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ANGULAR LIMITS AND INFINITE ASYMPTOTIC VALUES OF ANALYTIC FUNCTIONS OF SLOW GROWTH

BY

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1. Introduction

Let Δ and C denote the unit disk $\{|z| < 1\}$ and the unit circle $\{|z| = 1\}$ in the complex plane $\mathscr{C}$. A continuous mapping $\Gamma : [0, 1) \to \Delta$ is called a *boundary path* provided $\lim_{t \to 1} |\Gamma(t)| = 1$, and it is said to *end at* $\zeta \in C$ if $\lim_{t \to 1} \Gamma(t) = \zeta$. If f is analytic on Δ and w is a point in the extended plane $\mathscr{C} \cup \{\infty\}$, then f is said to have an *asymptotic limit* (*of* w) *at* ζ when there exists a boundary path Γ ending at ζ such that $\lim_{t \to 1} f \circ \Gamma(t)$ exists (and is equal to w). In case $\Gamma(t) = t\zeta$ for $t \in [0, 1)$, the function f is said to have a *radial limit at* ζ and the value w is denoted by $f^*(\zeta)$. When f has the limit $f^*(\zeta)$ as z approaches ζ in every Stolz angle

$$\{z \in \Delta : |\mathrm{Im}(z\bar{\zeta})| < \alpha |z - \zeta|\}, \quad \alpha \in (0, 1),$$

the function f is said to have an *angular* (*or Fatou*) *limit at* ζ. We denote by $A(f, w)$, $R(f, w)$, and $F(f, w)$ the set of all points ζ in C where f has an asymptotic, radial, and angular limit value of w at ζ, respectively. The set of all points ζ where f has an asymptotic, radial, and angular limit (for any value $w \in \mathscr{C} \cup \{\infty\}$) is denoted by $A(f)$, $R(f)$, and $F(f)$, respectively. These concepts and notation are defined similarly for harmonic functions with values w occurring in the extended real line $\mathscr{R} \cup \{\pm\infty\}$.

Let $\mathscr{B}$ denote the class of nonconstant bounded analytic functions on Δ of modulus no greater than 1. For every Borel subset E of C, let $|E|$ denote the linear (Lebesgue) measure of E. Classical results for $f \in \mathscr{B}$ (see [4; Chapter 2]) assert that $|R(f)| = 2\pi$ (the Fatou radial-limit theorem) and that for each $w \in \overline{\Delta} = \{|z| \le 1\}$, we have $A(f, w) = R(f, w) = F(f, w)$ (Lindelöf's theorem) and $|R(f, w)| = 0$ (the Riesz uniqueness theorem). The following local versions of these theorems are derived using a conformal mapping. If J is an arc of C and f a nonconstant analytic function on Δ which is bounded near J, then

$$|R(f) \cap J| = |J|$$

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and for each $w \in \mathscr{C}$, we have

$$A(f,w) \cap J = R(f,w) \cap J = F(f,w) \cap J \quad \text{and} \quad |R(f,w) \cap J| = 0.$$

A much wider class of functions, the MacLane class $\mathscr{A}$, was introduced and studied by G.R. MacLane in [10]. By definition [10; p. 8], the class $\mathscr{A}$ consists of all nonconstant analytic functions f on Δ for which $A(f)$ is dense in C. MacLane obtained a variety of results concerning the boundary behavior of the functions in this class. The following is an analogue in terms of asymptotic limits of the local version of the Fatou radial-limit theorem.

THEOREM A [10; Theorem 11, p. 25]. *Let $f \in \mathscr{A}$ and let J be an open arc of C. If*

$$A(f,\infty) \cap J = \varnothing,$$

then

$$|A(f) \cap J| > 0.$$

The inequality $|A(f) \cap I| < |I|$ for every open subarc I of J is possible as [5; Theorem 3, p. 9] shows. In a surprising result recently obtained by Barth, Rippon, and Sons [1], the conclusion of Theorem A was significantly strengthened.

THEOREM B [1; Theorem 1]. *If $f \in \mathscr{A}$ and J is an open arc of C such that*

$$A(f,\infty) \cap J = \varnothing,$$

then $|F(f) \cap J| > 0$.

