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# GENERALIZATION OF MYERS' THEOREM ON A CONTACT MANIFOLD

BY

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## 1. Introduction

In 1941, Myers [4] proved that a complete Riemannian manifold for which $\operatorname{Ric} \geq \delta > 0$, is compact. In 1981, Hasegawa and Seino [3] generalized Myers' theorem for a Sasakian manifold by proving that a complete Sasakian (normal contact metric) manifold for which $\operatorname{Ric} \geq -\delta > -2$, is compact. Actually their proof uses only that the structure is $K$-contact and not the full strength of the Sasakian condition. A *K-contact structure* is a contact metric structure such that the characteristic vector field of the contact structure is Killing.

Now a contact metric structure is $K$-contact if and only if all sectional curvatures of plane sections containing the characteristic vector field are equal to 1 (see e.g. [1], p. 65) and hence there is a lot of positive curvature involved in the problem from the outset. The question then arises for a general contact metric structure: Can we relax the condition that the sectional curvature $K(\xi, X)$ of any plane section containing the characteristic vector field $\xi$ be equal to 1; even if we must increase $-\delta$ from near $-2$ to near 0 to compensate? In general, the notion of a contact metric structure is quite weak; in fact, the set of all such structures associated to a given contact structure is infinite dimensional. So we seemingly must assume some condition generalizing the $K$-contact structure, then we can study $K(X, \xi) \geq \varepsilon > \delta' \geq 0$ and $\operatorname{Ric} \geq -\delta > -2$ where $\delta'$ is a function of $\delta$.

Let $M$ denote a $(2n + 1)$-dimensional contact metric manifold with structure tensors $(\varphi, \xi, \eta, g)$; i.e., $\eta$ is a globally defined contact form

$$\left(\eta \wedge (d\eta)^n \neq 0\right),$$

$\xi$ its characteristic vector field $(d\eta(\xi, X) = 0,\ \eta(\xi) = 1)$, $g$ a Riemannian metric, and $\varphi$ a skew-symmetric field of endomorphisms satisfying

$$\varphi^2 = -I + \eta \otimes \xi, \qquad \eta(X) = g(X, \xi), \qquad (d\eta)(X, Y) = g(X, \varphi Y).$$

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Following [1] we denote the operator $\frac{1}{2}\mathscr{L}_\xi\varphi$ by $h$ where $\mathscr{L}$ denotes Lie differentiation. It is well known [1] that $M$ is $K$-contact (i.e., $\xi$ is Killing) if and only if $h = 0$. We also define the strain tensor $\tau$ of $M$ along $\xi$ by

$$g(\tau X, Y) = (\mathscr{L}_\xi g)(X, Y).$$

Then using the relation ([1], p. 66)

$$\nabla_X \xi = -\varphi X - \varphi h X, \tag{1.1}$$

one obtains $\tau = 2h\varphi$. As a generalization of the $K$-contact condition we suppose that $\operatorname{div}\tau = \sigma\eta$. Recall that for a contact metric structure

$$\nabla_t \varphi^t_j = -2n\eta_j, \tag{1.2}$$

$$R^j_r \xi^r = \nabla^r \nabla_r \xi^j + 4n\xi^j. \tag{1.3}$$

The first identity can be found in Olszak [5] and the second one in Tanno [7]. Thus, using equations (1.1) and (1.2) in (1.3) we obtain

$$R^j_r \xi^r = \nabla_r\left(h^r_m \varphi^{mj}\right) + 2n\xi^j$$

and hence if $\operatorname{div}\tau = \sigma\eta$, $\xi$ is an eigenvector of the Ricci operator.

We also note the following example. Consider $\mathbf{R}^3$ with the contact structure

$$\eta = \frac{1}{2}(\cos x^3\, dx^1 + \sin x^3\, dx^2)$$

and the associated metric $g_{ij} = \frac{1}{4}\delta_{ij}$. Since $\eta$ is invariant by the translations in the coordinate directions by $2\pi$, the torus $T^3$ is a compact manifold also carrying this structure. For this contact metric structure, $\operatorname{div}\tau = 2\eta$. Thus there are both compact and non-compact contact metric manifolds satisfying $\operatorname{div}\tau = \sigma\eta$.

We present two theorems generalizing Myers' theorem for contact metric manifolds as follows:

THEOREM 1. *Let $M$ be a $(2n+1)$-dimensional complete contact metric manifold with $\operatorname{div}\tau = \sigma\eta$. If $\operatorname{Ric} \geq -\delta > -2$ and the sectional curvatures of plane sections containing $\xi$ are $\geq \varepsilon > \delta' \geq 0$ where*

$$\delta' = 2\sqrt{n(\delta - 2\sqrt{2\delta} + n + 2)} - (\delta - 2\sqrt{2\delta} + 1 + 2n)$$

*then $M$ is compact.*

In dimension 3 we can obtain a better estimate for $\delta'$ and have the following result.

