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TRANSFER OF INFORMATION ABOUT βN − N VIA OPEN REMAINDER MAPS[1]

BY

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0. Conventions

All spaces are completely regular, and Hausdorff of course. We use X^* to denote $\beta X - X$, and **N**, **Q** and **R** to denote the nonnegative integers, the rationals and the reals.

A *map* is a continuous function. The *Stone extension* of a map f: $X \to Y$ is the function $\beta X \to \beta Y$ which extends f; it will be denoted by βf. We use f^*, the *remainder map*, to denote the restriction $\beta f \upharpoonright X^*$. Recall from [G1] that f^* maps X^* into Y^* if (and only if) f is perfect ($\equiv$ closed + compact fibers); hence f^* maps X^* onto Y^* if f is a perfect map from X onto Y.

The closure operators in X, βX and X^* are denoted by cl, Cl and Cl*. We use a similar convention for the interior operators int, Int and Int*.

We remind the reader that a space X is *realcompact* if for each $x \in X^*$ there is a G_δ-subset G of βX with $x \in G \subseteq X^*$. (This is equivalent to the original definition). Clearly Lindelöf spaces are realcompact.

1. Introduction

$\mathbf{N}^*$ is one of the most intensely studied spaces; so it is worthwhile to have tools available to transfer information about $\mathbf{N}^*$ to information about other Čech-Stone remainders. Two such tools, which are available already, are:

T_1. *C-embedded copies of* **N**. Assume **N** can be embedded in X as a C-embedded subspace (this happens iff X is nonpseudocompact). Then **N** is closed in X, and $\mathrm{Cl}\,\mathbf{N} = \beta\mathbf{N}$, so $X^* \cap \mathrm{Cl}\,\mathbf{N} = \mathbf{N}^*$. The fact that **N** is C-embedded in X gives information about the way $\mathbf{N}^*$ fits inside X^*; cf. [R, 4.5(d)], [I], [GJ, 9M], [F]. (An example in [vD$_2$, 3] shows that it is not sufficient to know that **N** is closed and C*-embedded in X.) Rudin's proof of the implication (a) ⇒ (b) in Theorem 4.1 is an early example.

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[1]This paper was originally accepted subject to revision in 1978. The revised manuscript was found in the author's files after his death on July 28, 1987. His obituary appeared in Topology and its Applications, vol. 31, number 1.

T_2. *Generalizations of proofs*. A proof that $\mathbf{N}^*$ has a certain property often can be generalized so as to show that X^* has that property for noncompact locally compact X which is σ-compact or at least real-compact. The result of Fine and Gillman, quoted as Corollary 6.3, is an early example. Most of §6 is an example of this technique.

These two tools pass the information directly from $\mathbf{N}^*$ to other remainders. The purpose of this paper is to give an essentially different tool, in which the information is passed indirectly from $\mathbf{N}^*$ to other remainders since most of the time it "really" goes the other way. (Theorem 9.3 is an exception.)

1.1. OPEN RETRACTION LEMMA. *Let X be a noncompact locally compact space.*

(A) *If X is σ-compact, then X^* has regularly closed subspaces F_0 and F_1, and there are homeomorphs H_0 and H_1 of $\mathbf{N}^*$ in X^*, such that:*
  (1) $H_i \subseteq \operatorname{Int}^* F_i$ *for* $i = 0, 1$;
  (2) *there is an open retraction* $F_i \to H_i$ *for* $i = 0, 1$; *and*
  (3) $\operatorname{Int}^* F_0 \cup \operatorname{Int}^* F_1 = X^*$.

(B) *If X is realcompact, then for each $x \in X^*$ there are a regularly closed F in X^* and a homeomorph H of $\mathbf{N}^*$ in X^* such that:*
  (1) $H \subseteq \operatorname{Int}^* F$;
  (2) *there is an open retraction* r: $H \to F$; *and*
  (3) $x \in F$.

(*Given any neighborhood U of x in X^* one may require $F \subseteq U$.*)

(C) *If X is not pseudocompact, then there are a regularly closed F in X and a homeomorph H of $\mathbf{N}^*$ in X^* such that* (1) *and* (2) *of* (B) *hold.*

For nonlocally compact spaces we also have information; the following is an obvious corollary to the proof of 1.1.

1.2. OPEN MAPPING LEMMA. *If X is noncompact and realcompact, then for each $x \in X^*$ there is a compact subspace of X^* which contains x and which admits an open map onto* $\mathbf{N}^*$. □

We use these two lemmas to obtain results which on the one hand are simple, if not trivial corollaries, but on the other hand give significant new information about Čech-Stone remainders not otherwise available. Most of our results use the fact that we give open maps onto $\mathbf{N}^*$. (In some cases it is useful to remember that a map f: $X \to Y$ is open iff $\overline{f^{\leftarrow} B} = f^{\leftarrow} \overline{B}$ for each $B \subseteq Y$ [E, 1.4C].) Only one result uses the fact that we give retractions (see Theorem 4.3) and none uses the fact that we give open retractions. However, all our results use the fact that we have open maps onto $\mathbf{N}^*$ in an essential way. In this context we record one simple result which only uses the fact that we give a map onto $\mathbf{N}^*$:

$(*)$ *If X is nonpseudocompact and locally compact, then X^* has a pairwise disjoint open family of cardinality* $\mathfrak{c}$.

Indeed, some open subspace can be mapped onto $\mathbf{N}^*$ by 1.1C, and it is known that $(*)$ holds for $X = \mathbf{N}$. ($(*)$ was first proved in [CG, 3.2], as an application of T_2.)

We conclude this introduction by mentioning the analogue of the Open Retraction Lemma for Čech-Stone compactifications, which is a trivial corollary to the proof of 1.1A.

1.3. LEMMA. *If X is not pseudocompact, then βX has closed subspaces G_0 and G_1, and there are homeomorphs B_0 and B_1 of βN in βX, such that:*

(1) $B_i \subseteq \operatorname{Int} G_i$ *for* $i = 0, 1$;

(2) *there is an open retraction* $G_i \to B_i$ *for* $i = 0, 1$; *and*

(3) $\operatorname{Int} G_0 \cup \operatorname{Int} G_1 = \beta X$.

2. A Lemma

2.1. LEMMA. *If $f\colon X \to \mathbf{N}$ is a perfect surjection, then $f^*\colon X^* \to \mathbf{N}^*$ is an open surjection.*

This is an immediate consequence of the implication (a) $\Rightarrow$ (b) of our next result, which is of some independent interest since it is an antipode (but not the dual) of the theorem, essentially due to Gleason, [G1] that a space is projective for perfect maps iff it is extremally disconnected.

2.2. THEOREM. *The following conditions on a space E are equivalent:*

(a) *E is extremally disconnected*;

(b) *if X is any space and $f\colon X \to E$ is any open map, then $\beta f\colon \beta X \to \beta E$ is open*; *and*

(c) *if X is any space and $f\colon X \to E$ is any open surjection, then $\beta f\colon \beta X \to \beta E$ is open.*

Proof. (a) $\to$ (b). Let U be open in βX, and let $x \in U$ be arbitrary. There is an open V in βX with $x \in V$ and $\operatorname{Cl} V \subseteq U$. Since βf is closed, and since $\operatorname{Cl} V = \operatorname{Cl}(X \cap V)$, we have

$$\beta f(x) \in \operatorname{Cl} f^{\rightarrow}(X \cap V) \quad \text{and} \quad \operatorname{Cl} f^{\rightarrow}(X \cap V) \subseteq \beta f^{\rightarrow} \operatorname{Cl} V \subseteq \beta f^{\rightarrow} U$$

Now E is extremally disconnected, and $f^{\rightarrow}(X \cap V)$ is open in E, hence

$$\operatorname{Cl} f^{\rightarrow}(X \cap V) = \operatorname{Cl} \operatorname{cl} f^{\rightarrow}(X \cap V)$$

is open in βE. Consequently $\beta f^{\rightarrow} U$ is a neighborhood of $\beta f(x)$. Since $x \in U$ was arbitrary, it follows that $\beta f^{\rightarrow} U$ is open in βE.

(b) → (c). Trivial.

