alpha}(M)$, it remains to show that

$$\int_{H^{-\alpha}(M)} \exp\{-\beta\|l\|^2_{H^{-\alpha}(M)}\}\, dN_G^{-\alpha}(l) = [\Phi_\alpha(\beta)]^{-1/2}. \tag{4.28}$$

There is a CON system $\{1, \Psi_1, \dots\}$ in $H^\alpha(M)$ such that

$$\left(-\tfrac{1}{2}\Delta\Psi_k\right) = \lambda_k \Psi_k, \quad G_\beta \Psi_k = \lambda_k^{-\beta}\Psi_k, \quad k \geq 1, \quad \beta > 0.$$

Moreover,

$$\|l\|^2_{H^{-\alpha}(M)} = [l(1)]^2 + \sum_{i=1}^{\infty} [l(\Psi_i)]^2.$$

Now for $n \in \mathbf{N}$ the image measure in $\mathbf{R}^{n+1}$ of $N_G^{-\alpha}$ under the mapping

$$l \mapsto (l(1), l(\Psi_1), \dots, l(\Psi_n))$$

is normal with mean 0 and covariance matrix $a_{0j} = a_{j0} = 0$ for $j = 0, \dots, n$, $a_{ij} = 2\lambda_i^{-(\alpha+1)}\delta_{ij}$ for $i, j = 1, \dots, n$. It follows that

$$\int_{H^{-\alpha}(M)} \exp\left(-\beta \sum_{k=0}^{n} [l(\Psi_k)]^2\right) dN_G^{-\alpha}(l) = \prod_{k=1}^{n} \frac{1}{\sqrt{1 + \beta\lambda_k^{-(\alpha+1)}}},$$

and we obtain (4.28) by letting $n \to \infty$.

Proof of (1.25). We apply Theorem (1.15) again. For $n \in \mathbf{N}$,

$$l \mapsto \|l\|^2_{H^{-\alpha}(M)} \wedge n$$

is a bounded continuous function, so for all $x \in M$, P^x-a.e. we have

$$\begin{aligned} \lim_{t\to\infty} \frac{1}{\log t} \int_1^t \frac{ds}{s} \frac{\|\tilde{L}_s - sm\|^2_{H^{-\alpha}(M)}}{s} \wedge n \\ = \int_{H^{-\alpha}(M)} \|l\|^2_{H^{-\alpha}(M)} \wedge n\, dN_G^{-\alpha}(l), \end{aligned} \tag{4.29}$$

for all $\alpha > \frac{1}{2}d - 1$, $n \in \mathbf{N}$. As $n \to \infty$, the right side of (4.29) converges to

$$\int_{H^{-\alpha}(M)} \|l\|^2_{H^{-\alpha}(M)}\, dN_G^{-\alpha}(l) = \operatorname{trace} G_{\alpha+1}.$$

Now, (1.25) follows from the fact that for $x \in M$, P^x-a.e. for $\alpha > \frac{1}{2}d - 1$,

$$\lim_{n\to\infty} \sup_{t \geq e} \frac{1}{\log t} \int_1^t \frac{ds}{s} \left\{ \frac{\|\tilde{L}_s - sm\|^2_{H^{-\alpha}(M)}}{s} - \frac{\|\tilde{L}_s - sm\|^2_{H^{-\alpha}(M)}}{s} \wedge n \right\} = 0,$$

which follows immediately from Remark (4.23).

Proofs of (1.26) *and* (1.27). It suffices to prove (1.27) for $f_1 = f_2$. The proofs of (1.26) and (1.27) then follow the lines of the proof for (1.25). They use again truncation, Theorem (1.15) and Remark (4.23).

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