

ON THE DUALITY OF THE UNIFORM APPROXIMATION PROPERTY IN BANACH SPACES

BY

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In this paper X will always denote a Banach space with norm $\|\cdot\|$, X^* its dual, I_X the identity on X . $k(n)$ will denote a function $N \rightarrow N$.

X is said to have the uniform approximation property (U.A.P.) with uniformity function (u.f.) $k(n)$ if there is a constant $K > 1$ such that, for any finite dimensional subspace E of X we can find an operator $T \in \mathcal{L}(X)$ satisfying $\|T\| \leq K$, $\text{rank } T \leq k(\dim E)$ and $T|_E = I_E$. U.A.P. was defined by A. Pełczyński and H. Rosenthal in [7] and it was shown by S. Heinrich in [5] that it is a self-dual property. Heinrich's proof is straightforward but the problem is that it does not give any estimate of the u.f. of X^* in terms of the u.f. of X . Before Heinrich's paper, self-duality of U.A.P. was established for superreflexive spaces by J. Lindenstrauss and L. Tzafriri [4] using a constructive proof which yields an estimate of the u.f. of X^* in terms of the u.f. of X and of the modulus of convexity of an equivalent norm on X . However, this estimate is too bad to be used in some applications. A much better estimate (but still worse than $\exp k(n)$) easily follows from an unpublished argument of Bourgain (see below). In the following pages we will show that a factorization trick can be used to deduce a very good asymptotic estimate for the u.f. of X^* (see Theorem 4). For asking the right questions, and for many stimulating discussions, I would like to thank Professors W.B. Johnson and G. Pisier.

Let us recall some definitions. Given Banach spaces X and Y , and an operator $T: X \rightarrow Y$, the 2-summing norm $\pi_2(T)$ of T is defined to be the smallest constant c satisfying

$$\left(\sum_i \|Tx_i\|^2 \right)^{1/2} \leq c \sup_{y \in B_{X^*}} \left(\sum_i (y, x_i)^2 \right)^{1/2}$$

for all finite sequences x_1, \dots, x_n in X .

Received September 15, 1989.

1980 Mathematics Subject Classification (1985 Revision). Primary 46B10; Secondary 46B25.

¹Supported by the Swiss National Research Foundation.

If T is as above and $k \in \mathbb{N}$, the k -th approximation number $a_k(T)$ of T is defined by

$$a_k(T) = \inf\{\|T - L\| : L \in \mathcal{L}(X, Y), \text{rank } L < k\}.$$

The main properties of the 2-summing norm and of the approximation numbers can be found in [8].

Let α be an operator ideal norm (in the sense of [8]). Then we say that X has the α -U.A.P. if there are a constant $K > 1$ and a function $k(n)$ such that, for any n -dimensional subspace E of X , we can find a finite rank operator $T \in \mathcal{L}(X)$ such that $\|T\| \leq K$, $\alpha(T) \leq k(n)$ and $T|_E = I_E$. Notice that $\|\cdot\|$ -U.A.P. is nothing but B.A.P. (bounded approximation property).

Finally, let us recall that X has the convex U.A.P. if there are a constant $K > 1$ and a function $k(n)$ such that, for any n -dimensional subspace E of X , we can find operators $(T_i)_{i=1}^m$, $m \in \mathbb{N}$, $T_i \in \mathcal{L}(X)$, such that $\|T_i\| \leq K$, $\text{rank } T_i \leq k(n)$, and such that some convex combination $\sum_i \gamma_i T_i$ extends the identity on E .

To avoid annoying repetitions, let us agree that a space X has $(K, k(n))$ -U.A.P. (resp. $(K, k(n))$ - α -U.A.P., $(K, k(n))$ -convex U.A.P.) if the desired U.A.P. property holds for a constant K and a u.f. $k(n)$.

It is immediately seen that $(K, k(n))$ -U.A.P. implies $(K, k(n))$ -convex U.A.P., which in turn implies $(K, Kk(n))$ - α -U.A.P. for any norm α (in fact, we always have $\alpha \leq \nu$, where ν is the so-called nuclear norm, and $\nu(I_E) = n$ for every n -dimensional Banach space E). Conversely, if one does not pay attention to the constants and u.f.'s involved, to show that α -U.A.P. implies U.A.P. you need (for example) ultrastability of α (see [8] for the definition) and a fixed m such that the quasi-norm $\beta = \alpha \circ \cdots \circ \alpha$ (m -times) defines an operator ideal consisting exclusively of operators which are approximable (with respect to β) by finite rank operators. For instance, π_2 will do the trick.

Some years ago, Bourgain [unpublished] proved that convex U.A.P. passes over to the dual with the same u.f. Actually, Bourgain proved that convex U.A.P. is equivalent to U.A.P. (with possibly different u.f.), and so his proof produces an estimate of the U.A.P.-u.f. of X^* (which still lies somewhere near $\exp[k(n) \log k(n)]$, where $k(n)$ is the convex U.A.P.-u.f. of X).

To obtain a better estimate, we are going to exploit an idea which appears in Bourgain's proof of the duality of convex U.A.P. (see Lemma 1). Afterwards, a factorization argument combined with π_2 -U.A.P. will give us what we need (Proposition 2) to prove the main Theorem. Lemma 1 is stated for general α -U.A.P. because we feel that this property has independent interest (at least for the 2-summing and nuclear norms). Given the ideal norm α , we write α^d for the dual norm, which is defined by $\alpha^d(T) = \alpha(T^*)$. Let us agree to say that an ideal norm α satisfies property (*) if the following holds:

(*) For any finite rank operator $u: X \rightarrow Y$ we have

$$\alpha(u) = \inf \alpha(u_F),$$

where the inf runs over all finite dimensional subspaces F of Y containing $u(X)$, and u_F is u considered as an operator $X \rightarrow F$.