(c) → (a). Let U be open in E, and let X be the subspace $E \times \{0\} \cup U \times \{1\}$ of $E \times \{0, 1\}$, and let $f\colon X \to E$ be the (restriction of the) projection. Then f is open, hence $\beta f\colon \beta X \to \beta E$ is open. Since βf also is closed, it follows that U has open closure in βE. Then U also has open closure in E. (The idea of the proof of (c) → (a) is due to Gleason, [G, 1.2].) □

3. Proof of the open retraction and mapping lemmas

We begin with proving an easy special case of 1.1A.

3.1. LEMMA. *Let X be a noncompact σ-compact locally compact zero-dimensional space. Then there is a homeomorph H of* $\mathbf{N}^*$ *in X^* such that X^* admits an open retraction onto H.*

(There is an analogous version of 1.3.)

Proof. The conditions on X imply that there is a sequence $\langle X_n \rangle_n$ of compact open subsets of X such that

$$X_0 = \varnothing, \qquad X_n \subset X_{n+1} \text{ for } n, \quad \text{and} \quad X = \bigcup_n X_n.$$

We may assume that $n \in X_{n+1} - X_n$ for each n. Then

$$r = \bigcup_n (X_{n+1} - X_n) \times \{n\}$$

is a retraction from X onto $\mathbf{N}$ which is perfect. It easily follows that $r^*\colon X^* \to \mathbf{N}^*$ is a retraction. This retraction is open by Lemma 2.1. □

The proof of Lemma 1.1 is very similar, but is made more complicated since we have less information about X. We will prove the following extended version of part (A):

3.2. LEMMA. *Let X be a (noncompact) σ-compact locally compact space, and let U be a nonempty open subset of X^*. Then for $i = 0, 1$ there are a regularly closed subspace F_i of X^* and a homeomorph H_i of* $\mathbf{N}^*$ *such that*:

(1) $H_i \subseteq U \cap \operatorname{Int} F_i$ *for* $i = 0, 1$;

(2) F_i *admits an open retraction* r_i *onto* H_i *for* $i = 0, 1$; *and*

(3) $\operatorname{Int} F_0 \cup \operatorname{Int} F_1 = X$.

For the proof that this implies parts (B) and (C) of the Open Retraction Lemma; we need the following lemma, the proof of which is implicit in the proof of [FG_1, 3.1], as pointed out in [vD_4, 20.3].

3.3. LEMMA. *If X is locally compact, and if G is a closed G_δ-subset of βX with $G \subseteq X^*$, then* $G = \operatorname{Cl}\operatorname{Int}^* G$.

3.4. *Proof of (B) and (C) of Lemma* 1.1. First, let Y be noncompact, locally compact and realcompact, and let $y \in Y^*$. Since Y is realcompact there is a closed G_δ-subset G of βY with $y \in G \subseteq Y^*$. Let $U = \operatorname{Int}^* G$, (interior in Y^*), let $X = \beta Y - G$. As $Y \subseteq X \subseteq \beta Y$ we have $\beta X = \beta Y$, hence $X^* = G$. Clearly U is open in X^*, and $U \neq \varnothing$ by Lemma 3.3. Let F_i, H_i, and r_i, for $i = 0, 1$, be as in Lemma 3.2. We may assume that $y \in \operatorname{Int}^* F_0$ (interior in X^*). Clearly $U \cap \operatorname{Int}^* F_0$ is open in Y^*. It follows that $r_0^{\rightarrow}(\operatorname{Int}^* F_0) = H_0$ (interior in Y^*), and, since $y \in \operatorname{Cl} U$ by Lemma 3.3, also that $y \in \operatorname{Cl}\operatorname{Int}^* F_0$ (interior in Y^*).

This proves (B). The proof of (C) is entirely similar since a space Y is nonpseudocompact (if and) only if some nonempty closed G_δ-subset of βY is included in Y^*. □

3.5. *Proof of Lemma* 3.2. We proceed in four steps. Our first step would be trivial if $U = X^*$.

Step 1. *We embed* $\mathbf{N}$ *as a closed subspace in X so that* $\mathbf{N}^* \subseteq U$.

Since X^* is a G_δ in βX, so is U. We therefore can find a sequence $\langle G_n \rangle_n$ of nonempty open sets in βX such that $\bigcap_n G_n \subseteq U$ and $\operatorname{Cl} G_{n+1} \subseteq G_n$ for each n. We may assume that $\mathbf{N}$ is a subset of X with the property that $n \in G_n$ for each n. Then each cluster point of $\mathbf{N}$ must be in $\bigcap_n \operatorname{Cl} G_n = \bigcap_n G_n \subseteq U$.

Now, let $\mathbf{N}_0$ and $\mathbf{N}_1$ be the even and the odd integers.

Step 2. *For $i = 0, 1$ we find closed subspaces P_i and Q_i of X and a perfect retraction ρ_i: $Q_i \to \mathbf{N}_i$ such that*:

(1) $P_0 \cup P_1 = X$; *and*

(2) $P_i \subseteq \operatorname{int} Q_i$ *for* $i = 0, 1$.

Since X is σ-compact and locally compact, and since $\mathbf{N}$ is closed in X, we can find a sequence $\langle X_n \rangle_n$ of compact subsets of X such that $X_0 = X_1 = \emptyset$; and

$$X_n \subseteq \operatorname{int} X_{n+1} \quad \text{and} \quad n \in X_{3n+4} - X_{3n+3}, \quad \text{for } n \in \mathbf{N}.$$

For $i = 0, 1$ define

$$P_{i,n} = X_{6n+3i+4} - \operatorname{int} X_{6n+3i+1} \quad \text{for } n \in \mathbf{N}; \quad \text{and} \quad P_i = \bigcup_n P_{i,n};$$

and

$$Q_{i,n} = X_{6n+3i+5} - \operatorname{int} X_{6n+3i} \quad \text{for } n \in \mathbf{N}; \quad \text{and} \quad Q_i = \bigcup_n Q_{i,n};$$

and

$$\rho_i = \bigcup_n Q_{i,n} \times \{2n + i\}.$$

We omit the straightforward verification that this works.

Step 3. *We complete the proof, almost.*

For $i = 0, 1$ we have $\beta F_i = \operatorname{Cl} F_i$, since X is normal, hence we can define

$$F_i = Q_i^*, \qquad r_i = \rho_i^*, \quad \text{and} \quad H_i = \mathbf{N}_i^*.$$

The only thing to be checked is that $\operatorname{Int}^* F_0 \cup \operatorname{Int}^* F_1 = X^*$. Since $P_0^* \cup P_1^* = X^*$, by (1), it suffices to prove that $P_i^* \subseteq \operatorname{Int}^* Q^*$ for $i = 0, 1$, to which end we show that $\operatorname{Cl} P_i \subseteq \operatorname{Int} \operatorname{Cl} Q_i$ for $i = 0, 1$: P_i and $X - Q_i$ have disjoint closures in X, by (2), hence in βX since X is normal; so that

$$\operatorname{Cl} P_i \subseteq \beta X - \operatorname{Cl}(X - Q_i).$$

But $\beta X - \operatorname{Cl}(X - Q_i) \subseteq \operatorname{Int} \operatorname{Cl} Q_i$ since $\operatorname{Cl}(X - Q_i) \cup \operatorname{Cl} Q_i = \beta X$.

Step 4. *We take care of the F_i's being regularly closed.*

Our construction does not necessarily produce regularly closed (in X^*) F_i's, but this is easily taken care of. Since $Q_{i,n} \subseteq \operatorname{int} P_{i,n}$, for $i = 0, 1$ and $n \in \mathbf{N}$ we can replace each $P_{i,n}$ by $\operatorname{cl} \operatorname{int} P_{i,n}$, this makes P_i regularly closed, for $i = 0, 1$. But if A is regularly closed in X, then $A^* = X \cap \operatorname{Cl} A$ is regularly closed in X^* since X is σ-compact and locally compact, [Wo$_1$, 2.8], see also [Wo$_2$, 2.9]. □

4. P-Points

A point p of a space X is called a (*weak*) **P**-*point* of X if for every F_σ-subset (or: countable subset) A of X, if $p \notin A$ then $p \notin \operatorname{cl} A$. Note that not every point in an infinite compact space is a weak **P**-point.