MacLane [10; pp. 35–37] gave a sufficient condition for a nonconstant analytic function f to be in the class $\mathscr{A}$ in terms of the growth of its maximum modulus

$$M(f;r) = \max\{|f(r\zeta)| : \zeta \in C\}, \quad r \in [0,1).$$

An improvement of this condition by Hornblower [8] is

$$\int_0^1 \log^+ \log^+ M(f;r)\, dr < +\infty. \tag{1}$$

COROLLARY B. *If f is a nonconstant analytic function on Δ that satisfies* (1) *and J is an open arc of C, then $A(f,\infty) \cap J = \varnothing$ implies $|F(f) \cap J| > 0$.*

Let $\nu : [0, 1) \to (1, +\infty)$ be an increasing, continuous function with

$$\lim_{r \to 1} \nu(r) = +\infty.$$

We call any such function ν an *admissible growth function*. Define $\mathscr{A}_\nu$ to be the class of nonconstant analytic functions on Δ for which $M(f; r) \leq \nu(r)$ for all $r \in [0, 1)$. Clearly, $\mathscr{B}$ is the limiting class for the classes $\mathscr{A}_\nu$ as ν grows more slowly. From results just mentioned, $A_\nu \subset \mathscr{A}$ for sufficiently slowly growing ν. Therefore, it is natural to ask whether improvements of theorems for the class $\mathscr{A}$ are possible for $\mathscr{A}_\nu$ when ν grows more slowly. MacLane [9] proved a result which demonstrated that the assumption on $A(f, \infty) \cap J$ in Corollary B cannot be omitted when $\mathscr{A}$ is replaced by any class $\mathscr{A}_\nu$.

THEOREM C. *For every admissible growth function ν, there exists a function $f \in \mathscr{A}_\nu$ such that $R(f) = \varnothing$.*

However, Barth and Rippon [2] have proved the following result.

THEOREM D. *Suppose that f is a nonconstant analytic function on Δ such that*

$$\log M(f; r) = o[1/(1 - r)] \quad \textit{as } r \to 1.$$

If J is an open arc such that $A(f, \infty) \cap J$ is countable, then $|F(f) \cap J| > 0$.

Theorem 1 below fills in the picture for admissible ν satisfying

$$\log \nu(r) = o[1/(1 - r)] \quad \text{as } r \to 1,$$

and yields Theorem D as a corollary. We shall call a function ω an *admissible generating function* provided it is an increasing, continuous, concave-downward function on $[0, 2\pi]$ vanishing at 0 with $\omega'(0) = +\infty$. We associate with such a function ω the admissible growth function

$$\nu_\omega(r) = \exp[\omega(1 - r)/(1 - r)], \quad r \in [0, 1),$$

and the Hausdorff measure H_ω. Recall that for each Borel subset E of C, we have

$$H_\omega(E) = \lim_{t \to 0} \left\{ \inf \sum_{J \in \mathscr{O}} \omega(|J|) \right\},$$

where the infimum is taken over all covers $\mathscr{O}$ of E by open arcs J for which

$|J| \le t$ for every $J \in \mathscr{O}$. See [11] for background concerning Hausdorff measures.

THEOREM 1. *Let ω be an admissible generating function and $\nu = \nu_\omega$. If $f \in \mathscr{A}_\nu$ and J is an open arc such that $E = A(f, \infty) \cap J$ has Hausdorff measure $H_\omega(E) = 0$, then $|F(f) \cap J| > 0$.*

Theorem D follows from Theorem 1 since for every prescribed admissible growth function ν with $\log \nu(r) = o[1/(1-r)]$ as $r \to 1$, there exists an admissible generating function

$$\omega(r) \ge r \log \nu(1-r) \quad \text{for } r \in (0,1].$$

In fact, we can take ω to be the sum of any fixed admissible generating function with the infimum over all increasing, concave-downward functions on $[0, 2\pi]$ that are larger than $r \log \nu(1-r)$ on $(0,1]$. In connection with Theorem 1, we also note that for any prescribed admissible growth function ν, there exists an admissible generating function ω such that $\nu_\omega \le \nu$ (see Proposition 3 of §2).

We do not know if Theorem 1 is sharp; for example, whether or not the weaker hypothesis $H_\omega(E) < +\infty$ leads to the same conclusion. A more basic question (raised specifically by K. Barth in a written communication to the author) is whether there exist analytic functions of arbitrarily slow growth such that $|A(f)| = 0$. Our second theorem answers this question in the affirmative.