THEOREM 2. *Let $M$ be a 3-dimensional complete contact metric manifold with* $\operatorname{div}\tau = \sigma\eta$. *If* $\operatorname{Ric} \geq -\delta > -2$ *and the sectional curvatures of plane sections containing* $\xi$ *are* $\geq \varepsilon > \delta' \geq 0$ *where*

$$\delta' = -\frac{\delta^2}{4} + \sqrt{2}\,\delta^{3/2} - 3\delta + 2\sqrt{2}\,\delta^{1/2},$$

*then $M$ is compact.*

Before turning to the proofs let us review a few properties of the tensor field $h$ on a contact metric manifold $M$:

(1) $h$ is a symmetric and trace-free field of endomorphisms such that

$$h\varphi + \varphi h = 0 \quad \text{and} \quad h\xi = 0.$$

If $\lambda$ is an eigenvalue of $h$, so is $-\lambda$ and hence in dimension $3(n = 1)$ the eigenvalues of $h$ are $0, \lambda, -\lambda$ and we adopt the convention that $\lambda$ will denote the non-negative eigenvalue.

(2) $\operatorname{Ric}(\xi) = 2n - \operatorname{tr} h^2$ (see [1], p. 67), and therefore in dimension 3 (i.e., $n = 1$) we have

$$\operatorname{Ric}(\xi) = 2(1 - \lambda^2)$$

(3) In dimension 3, $h^2$ acts on the contact subbundle $\{\eta = 0\}$ as $\lambda^2 I$, thus for a unit vector $X$ orthogonal to $\xi$,

$$|hX|^2 = \lambda^2.$$

Also for any such $X$ we have

$$g(hX, X) \leq \lambda,$$

where, again by our convention, $\lambda$ denotes the non-negative eigenvalue of $h$.

The main idea of the proofs of our theorems is to use a $D$-homothetic deformation of the structure. This technique was introduced by Tanno [6] and used by Goldberg and Toth [2] as well as by Hasegawa and Seino [3]. Given a contact metric structure $(\varphi, \xi, \eta, g)$ let

$$\bar{\eta} = \alpha\eta, \qquad \bar{\xi} = \frac{1}{\alpha}\xi, \qquad \bar{\varphi} = \varphi \quad \text{and} \quad \bar{g} = \alpha g + \alpha(\alpha - 1)\eta \otimes \eta$$

for some positive constant $\alpha$. Then $(\bar{\varphi}, \bar{\xi}, \bar{\eta}, \bar{g})$ is again a contact metric structure. Such a change of structure is called a *D-homothetic deformation*. A $D$-homothetic deformation preserves many basic properties like being $K$-contact or Sasakian, but most notably completeness for our purpose [3], [6]. Computing the Ricci tensor $\bar{R}_{jk}$ of $\bar{g}_{jk}$ on a contact metric manifold $M$ we have

$$\bar{R}_{jk} = R_{jk} - \frac{(2\alpha-1)(\alpha-1)}{\alpha} g_{jk} + \left[2n(\alpha^2-1) + \frac{(2\alpha-1)(\alpha-1)}{\alpha}\right]\eta_j\eta_k + \frac{\alpha-1}{\alpha}\left(2h_{jk} - h_{jm}h^m_k - R_{jlik}\xi^l\xi^i\right)$$

## 2. Proof of Theorem 1

We now turn to the proof of Theorem 1, which is to seek a number $\alpha$ $(0 < \alpha < 1)$ such that for the structure obtained by the $D$-homothetic deformation, the new associated metric tensor has its Ricci curvature bounded below by a positive constant and hence by Myers' theorem $M$ must be compact.

Let $(\varphi, \xi, \eta, g)$ denote the contact metric structure satisfying the hypothesis. Let $X$ be a unit vector and decompose $X$ into the form $aX_D + b\xi$ where $X_D$ is a unit vector orthogonal to $\xi$ and, of course, $a^2 + b^2 = 1$. We now expand $\mathrm{Ric}(X)$ in terms of this decomposition. Since $\operatorname{div}\tau = \sigma\eta$, $\xi$ is an eigenvector of the Ricci operator as noted in section 1 and hence the coefficient of $ab$ is 0.