It is known that the statement that $\mathbf{N}^*$ has a **P**-point is consistent with ZFC [R, 4.2], but independent from ZFC, see [M] or [W] for proofs of this result of Shelah. (By contrast, it is true in ZFC that $\mathbf{N}^*$ has a weak **P**-point, [Ku$_2$].) It also is known that if X is locally compact and nonpseudocompact, then if

$\mathbf{N}^*$ has a **P**-point, then so has X^*, [R, 4.5]. (The argument also works for weak **P**-points, so it is true in ZFC that X^* has a weak **P**-point.) The **P**-point found lies in $X^\#$, defined by

$$X^\# = X^* \cap (\cup\{\mathrm{Cl}\, N: N \text{ is a discrete } \mathbf{C}\text{-embedded (hence closed) countable subset of } X\});$$

clearly the converse of this also is true, i.e., the statement that $X^\#$ contains a **P**-point of X^* (or of $X^\#$) implies the statement that $\mathbf{N}^*$ has a **P**-point. It is not immediately clear, however, that the weaker statement that X^* has a **P**-point, possibly not in $X^\#$, should tell us anything about $\mathbf{N}^*$. Put differently, one can ask if for sufficiently nice X we can construct a **P**-point in X^* using what we know about X, but without using the hypothesis that $\mathbf{N}^*$ has a **P**-point. This question was motivated by the facts that there is a simple argument, involving Lebesgue measure, that $\mathbf{R}^\# \neq \mathbf{R}^*$ [FG, 1.3] and that there is a more complicated argument, involving connectedness properties of $\mathbf{R}$, that $\mathbf{R}^*$ has a point that is topologically different (within $\mathbf{R}^*$) from all points of $\mathbf{R}^\#$ [vD_3]. The following result answers this question in the negative.

4.1. THEOREM. *The following statements are equivalent*:
(a) $\mathbf{N}^*$ *has a* **P**-*point*;
(b) *If X is any nonpseudocompact locally compact space, then X^* has a* **P**-*point*;
(c) $\mathbf{R}^*$ *has a* **P**-*point*;
(d) *There is a noncompact σ-compact locally compact space X such that X^* has a* **P**-*point*;
(e) *There is a noncompact realcompact* (*not necessarily locally compact*) *space X such that X^* has a* **P**-*point*.

Proof. (a) ⇒ (b). See [R, 4.5, p. 633].
(b) ⇒ (c) ⇒ (d). Obvious.
(d) ⇒ (e). σ-compact spaces are realcompact.
(e) ⇒ (a). Let p be a **P**-point of X^*. By the Open Mapping Lemma there is a closed $F \subseteq X^*$ with $p \in F$ which admits an open map onto $\mathbf{N}^*$. Clearly p is a **P**-point of F, hence $f(p)$ is a **P**-point of $\mathbf{N}^*$ since f is open. □

The analogous statement about weak **P**-points is true, but vacuously so, since $\mathbf{N}^*$ has a weak **P**-point, as mentioned above. In this context it is amusing that the above proof that (e) ⇒ (a) does not work for weak **P**-points.

4.2. *Example*. There is a compact space X which admits an open map f onto βN such that $f^{\leftarrow}\mathbf{N}^*$ contains a weak **P**-point of X.

Proof. $\mathbf{N}^*$ has a weak $\mathbf{P}$-point p that is not a $\mathbf{P}$-point [Ku$_2$]. There is a pairwise disjoint sequence $\langle Y_n \rangle_n$ of nonempty clopen subsets of $\mathbf{N}^*$ such that if $Y = \bigcup_n Y_n$ and $X = \operatorname{Cl} Y$ then $p \in X - Y$. Clearly p is a weak $\mathbf{P}$-point of X. Now $\beta Y = X$, e.g., since $\beta\mathbf{N}$ is extremally disconnected and since every σ-compact subspace of an extremally disconnected space is $\mathbf{C}^*$-embedded; cf. [GJ, 9H.1]. Hence the map $r = \bigcup_n Y_n \times \{n\}$ from Y onto $\mathbf{N}$ admits a continuous extension $\beta r: X \to \beta\mathbf{N}$. This extension is open by Theorem 2.2, and $f(p) \in \mathbf{N}^*$ since r is perfect. □

Another analogue of Theorem 4.1 is false, too: Call a point p of a space X a P_κ*-point* if $\bigcap \mathscr{U}$ is a neighborhood of p whenever $\mathscr{U}$ is a family of neighborhoods of p with $1 \le |\mathscr{U}| < \kappa$. It is consistent with ZFC that $\mathbf{N}^*$ have a P_{ω_2}-point, for $\mathbf{N}^*$ has a $\mathbf{P}_{\mathfrak{c}}$-point under MA. But the analogue of Theorem 4.1 is false for P_{ω_2}-points by Example 10.4.

5. Points of X^* are $\mathfrak{c}$-points

A point of a space X is called a γ*-point* of X if there is a pairwise disjoint open family $\mathscr{U}$ in X with $|\mathscr{U}| = \gamma$ such that $p \in \operatorname{Cl} U$ for each $U \in \mathscr{U}$. Note that X is extremally disconnected iff it has no 2-points. The following result of Balcar and Vojtás is the latest in a long chain of results [BV].

5.1. THEOREM. *Every point of* $\mathbf{N}^*$ *is a* $\mathfrak{c}$*-point.*

We use this theorem to generalize itself.

5.2. THEOREM. *If X is noncompact, realcompact and locally compact, then every point of X^* is a* $\mathfrak{c}$*-point.*

Let $x \in X^*$ be arbitrary. By the Open Retraction Lemma there is a regularly closed $F \subseteq X^*$ with $x \in F$ which admits an open map f onto $\mathbf{N}^*$. By Theorem 5.1, $f(x)$ is a $\mathfrak{c}$-point of $\mathbf{N}^*$; let $\mathscr{U}$ witness this. Then

$$\mathscr{V} = \{(\operatorname{Int}^* F) \cap f^{\leftarrow} U : U \in \mathscr{U}\}$$

witnesses that x is a $\mathfrak{c}$-point in X^*: Clearly $\mathscr{V}$ is a disjoint open family in X^*. For $U \in \mathscr{U}$ we have $f^{\leftarrow} \operatorname{Cl}^* U = \operatorname{Cl} f^{\leftarrow} U$ since $f\colon F = \operatorname{Cl}^* F \to \mathbf{N}^*$ is open, hence $x \in \operatorname{Cl}^* f^{\leftarrow} U$; since $f^{\leftarrow} U$ is open in F and $\operatorname{Int}^* F$ is dense in F it follows that $x \in \operatorname{Cl}^*((\operatorname{Int}^* F) \cap f^{\leftarrow} U)$. □

6. When nonempty $\mathbf{G}_\delta$-subsets of X^* have nonempty interior

We call a subset of a space Y a $\mathbf{G}_\kappa$-subset of Y if it is the intersection of a (nonempty) family consisting of less than κ open sets (so $\mathbf{G}_{\omega_1} \equiv \mathbf{G}_\delta$). Recall that a π*-base* for a space X is a family $\mathscr{B}$ of nonempty open sets such that each nonempty open set in X includes a member of $\mathscr{B}$.

6.1. THEOREM. *Let X be any noncompact locally compact realcompact space which has a countable π-base. Then for each infinite κ the following are equivalent*:

(A) *every nonempty $\mathbf{G}_\kappa$-subset of $\mathbf{N}^*$ has nonempty interior in $\mathbf{N}^*$*;

(B) *X^* has a base $\mathscr{E}$ such that* $\mathrm{Int}^* \cap \mathscr{W} \neq \emptyset$ *for each centered $\mathscr{W} \subseteq \mathscr{E}$ with $1 \leq |\mathscr{W}| < \kappa$; and*

(C) *every nonempty $\mathbf{G}_\kappa$-subset of X^* has nonempty interior in X^*.*

6.2. COROLLARY TO PROOF. *If X is any noncompact locally compact realcompact space, then X^* has a base $\mathscr{E}$ such that* $\mathrm{Int}^* \cap \mathscr{W} \neq \emptyset$ *for each nonempty centered countable $\mathscr{W} \subseteq \mathscr{E}$.*

6.3. COROLLARY TO COROLLARY [FG_1, Lemma 3.1]. *If X is any noncompact locally compact realcompact space, then nonempty $\mathbf{G}_\delta$'s in X^* have nonempty interior.*

The implication (A) $\Rightarrow$ (B) in 6.1 was proved in [vD_1], which will not be published; we find this paper a natural place to publish the proof since the Open Retraction Lemma is the tool for proving the implication (C) $\Rightarrow$ (A), and the remaining implication is trivial.

