ILLINOIS JOURNAL OF MATHEMATICS
Volume 34, Number 4, Winter 1990

MANIFOLDS WITH INFINITELY MANY ACTIONS OF AN ARITHMETIC GROUP[1]

BY

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It is well known that if Γ is a lattice in a simple Lie group of higher split rank then in any finite dimension Γ has only finitely many inequivalent linear representations. This is one manifestation of the strong linear rigidity properties that such groups satisfy. When one considers non-linear representations, say smooth actions of Γ on compact manifolds, one still sees a large number of rigidity phenomena [7]. This is particularly true for actions preserving a connection. On the other hand, the point of this note is to establish the following result.

THEOREM 1. *Let G be the Lie group $SL(n, \mathbf{R})$, $n \geq 3$, or $SU(p, q)$, $p, q \geq 2$. Then there is a cocompact discrete subgroup $\Gamma \subset G$ and a smooth compact manifold M such that there are infinitely many actions of Γ on M with the following properties*:

i) *The actions are mutually non-conjugate in* Diff(M), Homeo(M), *and* Meas(M), *where the latter is the group of measure class preserving automorphisms of M as a measure space*;

(ii) *Each action leaves a smooth metric on M invariant, is minimal* (*i.e., every orbit is dense*), *and ergodic* (*with respect to the smooth measure class.*)

Theorem 1 is easily deduced from a certain non-rigidity phenomenon for tori in compact semisimple groups. Namely, fix a compact semisimple Lie group C and call closed subgroups H_1 and H_2 equivalent if there is an automorphism α of C such that $\alpha(H_1) = H_2$. We can then ask to what extent the diffeomorphism class of C/H determines the equivalence class of H. (The natural question is under what circumstance the map from equivalence classes of (a class of) closed subgroups to diffeomorphism classes of manifolds is finite-to-one.) Here we show:

THEOREM 2. *Let $C = SU(n) \times SU(n)$, $n \geq 2$. Then there is a family of mutually non-equivalent tori T_k, $k \in \mathbf{Z}^+$, such that C/T_k are all diffeomorphic.*

Received July 7, 1988.

[1]Research partially supported by the National Science Foundation.

We have not determined whether or not a similar phenomenon can occur if C is a simple Lie group of higher rank. It has been shown by A. Borel (private communication) that this cannot happen for one dimensional tori in $SU(4)$.

To prove Theorem 1 from Theorem 2, we first claim that if G is $SL(n, \mathbf{R})$ or $SU(p, q)$ where $p + q = n$, there is a cocompact lattice $\Gamma \subset G$ and a dense embedding of Γ into $SU(n) \times SU(n)$. This follows via the standard arithmetic construction of cocompact lattices. (See, e.g., [2], [3], [5], [6].) For completeness, we indicate the construction for $SU(p, q)$. The case of $SL(n, \mathbf{R})$ is similar, but a bit more complicated. Let $p \in \mathbf{Q}[X]$ be a cubic with 3 real irrational roots a, b, c with $a < 0 < b, c$. Let k be the splitting field. We assume $[k : \mathbf{Q}] = 3$, and σ, τ the non-trivial elements of $\mathrm{Gal}(k/\mathbf{Q})$. Let B be the Hermitian form on $\mathbf{C}^n$ given by

$$B(z, w) = a\left(\Sigma_1^p z_i \overline{w}_i\right) + \Sigma_{p+1}^{p+q} z_i \overline{w}_i$$

Then $SU(B)$ can be identified with the set of real points of an algebraic group $\mathscr{G}$ defined over k, and as a Lie group it is ismorphic to $SU(p, q)$. Let $\mathscr{O} \subset k$ be the algebraic integers in k, and $\Gamma = \mathscr{G}_{\mathscr{O}} \subset SU(B)$. The real points of the twisted groups $\mathscr{G}^\sigma, \mathscr{G}^\tau$ will be identified with $SU(B^\sigma), SU(B^\tau)$ respectively, which are both isomorphic as Lie groups to $SU(n)$ since these forms are now positive definite, due to the fact $b, c > 0$. It follows from [3] (see also [6] for a discussion) that Γ is a lattice in $SU(B)$ and that $(\sigma, \tau){:}\Gamma \to SU(B^\sigma) \times SU(B^\tau)$ is a dense embedding.

Now choose T_k as in Theorem 2, and let M be the manifold C/T_k (which is independent of k.) For each k, let Γ act on M via the embedding in C, and the action of C on C/T_k. Since Γ is dense in C, if two of these Γ actions are conjugate in $\mathrm{Homeo}(M)$, then the C actions are conjugate in $\mathrm{Homeo}(M)$, which implies conjugacy of the corresponding tori. In fact, the same is true for a measurable conjugacy. Namely, the set of measurable maps (mod null sets) $C/T_1 \to C/T_2$ is a standard Borel space with the topology of convergence in measure. There is a natural Borel C action on this space (namely $(gf)(x) = gf(g^{-1}x)$), and the stabilizers of points in such actions are closed subgroups [6]. It follows that any Γ-map (which is a fixed point in this function space) is also a C-map. This map must then be a.e. equal to a continuous C-map. This establishes (i) of Theorem 1 and (ii) is obvious from the construction.