THEOREM 2. *For every admissible growth function ν, there exists a function $f \in \mathscr{A}_\nu$ for which $|A(f)| = 0$.*

The remainder of this paper is organized as follows. In §2 we establish notation, state several standard results, and prove some lemmas and propositions. The proofs of Theorems 1 and 2 are given in §3. Throughout the paper, we shall use the convention that c is a positive constant, independent of the relevant parameters, that may vary in its value within a sequence of inequalities.

2. Preliminary results

Corresponding to each finite, positive Borel measure μ on C, we denote by $P[d\mu]$ the Poisson integral of μ. Recall that

$$P[d\mu](z) = \frac{1}{2\pi} \int_C \frac{1 - |z|^2}{|\zeta - z|^2} d\mu(\zeta), \quad z \in \Delta, \tag{2}$$

is a positive harmonic function and the integrand in (2) is called the Poisson kernel.

For each arc J in C of length $|J| < \pi/2$, let $C(J)$ denote the intersection with Δ of the circle passing through the endpoints of J having the following property: The tangents to the circles at each endpoint of J form an angle of $\pi/4$ relative to the region R which is the intersection of the two disks determined by the circles. It is elementary to check that

$$\min\{|z| : z \in C(J)\} > 1 - |J|,$$

a fact that will be used later.

We prove two propositions—the first provides an auxiliary function used in the proof of Theorem 1, and the second an auxiliary function used in the proof of Theorem 2. We start with a lemma which can be obtained by using an easy geometric estimate of the Poisson kernel on each half of the arc $C(J)$.

LEMMA 1. *Let J be an open arc of C with $|J| < \pi/2$. Let μ be the measure on C consisting of the sum of two positive point masses each of magnitude M located at the endpoints of J. Then*

$$P[d\mu](z) \geq \frac{cM}{1 - |z|}, \quad z \subset C(J).$$

We proceed now to the proposition.

PROPOSITION 1. *Let ω be an admissible generating function and E a Borel subset of C with $H_\omega(E) = 0$. Then there exists a nonvanishing function $g \in \mathscr{B}$ with the following property: For each $\zeta \in E$, there is a sequence of open arcs $\{I_m(\zeta)\}$ each containing ζ, for which the sequence of positive numbers*

$$\sup\{|g(z)|\nu_\omega(|z|) : z \subset [I_m(\zeta)]\}, \quad m = 1, 2, \ldots$$

converges to 0.

In the proof we shall use the same functions that were used to prove a related but weaker radial result [3; Corollary 3.1].

Proof. Since $H_\omega(E) = 0$, there exist covers $\{J_{mk}\}_{k=1}^{\infty}$ of E by open arcs each of length less than $\pi/2$ for $m = 1, 2, \ldots$ such that $\sum_{m,k} \omega(|J_{mk}|) < +\infty$. Let $\{\alpha_m\}$ be a sequence of positive numbers for which $\lim \alpha_m = +\infty$ and

$$\sum_{m=1}^{\infty} \alpha_m \sum_{k=1}^{\infty} \omega(|J_{mk}|) < +\infty. \tag{3}$$

For each pair of positive integers m and k, let μ_{mk} be the measure of Lemma 1 with $J = J_{mk}$ and $M = \alpha_m \omega(|J_{mk}|)$. Then

$$\text{(4)} \quad \frac{1 - |z|}{\omega(1 - |z|)} P[d\mu_{mk}](z) \geq c \frac{\alpha_m \omega(|J_{mk}|)}{\omega(1 - |z|)} \geq c\alpha_m, \quad z \in C(J_{mk}).$$

By (3), the sum $\sum_{m,k} \mu_{mk}$ converges to a finite positive Borel measure and it follows from (4) that

$$\text{(5)} \qquad P[d\mu](z) \geq c \frac{\alpha_m \omega(1 - |z|)}{1 - |z|}, \quad z \in C(J_{mk}).$$

Let v be a harmonic conjugate of $P[d\mu]$. Then the function

$$g = \exp(-P[d\mu] - iv)$$

is contained in the class $\mathscr{B}$. Let m be a positive integer. By (5), we have

$$\text{(6)} \qquad |g(z)| \leq \exp\left[-c \frac{\alpha_m \omega(1 - |z|)}{1 - |z|}\right], \quad z \in C(J_{mk}),$$

for each positive integer k. Corresponding to each $\zeta \in E$, there is an index $k(m)$ such that $\zeta \in J_{m,k(m)}$ since $\{J_{mk}\}_{k=1}^{\infty}$ is a cover of E. Letting $I_m(\zeta) = J_{m,k(m)}$, the required conclusion for $\{I_m(\zeta)\}_1^{\infty}$ follows from (6). This completes the proof.