Before we proceed to the proof we point out that the conditions on X are essential: By 6.3, (A) holds in ZFC with $\kappa = \omega_1$, but Examples 9.1 and 9.2 show that (C) fails if X is not assumed to be locally compact or realcompact. Also, it is consistent with ZFC that (A) holds for $\kappa = \omega_2$ (see Remark 6.8) and Example 9.4 shows that (C) is false for $\kappa = \omega_2$ for a suitable X which has no countable π-base. (By 6.3 this cannot happen if $\kappa = \omega_1$.)

Condition (B) in Theorem 6.1, and Corollaries 6.2 and 6.3 suggest the following question.

6.4. *Question*. Does there exist a compact space in which nonempty $\mathbf{G}_\delta$'s have nonempty interior, but which does not have a base $\mathscr{E}$ such that $\cap \mathscr{W} \neq \emptyset$ for each nonempty countable centered $\mathscr{W} \subseteq \mathscr{E}$?

An easy noncompact example would be any **P**-space ($\equiv$ each $\mathbf{G}_\delta$ is open) which is not Baire, e.g., [CR, 3.2].

For the proof of Theorem 6.1 we need, for a fixed space X, information about the function

$$\mathrm{Ex}: \{\text{open sets of } X\} \to \{\text{open sets of } \beta X\}$$

defined by

$$\mathrm{Ex}(U) = \beta X - \mathrm{Cl}(X - U).$$

6.5. LEMMA. *Let U, V be open in X. Then*
(A) $X \cap \mathrm{Ex} U = U$;
(B) $\mathrm{Cl}\,\mathrm{Ex}(U) = \mathrm{Cl}\, U$;
(C) $\mathrm{Ex}(U \cap V) = \mathrm{Ex}(U) \cap \mathrm{Ex}(V)$;
(D) *if $K \subseteq X$ is compact, then* $\mathrm{Ex}(U) - K = \mathrm{Ex}(U - K)$;
(E) *if* $\mathrm{Cl}\, U$ *is not compact and X is realcompact, then* $X^* \cap \mathrm{Ex}(U) \neq \emptyset$.

Proof. (A) is obvious, (B) follows from (A), and (C) and (D) require only a straightforward computation. This is essentially known. We now prove (E). Consider any $p \in X^* \cap \mathrm{Cl}\, U$. Since X is realcompact there is a sequence $\langle V_n : n < \omega \rangle$ of open sets in βX such that

$$p \in \bigcap_n V_n \subseteq X^* \quad \text{and} \quad \mathrm{Cl}\, V_{n+1} \subseteq V_n \quad \text{for each } n$$

Without loss of generality, $U \cap (V_n - \mathrm{Cl}\, V_{n+1}) \neq \emptyset$ for each n, so pick

$$p_n \in U \cap (V_n - \mathrm{Cl}\, V_{n+1}) \quad \text{for } n < \omega.$$

It is easily checked that $\{p_n : n < \omega\}$ is discrete and **C**-embedded in X, hence

$$\mathrm{Cl}\{p_n : n < \omega\} \cap \mathrm{Cl}(X - U) = \emptyset;$$

cf. [GJ, 9M.1]. It follows that

$$X^* \cap \mathrm{Ex}(U) \supseteq X^* \cap \mathrm{Cl}\{p_n : n < \omega\} \neq \emptyset. \quad \square$$

6.6. *Proof of Theorem* 6.1. It is convenient to consider the following additional conditions:

(A′) *For every $\mathscr{A} \subseteq \mathscr{P}(\mathbf{N})$, if $|\mathscr{A}| < \kappa$, and if $|\bigcap \mathscr{F}| = \omega$ for each finite nonempty $\mathscr{F} \subseteq \mathscr{A}$, then there is an infinite $I \subseteq \mathbf{N}$ such that $|I - A| < \omega$ for each $A \in \mathscr{A}$;*

(B′) *For every collection $\mathscr{U}$ of open sets in X, if $|\mathscr{U}| < \kappa$ and if* $\mathrm{cl} \bigcap \mathscr{F}$ *is noncompact for each finite nonempty $\mathscr{F} \subseteq \mathscr{U}$, then there is an open $V \subseteq X$ with* $\mathrm{cl}\, V$ *noncompact such that* $\mathrm{cl}(V - U)$ *is compact for each $U \in \mathscr{U}$.*

We will prove

$$(\mathrm{A}) \Rightarrow (\mathrm{A}') \Rightarrow (\mathrm{B}') \Rightarrow (\mathrm{C}) \Rightarrow (\mathrm{B}) \Rightarrow (\mathrm{A}).$$

Since (1) trivially holds for $\kappa = \omega$ we assume $\kappa \geq \omega_1$.

(A) $\Leftrightarrow$ (A′). It is well known, and easy to see, that (A′) is a translation of (A).

(A′) $\Rightarrow$ (B′). Let $\mathscr{U}$ be as in (B′), and assume $\mathscr{U} \neq \emptyset$. Define

$$\mathscr{U}' = \mathscr{U} \cup \{X - W : W \text{ is open and } \mathrm{cl}\, W \text{ is compact}\}.$$

The condition on $\mathscr{U}$ implies that $\mathscr{U}'$ is centered. Also, $\mathscr{U}'$ is free since X is locally compact. It follows that we can pick

$$p \in X^* \quad \text{with } p \in \bigcap_{U \in \mathscr{U}} \operatorname{Cl} U.$$

Since X is realcompact there is a sequence $\langle U_n \rangle_n$ of open sets in βX such that

(1) $p \in \bigcap_n U_n \subseteq X^*$, and $\operatorname{Cl} U_{n+1} \subseteq U_n$ for each n.

Since $\kappa \geq \omega_1$ we may assume that $X \cap U_n \in \mathscr{U}$ for each n. Note that this does not affect the information that

(2) $\operatorname{Cl} \bigcap \mathscr{F}$ is noncompact for each finite nonempty $\mathscr{F} \subseteq \mathscr{U}$.

Since X is locally compact and has a countable π-base, X has a countable π-base $\mathscr{B}$ such that

(3) $\operatorname{cl} B$ is compact for each $B \in \mathscr{B}$.

For $U \in \mathscr{U}$ define

$$\mathscr{B}_U = \{B \in \mathscr{B} : B \in U\}.$$

From (2) we see that $|\bigcap_{U \in \mathscr{F}} \mathscr{B}_U| = \omega$ for each finite nonempty $\mathscr{F} \subseteq \mathscr{U}$. Since $|\mathscr{B}| = |\mathbf{N}|$, it now follows from (A') that there is an infinite $\mathscr{V} \subseteq \mathscr{B}$ such that $|\mathscr{V} - \mathscr{B}_U| < \omega$ for each $U \in \mathscr{B}$. Let $V = \bigcup \mathscr{V}$. Then

(4) $V \cap U \neq \emptyset$ for each $U \in \mathscr{U}$

since $\mathscr{V}$ is infinite, and $\operatorname{cl}(V - U)$ is compact for each $U \in \mathscr{U}$, because of (3). Clearly (1) and (2) imply that $\operatorname{cl} V$ is noncompact, since by assumption $X \cap U_n \in \mathscr{U}$ for each n.

(B') $\Rightarrow$ (C). We will show that

$$\mathscr{E} = \{X^* \cap \operatorname{Ex}(U) : U \text{ is open in } X\}$$

is as required. First perform an easy calculation and use Lemma 6.5A to see that $U \subseteq \operatorname{Ex}(X \cap U) \subseteq \operatorname{Cl} U$ for each open U in βX. Hence $\mathscr{E}$ is a base for X^*.

Next, let $\mathscr{U}$ be a family of open sets in X^* with $|\mathscr{U}| < \kappa$ such that

$$\mathscr{W} = \{X^* \cap \operatorname{Ex}(U) : U \in \mathscr{U}\}$$

is centered. Lemmas 6.5C and 6.5B imply that $\operatorname{cl} \bigcap \mathscr{F}$ is noncompact for each nonempty finite $\mathscr{F} \subseteq \mathscr{U}$. Let V be as in (B'). Then $X^* \cap \operatorname{Ex}(V) \neq \emptyset$ by

Lemma 6.2E. For each $U \in \mathscr{U}$ the set $K = \text{cl}(V - U)$ is compact. As $V - K \subseteq U$ we see from Lemma 6.5C.D that

$$\text{Ex}(V) - K = \text{Ex}(V - K) \subseteq \text{Ex}(U),$$

hence $X^* \cap \text{Ex}(V) \subseteq X^* \cap \text{Ex}(V)$ since $K \subseteq X$. It follows that

$$\text{Int}^* \cap \mathscr{W} \supseteq X^* \cap \text{Ex}(V) \neq \emptyset.$$

(C) $\Rightarrow$ (B) Obvious.