From property (2), we have

$$\mathrm{Ric}(\xi) = 2\left(n - \sum_{i=1}^{n}\lambda_i^2\right) \geq 2n\varepsilon \tag{2.1}$$

where the $\lambda_i$'s are the non-negative eigenvalues of $h$, and hence for the coefficient of $b^2$,

$$\mathrm{Ric}(\xi) + 2n(\alpha^2 - 1) \geq 2n\left(\alpha^2 - (1-\varepsilon)\right).$$

Thus, one of the requirements on the number $\alpha$ that we seek is

$$\alpha > \sqrt{1-\varepsilon}\,.$$

It follows immediately from (2.1) that $\sum_{i=1}^{n}\lambda_i^2 \leq n(1-\varepsilon)$ and hence

$$\lambda_i \leq \sqrt{n}\sqrt{1-\varepsilon} < \sqrt{n}\,\alpha. \tag{2.2}$$

Finally, the coefficient of $a^2$:

$$\begin{aligned}\operatorname{Ric}(X_D) &+ 2 - 2\alpha + \frac{1-\alpha}{\alpha}\left[-1 - 2g(hX_D, X_D) + |hX_D|^2 + K(\xi, X_D)\right]\\ &\geq -\delta + 2 - 2\alpha + \frac{1-\alpha}{\alpha}(-1 - 2\lambda + \varepsilon)\end{aligned}$$

where $\lambda$ stands for the maximum of the $\lambda_i$'s over $i = 1, 2, \ldots, n$. As per our requirements we must seek $\alpha$ such that

$$\frac{\alpha}{1-\alpha}(2 - \delta - 2\alpha) + \delta' - 2\lambda - 1 > 0$$

where $\delta' < \varepsilon$. Hence

$$\lambda < \frac{1}{2}\left(\delta' - 1 + \frac{\alpha}{1-\alpha}(2 - \delta - 2\alpha)\right). \tag{2.3}$$

Consider the following curve in the $xy$-plane, thinking of $x$ as corresponding to $\alpha$ and $y$ to $\lambda$:

$$y = \frac{1}{2}\left(\delta' - 1 + \frac{x}{1-x}(2 - \delta - 2x)\right)$$

for $0 < x < 1$. $y$ has a positive maximum at $x = 1 - \sqrt{\delta/2}$, viz.,

$$y\left(1 - \sqrt{\delta/2}\right) = \frac{1}{2}(1 - 2\sqrt{2\delta} + \delta + \delta').$$

Thus, since $\lambda < \sqrt{n}\sqrt{1-\varepsilon} < \sqrt{n}\sqrt{1-\delta'}$ (from (2.2)), $\alpha = 1 - \sqrt{\delta/2}$ gives (2.3) if

$$\sqrt{n}\sqrt{1-\delta'} = \frac{1}{2}\left(\delta' - 1 + \frac{\alpha}{1-\alpha}(2 - \delta - 2\alpha)\right),$$

that is, if

$$\delta' = 2\sqrt{n(\delta - 2\sqrt{2\delta} + n + 2)} - (\delta - 2\sqrt{2\delta} + 1 + 2n)$$

for then

$$\frac{1}{2}(1 - 2\sqrt{2\delta} + \delta + \delta') < 1 - \sqrt{\delta/2}.$$

This completes the proof of Theorem 1.

## 3. Proof of Theorem 2

Here $n = 1$ and $h$ has only 3 eigenvalues 0, $\lambda$ and $-\lambda$ each of multiplicity 1. For the coefficient of $b^2$, we have

$$\alpha > \sqrt{1-\varepsilon} \tag{3.1}$$

as before.

For the coefficient of $a^2$ we have by property (3) that

$$\mathrm{Ric}(X_D) + 2 - 2\alpha + \frac{1-\alpha}{\alpha}\left[-1 - 2g(hX_D, X_D) + |hX_D|^2 + K(\xi, X_D)\right]$$
$$\geq -\delta + 2 - 2\alpha + \frac{1-\alpha}{\alpha}(-1 - 2\lambda + \lambda^2 + \varepsilon)$$

and we seek $\alpha$ ($0 < \alpha < 1$) such that, in addition to (3.1), we have this last expression bounded below by a positive constant. In particular, we study the inequality

$$\frac{\alpha}{1-\alpha}(2 - \delta - 2\alpha) + \delta' - 2 + (\lambda - 1)^2 > 0$$

where $\delta' < \varepsilon$. Solving for $\lambda$, the non-negative eigenvalue of $h$, we observe

$$\lambda < 1 - \sqrt{2 - \delta' - \frac{\alpha}{1-\alpha}(2 - \delta - 2\alpha)}\,. \tag{3.2}$$

Now consider the following curve in the $xy$-plane taking $x$ as $\alpha$ and $y$ as $\lambda$:

$$y = 1 - \sqrt{2 - \delta' - \frac{x}{1-x}(2 - \delta - 2x)}$$

on $0 < x < 1$. $y$ has a positive maximum at $x = 1 - \sqrt{\delta/2}$, viz.,

$$y\left(1 - \sqrt{\frac{\delta}{2}}\right) = 1 - \sqrt{2\sqrt{2\delta} - \delta' - \delta}\,.$$