To prove Proposition 2 we need to establish some notation and a collection of preliminary lemmas.

If μ denotes linear measure on an arc J of C, then the bounded, positive harmonic function $P[d\mu](z)$, which we will denote by $\gamma(z; J)$, is the *harmonic measure of the arc J with respect to z*. The following lemma contains standard facts that can be found, for example, in [6; pp. 466–7].

LEMMA 2. *Suppose $0 < \beta - \alpha < 2\pi$ and $e^{i\alpha}, e^{i\beta}$ are the endpoints of the arc J.*

(1) *If θ is the angle with vertex z whose initial and terminal sides are the segments joining z to $e^{i\alpha}$ and $e^{i\beta}$ respectively, then*

$$\gamma(z; J) = \frac{\theta}{\pi} - \frac{\beta - \alpha}{2\pi}, \quad z \in \Delta.$$

(2) *The function γ is constant on any set S which is the intersection with Δ of a circle or line in the plane $\mathscr{C}$ that passes through $e^{i\alpha}$ and $e^{i\beta}$.*

Two immediate corollaries provide useful facts for our constructions.

COROLLARY 1. *If J is an open arc of C (with $|J| < \pi/2$), then*

$$\gamma(z;J) - \gamma(z;C \setminus J) = 1/2, \quad z \in C(J).$$

COROLLARY 2. *If $\{J^m\}$ is a sequence of mutually disjoint open arcs each of length less than $\pi/2$, and $\delta(m) = \pm 1$ for each m, then for each positive integer l we have*

$$\gamma(z;J^l) + \sum_{m \neq l} \delta(m)\gamma(z;J^m) \geq 1/2, \quad z \in C(J^l).$$

We proceed now to develop a system for placing "alternating harmonic measure" on certain subarcs of a given open arc J of C. Let $\alpha = |J|/2$ and $\theta_0 \in [0, 2\pi)$ such that $e^{i\theta_0}$ is the midpoint of the arc J. Corresponding to each positive integer n, put $\theta_n = \theta_{n-1} + \alpha 2^{-n}$ and $\theta_{-n} = \theta_0 - (\theta_n - \theta_0)$. Define J_n (J_{-n}) to be the open arc with endpoints $e^{i\theta_{n-1}}$ and $e^{i\theta_n}$ ($e^{i\theta_{-n+1}}$ and $e^{i\theta_{-m}}$), respectively. Thus the arcs J_n result from a successive bisection procedure rotating counterclockwise (clockwise) as n ($-n$) increases through the positive integers.

Let n and k be positive integers. We now divide the arcs J_n and J_{-n} into $2(n+k)^2$ subarcs of equal length as follows. Set

$$\theta_{nj} = \theta_{n-1} + \frac{j}{2(n+k)^2}\frac{\alpha}{2^n}$$

and $\theta_{-nj} = \theta_0 - (\theta_{nj} - \theta_0)$ for $j = 0, 1, \ldots, 2(n+k)^2$. Observe that

$$\theta_{n0} = \theta_{n-1}, \theta_{-n0} = \theta_{-n+1}, \text{and } \theta_{\pm n, 2(n+k)^2} = \theta_{\pm n}.$$

For each $j \in \{1, \ldots, 2(n+k)^2\}$, let $J_{\pm n, j}$ be the arc with endpoints $e^{i\theta_{\pm n,j}}$ and $e^{i\theta_{\pm n, j-1}}$. Define

$$h(z;J,k,\pm n) = \sum_{j=1}^{2(n+k)^2} (-1)^j \gamma(z;J_{\pm n,j})$$

and

$$h(z;J,k) = \sum_{n=1}^{\infty} [h(z;J,k,n) + h(z;J,k,-n)], \quad z \in \Delta.$$

Let $\Omega(J)$ denote the simply-connected Jordan subregion of Δ bounded by $C \setminus J$ and $C(J)$. The following lemma ensures that $|h(z;J,k)|$ can be made as small as we want in $\Omega(J)$ by taking k sufficiently large.