(B) $\Rightarrow$ (A) Immediately clear from the Open Retraction Lemma, which implies that X^* has an open subspace which admits an open map onto $\mathbf{N}^*$. □

6.7. *Proof of* 6.2. In the proof of (A′) = (B′), if $|\mathscr{U}| \leq \omega$ then we can find $\mathscr{V}$ without having a countable π-base available.

6.8. *Remark.* Condition (A′) in 6.4 is known under the name $\mathbf{P}(\kappa)$. It is well known that MA (= Martin's Axiom) implies $\mathbf{P}(\mathfrak{c})$, [MS, p. 154], hence that $\mathbf{P}(\mathfrak{c})$ is consistent with $\neg$CH.

7. The Baire number of X^*

Define the *Baire number* $b(Y)$ of a space Y by

$$b(Y) = \min\{|\mathscr{S}|: \mathscr{S} \text{ consists of dense open sets in } Y, \text{ but } \bigcap \mathscr{S} \text{ is not dense}\}.$$

So Y is a Baire space iff $b(Y) > \omega$. An immediate consequence of the Fine-Gillman result 6.3 is the next proposition.

7.1. Proposition. *If X is a noncompact locally compact realcompact space, then $b(X^*) > \omega_1$.*

The conditions on X are essential by Examples 9.1 and 9.5.

We now consider upper bounds for $b(X^*)$.

7.2. Theorem. *If X is a nonpseudocompact locally compact space, then $b(X^*) \leq b(\mathbf{N}^*)$.*

It is easy to see that if U is a nonempty open subspace of a space Y, then $b(Y) \leq b(U)$. Also, the proof that open maps preserve the property of being a Baire space shows that $b(Z) \geq b(Y)$ whenever Z is an open continuous image of Y. The result now follows from the Open Retraction Lemma.

7.3. COROLLARY TO PROOF. *If X is a noncompact locally compact realcompact space, then $b(U) \leq b(\mathbf{N}^*)$ for each nonempty open subspace U of X^*.*

Proof. Use the parenthetical remark in 1.1B. (The corollary holds already if $\mathrm{Cl}^*(X^* - vX) = X^*$.) □

It is known that the size of $b(\mathbf{N}^*)$ depends on your set theory: By Remark 6.8, MA implies that $b(\mathbf{N}^*) > \mathfrak{c}$, and, more generally, that $b(X^*) > \mathfrak{c}$ if X is noncompact, locally compact and realcompact, and has countable π-weight. On the other hand, Hechler has shown that it is consistent with ZFC that $b(\mathbf{N}^*) = \mathfrak{c}$ or that $b(\mathbf{N}^*) = \omega_2$ and $\omega_2 < \mathfrak{c}$, [He].

Example 9.4 has $b(T^*) \leq 2^{\omega_1}$, hence shows that equality need not hold in Theorem 7.2, for MA implies $b(\mathbf{N}^*) > \mathfrak{c}$, as just noted, and MA + $\neg$CH implies that $2^{\omega_1} = \mathfrak{c}$. In view of Theorem 6.1 this suggest the following question.

7.4. *Question.* If X is noncompact, σ-compact and locally compact, and if X has a countable π-base, is $b(X^*) = b(\mathbf{N}^*)$?

(It is no gain of generality to replace "σ-compact" by "realcompact.)

We conclude this section with pointing out that trivially $b(\mathbf{N}^*) \leq 2^{\mathfrak{c}}$ since $|\mathbf{N}^*| = 2^{\mathfrak{c}}$ [GJ, 9.2 or 90.2], hence that MA + $2^{\mathfrak{c}} = \mathfrak{c}^+$ implies that $b(\mathbf{N}^*) = 2^{\mathfrak{c}}$. But it is consistent with $2^{\mathfrak{c}} > \mathfrak{c}^+$ that $b(\mathbf{N}^*) = 2^{\mathfrak{c}}$: BACH (= Baumgartner's Axiom + CH), which is consistent with $2^{\mathfrak{c}} > \mathfrak{c}$, implies that $b(Y) \geq 2^{\mathfrak{c}}$ whenever Y is a compact space with $d(Y) \leq \mathfrak{c}$ ($d \equiv$ density) in which nonempty G_δ's have nonempty interior [T, 4.1]. Since $d(X^*) \leq d(X)^\omega$ for any realcompact X, as one can easily verify (cf. [C_1, 4.1]) it now follows from Theorem 7.2 and Corollary 6.3 that BACH implies that $b(X^*) = 2^{\mathfrak{c}}$ whenever X is a noncompact locally compact realcompact space with $d(X) \leq \mathfrak{c}$, in particular BACH implies that $b(\mathbf{N}^*) = 2^{\mathfrak{c}}$ [T], and in fact that $b(X^*) = 2^{\mathfrak{c}}$ if X is as in Question 7.4.

8. Minimum character

For a space X define the *character* of $F \subseteq X$, or of $x \in X$ by

$$\chi(F, X) = \min\{\kappa: F \text{ has a neighborhood base of cardinality } \kappa\};$$

$$\chi(x, X) = \chi(\{x\}, X).$$

Also, define the *character* and the *minimum character* of X by

$$\chi(x) = \sup\{\chi(x, X) : x \in X\};$$
$$m\chi(x) = \min\{\chi(x, X) : x \in X\}.$$

We begin with reminding the reader that $\chi(\mathbf{N}^*) = \mathfrak{c}$ (In fact there is $p \in \mathbf{N}^*$ with $\chi(p, \mathbf{N}^*) = \mathfrak{c}$ [P]; cf. [Ku$_1$, remark on p. 303].), and that trivially $m\chi(\mathbf{N}^*) > \omega$. Also, it is consistent with $\neg$CH that $m\chi(\mathbf{N}^*) = \mathfrak{c}$, by Remark 6.8, but it also is consistent with ZFC that $m\chi(\mathbf{N}^*) < \mathfrak{c}$ [Ku$_1$, remark on p. 303].

As a trivial application of the Open Mapping Lemma we have:

8.1. THEOREM. *If X is noncompact and realcompact, then $m\chi(X^*) \geq m\chi(\mathbf{N}^*)$.*

This result would not be available without the Open Mapping Lemma, for then we would only know that $\chi(x, X^*) \geq m\chi(\mathbf{N}^*)$ for those $x \in X^*$ for which there is a closed discrete $\mathbf{C}^*$-embedded $D \subseteq X$ with $x \in \operatorname{Cl} D$, but there may be other points in X^*; e.g., if $X = \mathbf{R}$ [FG, 1.3], or $X = \mathbf{Q}$ [vD$_4$], [vD$_5$].

Example 10.4 shows that there is for each $\kappa \geq \omega$ a noncompact σ-compact locally compact X with $m\chi(X^*) = \chi(X^*) = \kappa^\omega$, hence equality need not hold in Theorem 8.1. This suggests the question if equality holds for sufficiently nice X. Since trivially $m\chi(X^*) \leq m\chi(\mathbf{N}^*)$ if X has an infinite closed set of isolated points, we are interested in spaces without isolated points.

8.2. *Example.* There is a noncompact σ-compact locally compact space X without isolated points such that $m\chi(X^*) = m\chi(\mathbf{N}^*)$.

Proof. Let L be the long line, i.e. the space one gets from the ordinals $[0, \omega_1]$ by inserting a copy of the real interval $(0, 1)$ between α and $\alpha + 1$ for each $\alpha < \omega_1$. Then $X = \mathbf{N} \times L$ is a noncompact σ-compact locally compact space without isolated points in which $N = \mathbf{N} \times \{\omega_1\}$ is a countable $\mathbf{C}$-embedded discrete subset with $\chi(N, X) = \omega_1$. Now apply the lemma below to see that $m\chi(X^*) \leq m\chi(\mathbf{N}^*)$.