We now prove Theorem 2. Suppose more generally that $C = K \times L$ where K and L are simple. Let T be a torus in K and $\rho{:}T \to L$ a homomorphism. Then

$$T_\rho = \{(t, \rho(t) \in K \times L | t \in T\}$$

is a subgroup of C. Since T is abelian $\rho^*(t) = \rho(t)^{-1}$ is also a homomorphism and $E_\rho = G/T_\rho = K \times_T L, T$ acting on L via ρ^*, is an associated principal bundle to $q:K \to K/T$. The idea of the proof is to choose homomorphisms such that these bundles are equivalent. Also observe that automorphisms of C are of the form $\alpha = (\beta, \delta)$, where β and δ are automorphisms of K and L respectively if $K \neq L$, and such an automorphism composed with a permutation of the factors if $K = L$. This makes it easy to tell when two such groups are not equivalent. As an illustration, consider $K = SU(2), L = SU(n), T$ a maximal torus of K. Since $K/T = S^2$ and $\pi_2(BL) = 0$, where BL is the classifying space for L-bundles, every L-bundle over K/T is trivial. But if $\rho:SU(2) \to SU(n)$ is a non-trivial representation, then T_ρ is not equivalent to T (which corresponds to the trivial representation.)

We now take $K = L = SU(n)$ as in the statement of the theorem. Let T be a maximal torus in K, e.g., the set of diagonal matrices with entries $d_j = \exp(i\theta_j)$ satisfying $\Sigma\theta_j = 0$. The Weyl group W of $SU(n)$ is the group of permutations of the factors d_j. Let $\rho_k:T \to L$ be the homomorphism of T onto the maximal torus $T' = T$ of $L, \rho_k(t) = t^k, k = 1, 2, \ldots$. Note that if $w \in W$ and $\rho = \rho_k$ for some k, then $\rho w = w\rho$. Let $\lambda:K/T \to BT$ be the classifying map for $q:K \to K/T$, let $i:BT' \to BL$ be the map induced by the inclusion of T' in L, and let $B_\rho:BT \to BT'$ be the map induced by ρ. Then the classifying map for E_ρ is $f_\rho = iB_\rho\lambda$.

Now in [1] it is shown that $H^*(SU(n))$ and $H^*(SU(n)/T)$ have no torsion and that this implies that $i^*:H^*(BSU(n)) \to H^*(BT)$ is an isomorphism onto $H^*(BT)^W$, the fixed set under the action of the Weyl group. In particular, this implies $\lambda^*:H^*(BT)^W \to H^*(K/T)$ is trivial. But $\rho w = w\rho$ implies $(B\rho)^*:H^*(BT')^W \to H^*(BT)^W$. Hence $f_\rho^* = \lambda^* \cdot (B\rho)^* \cdot i^*$ is trivial. We claim this implies there are only finitely many equivalence classes of bundles E_ρ for $\rho = \rho_k, k = 1, 2, \ldots$

First note that $SU(n)/T$ is a finite CW complex whose cohomology has no torsion. We will use the following theorem of F. Peterson [4].

THEOREM. *Let X be a CW complex of dimension $\leq 2n$ such that $H^*(X)$ has no torsion. Then a complex vector bundle over X is trivial iff all its Chern classes are trivial.*

To apply this result to $SU(n)/T$ we first note the next result.

LEMMA. *Let X be a CW complex and $X^{(n)}$ its n-skeleton. If $H^*(X)$ has no torsion, then $H^*(X^{(n)})$ has no torsion.*

Our claim is an immediate consequence of the following:

PROPOSITION. *Let X be a 1-connected finite CW complex such that $H^*(X)$ has no torsion. Then there are only a finite number of equivalence classes of $SU(n)$ bundles X with all Chern classes zero.*

Proof. By Peterson's theorem and the above lemma, if $f: X \to BU(n)$ is such that $f^* H^*(BU(n)) \to H^*(X)$ is zero, then $f|X^{(2n)}$ is homotopically trivial. Let

$$d: BU(n) \to BU(1)$$

be induced by

$$\det: U(n) \to U(1),$$

so that the fibre of d is $BSU(n)$. If $f = j \cdot g$, $g: \mathrm{X} \to BSU$ and $j: BSU(n) \to BU(n)$ induced by the inclusion, so that df is trivial, then the homotopy of $f|X^{(2n)}$ to the trivial map gives a map of $\Sigma(X^{(2n)})$ to $BU(1)$. Since $\Sigma(X^{(2n)})$ is 2-connected, this last is homotopically trivial rel endpoints. Hence $g|X^{(2n)}$ is homotopically trivial. Since the homotopy of $g|X^{(2n)}$ extends to a homotopy of g, we can assume $g|X^{(2n)}$ is trivial. Since $\pi_i(BSU(n))$ is finite for $i > 2n, [X/X^{(2n)}, BSU(n)]$ is finite. Thus up to equivalence there are only finitely many $SU(n)$ bundles over X with all Chern classes zero.

Thus an infinite number of the E_ρ for $\rho = \rho_k, k = 1, 2, \ldots,$ are equivalent. On the other hand, if $T_k = T_\rho$ for $\rho = \rho_k$, no automorphism of C sends T_j to T_k if $j \neq k$. This completes the proof of Theorem 2.

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