LEMMA 3. *For every open arc J of C having length less than $\pi/2$ and positive integer k, we have $h(z; J, k)$ is a harmonic function such that*

$$|h(z; J, k)| \le c/k, \quad z \in \Omega(J). \tag{7}$$

Proof. Clearly h is a harmonic function with $|h| \le 1$ that is continuously 0 at each point of $C \setminus \bar{J}$ (where $\bar{J}$ denotes the closure of J). By the Phragmén-Lindelöf maximum principle ([7; p. 76]), it suffices to prove that (7) holds at each point of $C(J)$ (and omit consideration of the endpoints ζ_1 and ζ_2 of the arc J).

We assume, without loss of generality, that the midpoint of J is 1 and $z \in C(J)$ has its principal argument $\operatorname{Arg} z > 0$. Observe first that for all allowed indices n and j, we have

$$\gamma(z; J_{nj}) \le c\frac{|J_{nj}|}{|J_n|} \le \frac{c}{(n+k)^2},$$

(where c is independent of all the allowed parameters). The first inequality depends on a trivial estimate of the Poisson integral and the fact that the distance of z from the arc J_{nj} is at least as large as the order of the length $|J_n|$.

We claim next that for each nonzero integer n, there is an index

$$l = l(n) \in \{1, \ldots, 2(|n| + k)^2\}$$

such that

$$|h(z; J, n, k)| \le \gamma(z; J_{nl}).$$

In fact, l is the index for which J_{nl} contains the point closest to z. The inequality is verified using elementary geometry to obtain monotonicity relationships on the Poisson kernel relative to the arcs J_{nj} of J_n, and the fact that the sum involved in the definition of $h(z; J, n, k)$ alternates.

Finally, inequality (7) is a consequence of the inequalities of the preceding two paragraphs on summing over nonzero integers n. This completes the proof.

We now define a harmonic function and an associated region relative to a sequence $\{J^m\}$ of mutually disjoint open arcs each of length less than $\pi/2$. Define the function

$$h(z; \{J^m\}, k) = \sum h(z; J^m, [k+m]^2)$$

along with the arcs J^m_{nj}, $n = \pm 1, \pm 2, \ldots,$ $j = 1, \ldots, 2[|n| + (k+m)^2]^2$

(defined relative to J^m as the J_{nj} are relative to J). Also, let

$$\Omega(\{J^m\}) = \bigcap_m \Omega(J^m).$$

The following proposition collects the facts concerning $h(z;\{J^m\},k)$ that will be needed in the proof of Theorem 2. We omit the proof since it is a straightforward consequence of Corollary 2, Lemma 3, and other observations made in this section. The indices m, n and j will be understood to belong to the appropriate index sets.

PROPOSITION 2. *Let k be a positive integer, $\{J^m\}$ a sequence of mutually disjoint open arcs of length less than $\pi/2$, $h(z) = h(z;\{J^m\},k)$ the function just defined, and $\Omega = \Omega(\{J^m\})$ the associated region. Then h is a harmonic function such that*

$$|h(z)| \le 1, \quad z \in \Delta, \tag{8}$$

$$|h(z)| \le c/k, \quad z \in \Omega, \tag{9}$$

$$(-1)^j h(z) > 1/2, \quad z \in C(J^m_{nj}), \tag{10}$$

$$\inf\{|z| : z \in \Delta \setminus \Omega\} > 1 - \max_m\{|J^m|\}, \tag{11}$$

and

$$\sum_{n,j} |J^m_{nj}| = |J^m|/2, \tag{12}$$

where the sum is taken over all indices n and j for which $h(z) > 1/2$, respectively $< -1/2$, for every $z \in C(J^m_{nj})$.

Note that the definition of the function h ensures that the constant c in (9) can be taken to be the same as the one in (7) of Lemma 3. Thus the constant is independent of the sequence $\{J^m\}$.

We conclude this section with a result, referred to in §1, which deals with arbitrarily slowly growing admissible functions ν.

PROPOSITION 3. *Let ν be an admissible growth function. Then there exists an admissible generating function ω such that $\nu_\omega(r) \le \nu(r)$ for $r \in [0,1)$.*

Proof. Let $\sigma(r) = \log \nu(1-r)$ for $r \in (0,1]$. We need to construct an increasing, continuous, concave-downward function $\omega : [0, 2\pi] \to [0,\infty)$ vanishing at 0 with $\omega'(0) = +\infty$, such that $\omega(r)/r \le \sigma(r)$ for $r \in (0,1]$.