The reverse inequality follows from Theorem 8.1. □

8.3. LEMMA. *If X has a countable discrete* $\mathbf{C}$*-embedded subset N with $\chi(N, X) = \kappa$, then $m\chi(X^*) \leq \kappa \cdot m\chi(\mathbf{N}^*)$.*

Proof. N will be closed discrete, and for $p \in X^* \cap \operatorname{Cl} N$ we have

$$\chi(p, X^*) \leq \chi(p, \beta X) \leq \chi(p, X^* \cap \operatorname{Cl} N) \cdot \chi(X^* \cap \operatorname{Cl} N, \operatorname{Cl} N) \cdot \chi(\operatorname{Cl} N, \beta X).$$

Since $X^* \cap \mathrm{Cl}\, N$ and $\mathbf{N}^*$ are homeomorphic and $\chi(X^* \cap \mathrm{Cl}\, N, \mathrm{Cl}\, N) = \omega$, it follows that

$$m\chi(X^*) \leq \chi(\mathrm{Cl}\, N, \beta X) \cdot m\chi(\mathbf{N}^*).$$

Since N is discrete and $\mathbf{C}$-embedded in X we have $\mathrm{Cl}\, N \cap \mathrm{Cl}\, F = \emptyset$ for each closed F in X with $N \cap F = \emptyset$ [GJ, 9 M.1], hence

$$\chi(\mathrm{Cl}\, N, \beta X) = \chi(N, X) = \kappa. \qquad \square$$

This gives no information about $m\chi(X^*)$ for more interesting spaces like $\mathbf{Q}$ and $\mathbf{R}$. These spaces are first countable, and have a countable π-base, hence we are led to the following question, especially in view of §6.

8.4. *Question.* If X is noncompact and realcompact, and is first countable or has a countable π-base, is $m\chi(X^*) = m\chi(\mathbf{N}^*)$?

We only can show that it is not true in ZFC that $m\chi(\mathbf{Q}^*) = \mathfrak{c}$ or $m\chi(\mathbf{R}^*) = \mathfrak{c}$.

8.5. THEOREM. *It is consistent with* ZFC *that* $m\chi(X^*) < \mathfrak{c}$ *for each nonpseudocompact first countable space* X.

X will have a countable discrete $\mathbf{C}$-embedded subset N. Let ${}^{\omega}\omega$ be the set of functions from ω to ω, and define $\leq$ on ${}^{\omega}\omega$ by

$$f \leq g \text{ if } f(n) \leq g(n) \quad \text{for all } n \in \omega.$$

Let

$$\mathbf{d} = \min\{|F|: F \subseteq {}^{\omega}\omega \text{ is dominant; i.e., } \forall g \in {}^{\omega}\omega\ \exists f \in F \quad (g \leq f)\}.$$

Then clearly $\chi(N, X) \leq \mathbf{d}$. (We have equality iff infinitely many points of N are nonisolated.) So it suffices to know that it is consistent with $\neg$CH that $\mathbf{d} = m\chi(\mathbf{N}^*) = \omega_1$; this is follows from the Simple Definable Forcing Axiom [vDF]. $\square$

8.6. *Remark and question.* Define $\leq^*$ on ${}^{\omega}\omega$ by

$$f \leq^* g \text{ if } f(n) \leq g(n) \quad \text{for all but finitely many } n.$$

Clearly $\mathbf{d}$ also is $\min\{|F|: F$ dominates in $\langle {}^{\omega}\omega, \leq^* \rangle\}$. Define $\mathbf{b}$ by

$$\mathbf{b} = \min\{|F|: F \subseteq {}^{\omega}\omega \text{ is unBounded under } \leq^*\}.$$

Then $\omega_1 \le \mathbf{b} \le \mathbf{d} \le \mathfrak{c}$, and nothing more can be said in ZFC [He]. Solomon has shown that $m\chi(\mathbf{N}^*) \ge \mathbf{b}$ [S], and Ketonen has (essentially) shown that

$$\min\{\chi(p, \mathbf{N}^*) : p \in \mathbf{N}^* \text{ is not a } \mathbf{P}\text{-point}\} \ge \mathbf{d} \quad [\mathrm{K}].$$

I don't know if these results can be improved to read $m\chi(\mathbf{N}^*) \ge \mathbf{d}$. (If so then the answer to Question 8.4 is affirmative for first countable X.) If not then it might be difficult to calculate $m\chi(\mathbf{Q}^*)$ or $m\chi(\mathbf{R}^*)$: if N is a countable discrete $\mathbf{C}$-embedded subset of a first countable space X and if N contains no isolated points, then $\chi(p, X^*) \ge \mathbf{d}$ for each $p \in X^* \cap \mathrm{Cl}\, N$.

9. Hausdorff gaps

If κ_0 and κ_1 are cardinals, we define a (κ_0, κ_1)*-gap* for a space X to be a pair $\langle\langle U_{i,\xi} : \xi < \kappa_i\rangle : i < 2\rangle$ such that:

(1) $U_{i,\xi}$ is open in X for $\xi < \kappa_i$, for $i < 2$;

(2) $\mathrm{cl}\, U_{i,\xi} \subseteq U_{i,\eta}$ whenever $\xi < \eta < \kappa_i$, for $i < 2$;

(3) $\bigcap_{i<2} \bigcup_{\xi<\kappa_i} U_{i,\xi} = \emptyset \ne \bigcap_{i<2} \mathrm{cl} \bigcup_{\xi<\kappa_i} U_{i,\xi}$.

The following is a classical result of Hausdorff [H, §1].

9.1. THEOREM. (A). $\mathbf{N}^*$ *has an* (ω_1, ω_1)*-gap.*

(B) $\mathbf{N}^*$ *has no* (ω, ω)*-gap.*

Part (B) has been generalized by Gillman and Henriksen [GH, 2.7]:

9.2. THEOREM. *If Y is σ-compact and locally compact, then every two disjoint open F_σ-subsets of Y^* have disjoint closures.*

See [N, 3.1] for a short proof. We generalize part (A); our proof uses the retraction rather than the open map provided by the Open Retraction Lemma.

9.3. THEOREM. *If X is nonpseudocompact and locally compact, then X^* has an (ω_1, ω_1)-gap.*

Proof. By the Open Retraction Lemma there are a closed $\mathbf{G}_\delta$-set F of βX with $F \subseteq X^*$ and a homeomorph H of $\mathbf{N}^*$ such that

(1) $H \subseteq \mathrm{Int}^* F$ and

(2) F admits a retraction onto H.

By Theorem 9.1 H has an (ω_1, ω_1)-gap $\langle\langle U_{i,\xi}: \xi < \omega_1\rangle: i < 2\rangle$. We prove our theorem by finding a pair $\langle\langle V_{i,\xi}: \xi < \omega_1\rangle: i < 2\rangle$ such that:
(3) $V_{i,\xi}$ is open in X^* for $\xi < \omega_1$ and $i < 2$;
(4) $\mathrm{Cl}^* V_{i,\xi} \subseteq V_{i,\eta}$ whenever $\xi < \eta < \omega_1$, for $i < 2$;
(5) $\bigcap_{i<2} \bigcup_{\xi<\omega_1} V_{i,\xi} = \emptyset$; and
(6) $U_{i,\xi} \subseteq V_{i,\xi}$ for $\xi < \omega_1$ and $i < 2$.
We construct $V_{i,\xi}$ with transfinite recursion on $\xi < \omega_1$, separately for $i < 2$; this causes no difficulties provided we impose the additional condition
(7) $V_{i,\xi} \subseteq r^{\leftarrow} U_{i,\xi}$, for $\xi < \omega_1$ and $i < 2$;
this we can do if we are able to ensure that
(8) $\mathrm{Cl}^* \bigcup_{\xi<\eta} V_{i,\xi} \subseteq \mathrm{Int}^* F$ for $\eta < \omega_1$ and $i < 2$.
To this end we note that $F = (\beta X - F)^*$, hence every two disjoint open $\mathbf{F}_\sigma$-subsets of the subspace F have disjoint closures by Theorem 9.2. So if W is an open $\mathbf{F}_\sigma$-subset of X^* such that

$$X^* - \mathrm{Int}^* F \subseteq W \quad \text{and} \quad \mathrm{Cl}^* W \cap F = \emptyset,$$

(there is such a W by (1)) then we ascertain (8) by demanding
(9) $W \cap \mathrm{Cl}^* V_{i,\xi} = \emptyset$ for $\xi < \omega_1$ and $i < 2$. □

This argument also shows that if $\mathbf{N}^*$ has an (ω, ω_1)-gap (this statement is consistent with ZFC (since it follows from CH—but not conversely) and independent from ZFC (since it is false under MA + $\neg$CH)), then so has X^* if X is nonpseudocompact and locally compact. However, the argument does not allow us to transfer a (κ, λ)-gap if $\kappa > \omega_1$. (It is consistent with ZFC that $\mathbf{N}^*$ have an (ω_2, ω_2)-gap; (Kunen—see [Ba].) I have not seriously investigated this, but point out that if X is a strongly zero-dimensional nonpseudocompact locally compact space, which implies that some clopen subspace of X^* admits an open retraction onto a homeomorph of $\mathbf{N}^*$, then there is no difficulty whatsoever because of the following simple result.