Thus, since $1 - \lambda^2 \geq \varepsilon$ so that $\lambda \leq \sqrt{1-\varepsilon} < \sqrt{1-\delta'}$, $\alpha = 1 - \sqrt{\delta/2}$ gives (3.2) if

$$\sqrt{1-\delta'} = 1 - \sqrt{2\sqrt{2\delta} - \delta' - \delta}\,,$$

that is, if

$$\delta' = -\frac{\delta^2}{4} + \sqrt{2}\,\delta^{3/2} - 3\delta + 2\sqrt{2}\,\delta^{1/2},$$

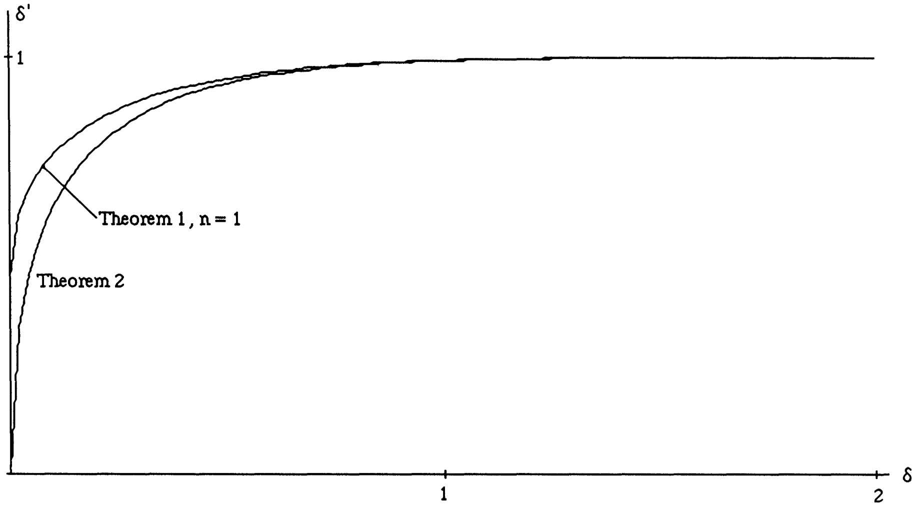


FIG. 1

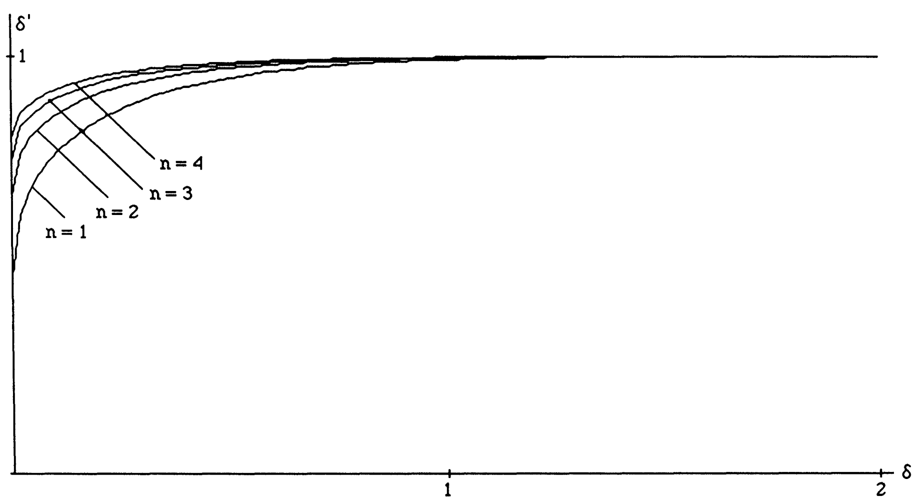


FIG. 2

for then

$$1 - \sqrt{2\sqrt{2\delta} - \delta' - \delta} < 1 - \sqrt{\delta/2}\,.$$

This completes the proof of Theorem 2.

*Remark.* In Theorem 2 ($n = 1$) we observe that when $\delta = 0$, $\delta' = 0$ and when $\delta = 2$, $\delta' = 1$. In Theorem 1 ($n \geq 1$) we observe that when $\delta = 0$, $\delta' = 2\sqrt{n(n+2)} - (2n+1)$ which approaches 1 as $n$ tends to $\infty$, and when $\delta = 2$, $\delta' = 1$. Thus we have a better estimate for $\delta'$ in Theorem 2 (for $n = 1$) than that provided by Theorem 1 (for $n = 1$). Figure 1 below shows the graph of $\delta'$ in Theorem 1 for $n = 1$ and the graph of $\delta'$ in Theorem 2. Figure 2 shows the graphs of $\delta'$ in Theorem 1 for $n = 1, 2, 3, 4$.

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