Let $\{r_n\}_1^\infty$ be a decreasing sequence in $(0,1)$ such that $r_n < \sigma(1)/n^2$ and $\sigma(r_n) \ge n+1$ for each positive integer n. We shall inductively select two decreasing sequences $\{t_n\}_0^\infty$ and $\{\omega(t_n)\}_0^\infty$ such that the function ω whose graph consists of the union of the origin with the line segments S_n joining the points $(t_{n-1}, \omega(t_{n-1}))$ and $(t_n, \omega(t_n))$ for $n = 1, 2, \ldots,$ has the desired properties.

Let $t_0 = 2\pi$ and $r_1 = t_1 = \omega(t_0) = \omega(t_1) = y_0 = y_1$. Let L_1 be the horizontal line with y-value equal to r_1 and let S_1 be the segment joining the points $(t_0, \omega(t_0))$ and $(t_1, \omega(t_1))$. Also observe that $\omega(t_1)/t_1 = 1$.

Assume that n is a positive integer and we have selected positive, decreasing sequences $\{t_k\}_1^n$, $\{y_k\}_1^n$, and $\{\omega(t_k)\}_1^n$ such that the point $(0, y_k)$ and the segment S_k joining the points $(t_{k-1}, \omega(t_{k-1}))$ and $(t_k, \omega(t_k))$ lie on the same line L_k, and $\omega(t_k)/t_k = k$ for $k = 1, \ldots, n$.

Choose $t_{n+1} \in (0, \min\{t_n, r_{n+1}\})$ such that the point (t_{n+1}, y) on the line L_n has $y/t_{n+1} \geq n + 2$. Select $y_{n+1} \in (0, y_n)$ such that if L_{n+1} is the line passing through the points $(0, y_{n+1})$ and $(t_n, \omega(t_n))$, the point (t_{n+1}, w) on L_{n+1} has $w/t_{n+1} = n + 1$. Notice that this is possible since as y_{n+1} decreases from y_n to 0, the associated quantity w/t_{n+1} decreases from y/t_{n+1} to n. Now take $\omega(t_{n+1}) = w$ and let S_{n+1} be the segment of L_{n+1} joining the points $(t_n, \omega(t_n))$ and $(t_{n+1}, \omega(t_{n+1}))$. This completes the inductive step.

Let ω be the function whose graph is composed of the segments S_n and the origin as described earlier. Since $t_n < r_n < \sigma(1)/n^2$ and $nt_n = \omega(t_n)$ for each n, we see that $\{\omega(t_n)\}$ converges to 0. Because the point $(t_n, \omega(t_n))$ is on both of the lines L_n and L_{n+1}, and we have $(0, y_n) \in L_n$ and $(0, y_{n+1}) \in L_{n+1}$ with $y_{n+1} < y_n$ for each n, it follows that the function ω is an increasing, continuous function on $[0, 2\pi]$ with $\omega(0) = 0$.

By the construction, it is evident that the slopes of the segments S_n are increasing as n increases so that ω is concave downward. This implies that $\omega(r)/r$ is a decreasing function on $(0, 2\pi]$. Because $\omega(t_n)/t_n = n$ and $\sigma(t_n) \geq n + 1$ for each n with σ decreasing, we conclude that $\omega(r)/r \leq \sigma(r)$ for $0 < r < t_1$. To ensure that this inequality also holds for $t_1 \leq r \leq 1$, we can multiply ω by a sufficiently small positive constant. The proposition is thereby established.

3. Proofs of the theorems

We start with Theorem 1. Suppose that $f \in \mathscr{A}_\nu$ and $H_\omega(E) = 0$ as stated in the theorem. Then by assumption,

$$|f(z)| \leq \nu_\omega(|z|), \quad z \in \Delta. \tag{13}$$

Let g be a function as in Proposition 1 and put $G = fg$. Then G is a nonconstant analytic function with $|G| \leq |f|$ so that

$$G \in \mathscr{A}_\nu \subset \mathscr{A} \quad \text{and} \quad A(G, \infty) \cap J \subseteq E.$$

From Proposition 1 and inequality (13), we also see that $A(G, \infty) \cap E = \varnothing$. Hence $A(G, \infty) \cap J = \varnothing$ and we may apply Corollary B to conclude that $|F(G) \cap J| > 0$. Since $|F(g) \cap J| = |J|$ and $|F(g, 0)| = 0$, we conclude upon dividing G by g that $|F(f) \cap J| > 0$. This completes the proof of Theorem 1.