9.4. Proposition. *If X admits an open map onto Y, or has a retract homeomorphic to Y, then X has a (κ, λ)-gap if Y has one.*

It is not generally true that if X is σ-compact and locally compact, then X^* has a (κ, λ)-gap iff $\mathbf{N}^*$ has one: This follows from Example 10.5 and the fact that $\mathbf{N}^*$ has no (κ, κ)-gaps if $\kappa > \mathfrak{c}$, or from the fact that it is consistent with $\mathfrak{c} = \omega_2$ that $\mathbf{N}^*$ has no (ω_2, ω_2)-gap [Ku_3]. Again this leaves open the following question.

9.5. *Question.* If X is noncompact, σ-compact and locally compact, and has a countable π-base, does X^* have a (κ, λ)-gap iff $\mathbf{N}^*$ has one?

We return to Hausdorff gaps in §12.

10. Examples

Since one of our results, 6.1, uses the assumption that the space considered has a countable π-base, we give our examples, when possible, a countable π-base.

10.1. *Example.* A realcompact space P with a countable base such that P^* is separable and is not Baire.

Let P be the irrationals. P is realcompact, being Lindelöf. Also,

(1) P^* is dense in βP

since no point of P has a compact neighborhood in P. Since a regular space has a countable π-base iff each dense subspace does, and P has a countable base, we see that βP, hence P^* too, has a countable π-base. Hence P^* is separable. Next, P is a G_δ in βP, being completely metrizable, hence P^* is not Baire because of (1).

10.2. *Example.* A noncompact locally compact space L with countable π-weight such that $m\chi(L^*) = \omega$ and $b(L^*) = \omega_1$.

By Theorem 11.1 it suffices to find a compact space X with weight ω_1 such that $m\chi(X) = \omega$ and $b(X) = \omega_1$. An easy example is the product of ω copies of the one-point compactification $\alpha\omega_1 = \omega_1 \cup \{\infty\}$ of ω_1: the family

$$\{\{x \in X : x_n \in \alpha \cup \{\infty\} \text{ for each } n\} : \alpha < \omega_1\}$$

witnesses that $b(X) \leq \omega_1$, and $b(X) \geq \omega_1$ since X is compact. Clearly $m\chi(X) = \omega$ (but $\chi(X) = \omega_1$). Another example, even with $\chi(X) = \omega$, would be an Aronszajn line.

10.3. *Remark.* The above example L is pseudocompact. A nonpseudocompact example is the topological sum $\omega + L$.

10.4. *Example.* For each $\kappa \geq \omega$ with $\kappa^\omega = \kappa$ there is a noncompact σ-compact locally compact space T such that

(1) T^* is covered by a collection of 2^{ω_1} nowhere dense sets, each of which is an intersection of ω_1 open sets; and

(2) $\chi(x, T^*) = \kappa$.

In the space T of [vDvM] replace $\mathfrak{c}$ by κ.

10.5. *Example.* For each regular $\kappa \geq \omega$ there is a zero-dimensional σ-compact locally compact space X such that X^* has (κ^+, κ^+)-gap.

By 12.2 there is a compact zero-dimensional space Y which has a (κ^+, κ^+)-gap $\langle\langle A_{i,\xi}: \xi < \kappa^+\rangle: i < 2\rangle$. We may assume the $A_{i,\xi}$'s are clopen. Let $X = Y \times \mathbf{N}$, and define

$$U_{i,\xi} = X^* \cap \mathrm{Cl}(A_{i,\xi} \times \mathbf{N}) \quad \text{for } \xi < \kappa^+, i < 2,$$
$$V_i = \bigcup_{\xi < \kappa^+} U_{i,\xi} \quad \text{for } i < 2.$$

It is easy to see that $U_{i,\xi} \subseteq U_{i,\eta}$ whenever $\xi < \eta < \kappa^+$, that the $U_{i,\xi}$'s are clopen in X^*, and that $V_0 \cap V_1 = \emptyset$. Suppose $\mathrm{Cl}\, V_0 \cap \mathrm{Cl}\, V_1 = \emptyset$. Then there are disjoint clopen sets W_0, W_1 in βX such that $\mathrm{Cl}\, V_i \subseteq W_i$ for $i < 2$, hence such that

$$\forall \xi < \kappa^+ \ \exists n \in \mathbf{N}\ \forall k \in \mathbf{N}\ (k \geq n \Rightarrow A_{i,\xi} \times \{k\} \subseteq W_i) \quad \text{for } i < 2.$$

It follows that there is $n \in \mathbf{N}$ and a cofinal $K \subseteq \kappa^+$ such that

$$\forall i < 2\ \forall \xi \in K (A_{i,\xi \times \{n\}} \subseteq W_i).$$

This, in turn, implies that

$$\bigcap_{i<2} \mathrm{cl} \bigcup_{\xi \in K} A_{i,\xi} \neq \emptyset \quad (\text{closure in } Y),$$

which contradicts $\langle\langle A_{i,\xi}: \xi < \kappa^+\rangle: i < 2\rangle$ being a (κ^+, κ^+)-gap in Y.

11. Appendix. A compactification of N

11.1. THEOREM. *For every space X with weight at most $\mathfrak{c}$ there is a space Y which has a countable dense subset of isolated points (hence has a countable π-base) such that Y^* is homeomorphic to X.*

Let $\mathbf{K}$ denote the Tychonoff cube of weight $\mathfrak{c}$, i.e., the product of $\mathfrak{c}$ copies of $[0, 1]$. Then a space has weight at most $\mathfrak{c}$ iff it can be embedded in $\mathbf{K}$ [E, 2.3.23]. Since $\beta T = \beta S$ whenever S and T are spaces such that $S \subseteq T \subseteq \beta S$, Theorem 11.1 follows from our next result.

11.2. THEOREM. $\mathbf{N}$ *has a compactification $b\mathbf{N}$ which has a homeomorph H of* $\mathbf{K}$, *(necessarily) with $H \subseteq b\mathbf{N} - \mathbf{N}$, such that $\beta(b\mathbf{N} - H) = b\mathbf{N}$.*

Proof. $\mathbf{K}$ is separable, so let $\{x_n: n \in \mathbf{N}\}$ be a countable dense subset with $x_m \neq x_n$ whenever $m < n$. Topologize $b\mathbf{N} = \mathbf{N} \cup \mathbf{K}$ as follows: points of $\mathbf{N}$ are isolated, and a basic neighborhood of $x \in \mathbf{K}$ in $b\mathbf{N}$ has the form

$$U \cup \{n \in \mathbf{N}: n > k \text{ and } x_n \in U\},$$

where U is a neighborhood of x in $\mathbf{K}$ and $k \in \mathbf{N}$; clearly $b\mathbf{N}$ is compact Hausdorff and $\mathbf{N}$ is dense in $b\mathbf{N}$. ($b\mathbf{N}$ is a subspace of the Alexandroff double of $\mathbf{K}$.) Since $\mathbf{K} \times \mathbf{K}$ is homeomorphic to $\mathbf{K}$, there is a pairwise disjoint family $\mathscr{H}$ of homeomorphs of $\mathbf{K}$ in $\mathbf{K}$ with $|\mathscr{H}| = |\mathbf{K}| = 2^{\mathfrak{c}} > \mathfrak{c} = |\mathscr{P}(\mathbf{N}) \times \mathscr{P}(\mathbf{N})|$. The following claim therefore implies that there is $H \in \mathscr{H}$ with $\beta(b\mathbf{N} - H) = b\mathbf{N}$.