We note in passing that the growth hypothesis $f \in \mathscr{A}_\nu$ of Theorem 1 can be weakened to the requirement that the set S of points ζ in J where $\log|f(r\zeta)| = O[\omega(1-r)/(1-r)]$ as $r \to 1$ is of second category in C. In fact, S is the countable union of the relatively closed sets

$$F_n = \{\zeta \in J : \log|f(r\zeta)| \leq n\omega(1-r)/(1-r) \quad \text{for } r \in [0,1)\},$$

over all positive integer n. Hence for at least one n, the set F_n must contain a subarc I of J. Observing that (for Borel sets), H_ω-measure 0 is equivalent to $H_{n\omega}$-measure 0, we can apply a localized version of the theorem to the arc I with $\nu = \nu_{n\omega}$ and obtain the same conclusion.

We turn now to the proof of Theorem 2. Let $\sigma = \log \nu$. We shall construct a harmonic function u on Δ with $|u(z)| \leq \sigma(|z|)$ for all $z \in \Delta$ such that $|A(u)| = 0$. Then taking v to be a harmonic conjugate of u, the analytic function $f = \exp(u + iv)$ will have the required properties. Additionally, we obtain $M(1/f; r) \leq \nu(r)$. In the subsequent proof, we shall use the notation (sometimes analogously defined with additional subscripts or superscripts) and the results of Proposition 2 without specific reference.

Let $u_0(z) = 0$ for all $z \in \Delta$. Select $r_1 \in (1 - 2^{-1}, 1)$, a positive constant c_1, and a positive integer k_1 such that $4 < 2c_1 < \sigma(r_1)$ and $c_1 c/k_1 < 2^{-1}\sigma(0)$, where c is the fixed constant (independent of the positive integer k and the allowed sequence of arcs) of inequality (9). Let $\{J^{1m}\}$ be a mutually disjoint sequence of open arcs such that $1 - \max\{|J^{1m}|\} > r_1$ and $|C \setminus \cup J^{1m}| = 0$. Define

$$u_1(z) = c_1 h\big(z; \{J^{1m}\}, k_1\big), \quad z \in \Delta,$$

and

$$\left\{J_{nj}^{1m} : n = \pm 1, \pm 2, \ldots, \; j = 1, \ldots, 2\big[|n| + (m + k_1)^2\big]^2\right\}$$

the associated arcs (as preceding Proposition 2). Observe that

$$|C \setminus \cup J_{nj}^{1m}| = 0,$$

u_1 is bounded, $|u_1(z)| \leq \sigma(|z|)(1 - 2^{-1})$ for $z \in \Delta$ (as is seen on considering the cases $|z| \le r_1$ and $r_1 < |z| < 1$ separately), and the quantity

$$c'_{11} = \inf\left\{|u_1(z)| - 1 : z \in \bigcup_{m,n,j} C\left(J_{nj}^{1m}\right)\right\}$$

is positive. Note also that $\sum |J_{nj}^{1m}| = |J^{1m}|/2$ when the sum is taken over all of the arcs J_{nj}^{1m} for which $u(z) > 1$, respectively $u(z) < -1$, for every $z \in C(J_{nj}^{1m})$.

For our inductive hypothesis, we assume that p is a positive integer and that we have defined sequences of mutually disjoint open arcs $\{J^{qm}\}_{m=1}^{\infty}$ along with bounded harmonic functions u_q, numbers $r_q \in (1 - 2^{-q}, 1)$, and positive integers k_q for $q = 1, \dots, p$ having the following properties:

$$\bigcup_m J^{pm} \subseteq \bigcup_{m,n,j} J_{nj}^{p-1,m} \quad (p > 1), \tag{14}$$

$$|u_p(z) - u_{p-1}(z)| \le \sigma(0)2^{-p}, \quad |z| \le r_p, \tag{15}$$

$$|u_p(z)| \le \sigma(|z|)(1 - 2^{-p}), \quad z \in \Delta, \tag{16}$$

$$\left| C \setminus \bigcup_{m,n,j} J_{nj}^{pm} \right| = 0, \tag{17}$$

$$c'_{pq} = \inf\left\{|u_p(z)| - q : z \in \bigcup_{m,n,j} C\left(J_{nj}^{qm}\right)\right\} > 0, \quad 1 \le q \le p, \tag{18}$$

and

$$\sum |J_{n'j'}^{pm'}| = |J_{nj}^{p-1,m}|/2 \quad (p > 1), \tag{19}$$

where the sum is taken over all indices m', n', and j' for which the arc $J_{n'j'}^{pm'}$ is contained in $J_{nj}^{p-1,m}$ and $u_p > p$, respectively $u_p < -p$, on $C(J_{n'j'}^{pm'})$.