Claim. For each $H \in \mathscr{H}$, if $\beta(b\mathbf{N} - H) \neq b\mathbf{N}$ then there are disjoint $A, B \subseteq \mathbf{N}$ with

$$\emptyset \neq \bar{A} \cap \bar{B} \subseteq H \text{ (closure in } b\mathbf{N}).$$

Indeed, if $\beta(b\mathbf{N} - H) \neq b\mathbf{N}$ there must be a continuous $f\colon b\mathbf{N} - H \to \mathbf{I} = [0, 1]$ and $x \in H$ such that f has positive oscillation at f. Then there are $a, b \in \mathbf{I}$ with $a < b$ such that $x \in (f^{\leftarrow}[0, a])^{-} \cap (f^{\leftarrow}[b, 1])^{-}$. Consider any $a', b' \in \mathbf{I}$ with $a < a' < b' < b$; then A and B can be defined by

$$A = \mathbf{N} \cap f^{\leftarrow}[0, a') \quad \text{and} \quad B = \mathbf{N} \cap f^{\leftarrow}(b, 1].$$

12. Appendix 2. Spaces of uniform ultrafilters

For an infinite cardinal κ we denote as usual the space of uniform ultrafilters on κ by $U(\kappa)$ [CN]. Consider the following statements, where cf $\equiv$ cofinality:

(1) *If* $\mathrm{cf}(\kappa) \neq \mathrm{cf}(\lambda)$ *then* $U(\kappa)$ *and* $U(\lambda)$ *are nonhomeomorphic.*
(1′) *If* $\kappa > \lambda \geq \omega$ *but* $\mathrm{cf}(\kappa) = \mathrm{cf}(\lambda)$, *then* $U(\kappa)$ *and* $U(\lambda)$ *are nonhomeomorphic.*

It is known, and easy to prove, that (1) holds in ZFC, but it is unknown if (1′) holds in ZFC [C, §4], [vDMR, §9]. Clearly the following are stronger statements of the special case $\lambda = \omega$:

(2) *If* $\mathrm{cf}(\kappa) > \omega$ *and if X is σ-compact then* $U(\kappa)$ *is not homeomorphic to* X^*.
(2′) *If* $\mathrm{cf}(\kappa) = \omega < \kappa$ *and if X is σ-compact, then* $U(\kappa)$ *is not homeomorphic to* X^*.

Since $U(\kappa)$ is zero-dimensional, and since each nonempty clopen subspace of $U(\kappa)$ is homeomorphic to $U(\kappa)$, the special case $\lambda = \omega$ of the following statements are again stronger, by the Open Retraction Lemma.

(3) *If* cf(κ) > cf(λ), *then* $U(\kappa)$ *does not admit an open map onto* $U(\lambda)$ *and has no retract homeomorphic to* $U(\lambda)$.

(3′) *If* $\kappa > \lambda \geq \omega$ *and* cf(κ) = cf(λ), *then* $U(\kappa)$ *does not admit an open map onto* $U(\lambda)$.

Now the proof of (1) also establishes (3), see below, so it is natural to ask if (3′) is true in ZFC, perhaps with the additional condition that the open map be a retraction (on a homeomorph of $U(\lambda)$). While (1′) is consistent with ZFC since it trivially follows from the GCH, I do not know if (3′) is consistent with ZFC. However, we will see below that the special case $\lambda = \omega$ of (3′) is consistent with ZFC, hence (2′) is consistent with ZFC, too. We will also show that while (1) and (1′), or (2) and (2′) only differ in their hypotheses, it is essential that the conclusion of (3′) is weaker than that of (3):

(4) $U(\kappa)$ *has a retract homeomorphic to* $U(\text{cf}(\kappa))$.

(I did not seriously investigate the case cf(κ) = cf(λ) and $\kappa > \lambda > \text{cf}(\kappa)$.)

12.1. *First proof of* (3). Call a space X *κ-basically disconnected* if it is zero-dimensional and if cl U is open whenever U is the union of a family of less than κ clopen sets. (So $|X|^+$-basically disconnected ≡ extremally disconnected.) It is known that $U(\kappa)$ is cf(κ)-basically disconnected but not cf$(\kappa)^+$-basically disconnected [CN, 14.7(a) and 14.11]. It is easy to show that the property of being κ-basically disconnected is preserved by retractions and by open maps onto zero-dimensional spaces. (This is known for extremally disconnected spaces [E, 6.2.H(b)].)

12.2. *Second proof of* (3). By Proposition 9.4 it suffices to prove that $U(\kappa)$ has no (λ, λ)-gap if $\lambda \leq \text{cf}(\kappa)$, which is easy, but has a $(\text{cf}(\kappa)^+, \text{cf}(\kappa)^+)$-gap. By Proposition 9.4 and (4) it suffices to show that $U(\kappa)$ has a (κ^+, κ^+)-gap for regular κ. This is similar to the proof of the case $\kappa = \omega$, due to Hausdorff [H, §2], but requires an additional idea if $\kappa > \omega$ since one wants to avoid that one has to take $A_{0,\xi}, A_{1,\xi}$ such that

$$A_{0,\xi} \cup A_{1,\xi} = U(\kappa)$$

for some $\xi < \kappa$ with cf(ξ) < κ. The additional idea is to take each $A_{i,\xi}$ of the form $U(\kappa) \cap \text{Cl}\, B_{i,\xi}$, with $B_{i,\xi}$ a nonstationary subset of κ. (Kunen has informed me that this sort of argument was used earlier by Herink.)

12.3. *Proof that the special case* $\lambda = \omega$ *is consistent with* ZFC. From the proof of the Shelah **P**-point Independence Theorem one deduces that it is consistent with ZFC to have

$$\mathbf{N}^* \text{ has no } \mathbf{P}\text{-points, and } 2^\kappa = \max\{\kappa^+, \omega_2\} \quad \text{for } \kappa \geq \omega$$

[M], [W]. Since if a space has a **P**-point, then so has every open continuous image of it, it now suffices to observe that

(δ) if $2^\kappa = \kappa^+$ and $\mathrm{cf}(\kappa) = \omega$, then $U(\kappa)$ has a **P**-point. We leave the proof as an exercise to the reader.

12.4. *Remark.* Comfort and Vaughan, independently, have shown that $U(\kappa)$ has no **P**-point if $\mathrm{cf}(\kappa) > \omega$ and if κ is not Ulam-measurable [C_2, 7.3b]. Observation (δ) above answers a natural question not asked in [C_2].

12.5. *Proof of* (4). We may assume that $\kappa > \mathrm{cf}(\kappa)$. Let $\langle D_\xi : \xi < \mathrm{cf}(\kappa)\rangle$ be a decomposition of κ such that

(ε) $\omega \le |D_\xi| < |D_\eta|$ whenever $\xi < \eta < \mathrm{cf}(\kappa)$.

Define a map $f: \kappa \to \mathrm{cf}(\kappa)$ by

$$f = \bigcup_{\xi < \mathrm{cf}(\kappa)} D_\xi \times \{\xi\}.$$

By the Axiom of Choice there is a map $g: \mathrm{cf}(\kappa) \to \beta\kappa$ such that

(ζ) $|U \cap D_\xi| = |D_\xi|$ for each neighborhood U of $g(\xi)$.

(In other words, $g(\xi)$ is uniform on D_ξ.) It is easy to check that

$$(\beta f)^{\to} U(\kappa) = U(\mathrm{cf}(\kappa));$$

and $(\beta g)^{\to} U(\mathrm{cf}(\kappa)) \subseteq U(\kappa)$, and βg embeds $\beta cf(\kappa)$ into $\beta\kappa$; and $(\beta g)\circ(\beta f)$ is a retraction.

It follows that $((\beta g)\circ(\beta f)) \restriction U(\kappa)$ is a retraction onto a homeomorph of $U(\mathrm{cf}(\kappa))$.

12.6. *Remark.* If $\mathrm{cf}(\kappa) = \omega$ then the retract found in 12.5 is easily seen to be a **P**-set.

12.7. *Remark.* (2) and (2′) are equivalent to the formally stronger statements one gets by substituting "nonpseudocompact" for "σ-compact": if X is nonpseudocompact, and X^* is compact and zero-dimensional, then X^* has a nonempty clopen subset K which is a $\mathbf{G}_\delta$ in βX. Then $Y = \beta X - K$ is σ-compact with $Y^* = K$. Now recall that every nonempty clopen set in $U(\kappa)$ is homeomorphic to $U(\kappa)$, for each $\kappa \ge \omega$.

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