Choose $r_{p+1} \in (1 - 2^{-p-1}, 1)$, the constant c_{p+1}, and the positive integer k_{p+1} such that

$$c_{p+1} > 2\left[(p + 1) + \sup\{|u_p(z)|\}\right],$$

$$\sigma(r_{p+1}) > 2\left[c_{p+1} + \sup\{|u_p(z)|\}\right],$$

and

$$c_{p+1}c/k_{p+1} < \min\left\{2^{-p-1}\sigma(0), c'_{pq}\ (1 \le q \le p)\right\}.$$

Let $\{J^{p+1,m}\}$ be a sequence of mutually disjoint open arcs such that

$$\bigcup_m J^{p+1,m} \subseteq \bigcup_{m,n,j} J_{nj}^{pm}, \quad \left| C \setminus \bigcup_m J^{p+1,m} \right| = 0$$

and

$$r_{p+1} < 1 - \max\{|J^{p+1,m}|\},$$

and define

$$u_{p+1}(z) = c_{p+1} h\left(z; \{J^{p+1,m}\}, k_{p+1}\right) + u_p(z), \quad z \in \Delta,$$

along with the associated arcs $\{J_{nj}^{p+1,m}\}$.

It is straightforward to check that the inductive hypothesis holds for $p = 1$ and with p replaced by $p + 1$, so the induction is complete. Therefore, there exist sequences $\{J^{pm}\}$, $\{J_{nj}^{pm}\}$, and $\{u_p\}$ so that the inductive hypothesis holds for all p. By (15) and (16) we conclude that $\{u_p\}$ converges, uniformly on compact subsets of Δ, to a harmonic function u such that $|u(z)| \leq \sigma(|z|)$ for all $z \in \Delta$. From (18) we see that for each positive integer p,

$$|u(z)| \geq p \quad \text{for all } z \in \bigcup_{m,n,j} C\left(J_{nj}^{pm}\right).$$

Also by (14) and (17), the set $E = \bigcap_p \bigcup_{m,n,j} J_{nj}^{pm}$ has $|E| = 2\pi$.

Suppose now that $\zeta \in E$. For each positive integer p, there is precisely one arc J_{nj}^{pm}, call it I^p, that contains ζ. Let $\mathscr{P}(\zeta)$ $(N(\zeta))$ be the set of all positive integers p such that $\zeta \in I^p$ and $u(z) \geq p$ $(u(z) \leq -p)$ on $C(I^p)$, respectively. If $\Gamma(t)$ is any boundary path ending at ζ, then

$$\limsup_{t \to 1} u \circ \Gamma(t) = +\infty \quad \left(\liminf_{t \to 1} u \circ \Gamma(t) = -\infty\right) \tag{20}$$

whenever $\mathscr{P}(\zeta)$ $(\mathscr{N}(\zeta))$ is an infinite set, respectively.

Consider the set

$$W^+ = \{\zeta \in E : \mathscr{P}(\zeta) \text{ is finite}\}.$$

We claim that $|W^+| = 0$. Now $W^+ = \bigcup W_n^+$, where for each positive integer n, we define

$$W_n^+ = \{\zeta \in E : \mathscr{P}(\zeta) \subseteq \{1, \ldots, n\}\}.$$

But $|W_n^+| = 0$ for each n by (19), and the claim follows since the union is over a countable collection of sets. A similar argument shows that

$$W^{-} = \{\zeta \in E : \mathcal{N}(\zeta) \text{ is finite}\}$$

has $|W^{-}| = 0$. Hence $|W| = 0$ where $W = W^{+} \cup W^{-}$, and $|E \setminus W| = 2\pi$.

For each $\zeta \in E \setminus W$ and every boundary path Γ ending at ζ, both conditions of (20) hold, so we have $\zeta \notin A(u)$. We conclude that $|A(u)| = 0$, and the theorem follows as indicated before.

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