It is clear that, if α is injective (in the sense of [8]), then α satisfies $(*)$ (this is the case for π_2 and we will use this later).

LEMMA 1. *Let α be an ideal norm satisfying property $(*)$. Then, if X^* has $(K, k(n))$ - α^d -U.A.P., X has $((1 + \varepsilon)K, (1 + \varepsilon)k(n))$ - α -U.A.P. for all $\varepsilon > 0$.*

Proof. Let α be an ideal norm satisfying $(*)$, and let X^* have $(K, k(n))$ - α^d -U.A.P. Fix an n -dimensional subspace E of X and an Auerbach basis $\{e_1, \dots, e_n\}$ of E , i.e., a basis such that

$$\max_i |\beta_i| \leq \left\| \sum_i \beta_i e_i \right\| \leq \sum_i |\beta_i|$$

for all scalars β_i . Fix $\varepsilon > 0$ and define

$$\mathcal{R} = \{T \in \mathcal{L}(X) : \|T\| \leq K(1 + \varepsilon)^{1/2}, \alpha(T) \leq (1 + \varepsilon)^{1/2}k(n), \text{rank } T < \infty\}$$

and

$$\mathcal{C} = \{(Te_i)_{i=1}^n : T \in \mathcal{R}\}.$$

We show first that $(e_i) \in \overline{\mathcal{C}}$, where the closure is taken in $l_\infty^n(X)$. In fact, supposing that $(e_i) \notin \overline{\mathcal{C}}$, by the convexity of \mathcal{C} we can find (z_i) in the unit ball of $l_1^n(X^*)$ such that

$$\sum_i (z_i, Te_i) < \sum_i (z_i, e_i) \quad \forall T \in \mathcal{R}. \tag{1}$$

Now, let $S \in \mathcal{L}(X^*)$ be a finite rank operator such that $\|S\| \leq K$, $\alpha^d(S) = \alpha(S^*) \leq k(n)$ and $Sz_i = z_i$ for all $i = 1, \dots, n$. Since α has property $(*)$, we can find a finite dimensional subspace F of X^{**} containing $S^*(X^{**})$ and such that, if S_F is S^* considered as an operator $X^{**} \rightarrow F$, we have

$$\alpha(S_F) \leq (1 + \varepsilon)^{1/4} \alpha(S^*).$$

By local reflexivity, let $J:F \rightarrow X$ be an isomorphism such that $\|J\| \leq (1 + \varepsilon)^{1/4}$ and $(z_i, Jf) = (z_i, f)$ for all $i = 1, \dots, n$ and all $f \in F$. In particular, we have $(z_i, JS^*e_i) = (z_i, S^*e_i)$ for all $i = 1, \dots, n$, since $S^*(X^{**}) \subset F$. Further, if we define $T_0 = JS_F|_X$, we see that

$$\begin{aligned} \alpha(T_0) &\leq (1 + \varepsilon)^{1/4} \alpha(S_F) \\ &\leq (1 + \varepsilon)^{1/2} \alpha(S^*) \\ &\leq (1 + \varepsilon)^{1/2} k(n), \end{aligned}$$

and $\|T_0\| \leq (1 + \varepsilon)^{1/4}\|S\|$. Consequently, $T_0 \in \mathcal{R}$ but, since T_0 satisfies

$$\sum_i (z_i, T_0 e_i) = \sum_i (z_i, e_i),$$

we have a contradiction with (1).

Having proved that $(e_i) \in \overline{\mathcal{L}}$, for an arbitrarily small δ we can find an operator $T \in \mathcal{R}$ such that

$$\max_{1 \leq i \leq n} \|Te_i - e_i\| \leq \frac{\delta}{n^{3/2}}.$$

The system $\{e_1, \dots, e_n\}$ was chosen to be Auerbach, and so it is easy to see that we actually have $\|T|_E - I_E\| \leq \delta/n^{1/2}$. This means that $T|_E$ is invertible with inverse V satisfying $\|V\| \leq (1 - \delta/n^{1/2})^{-1}$. Let P be a projection from X onto $T(E)$ with $\|P\| \leq n^{1/2}$ (such a projection exists by [8, 28.2.6]) and define $T_{00} = (I_X - (I_X - V)P)T$. We have then

$$\|I_X - (I_X - V)P\| \leq 1 + \frac{\delta}{1 - \delta/n^{1/2}}.$$

Choose δ such that the quantity on the right hand side of the last inequality does not exceed $(1 + \varepsilon)^{1/2}$. Then we get (recalling that $T \in \mathcal{R}$)

$$\begin{aligned} \|T_{00}\| &\leq (1 + \varepsilon)^{1/2}\|T\| \leq (1 + \varepsilon)K, \\ \alpha(T_{00}) &\leq (1 + \varepsilon)^{1/2}\alpha(T) \leq (1 + \varepsilon)k(n) \end{aligned}$$

and it is immediately verified that $T_{00}|_E = I_E$. Q.E.D.

Note that Lemma 1 can be seen as a generalization of Grothendieck's theorem stating that if X^* has B.A.P. then X has B.A.P. (just let α be the usual operator norm). This also shows that X and X^* in Lemma 1 cannot be interchanged in general: for this to hold you need (for instance) an ultra-stable norm α such that α -U.A.P. already implies U.A.P. (see above).

PROPOSITION 2. *If X has $(K, k(n))$ - π_2 -U.A.P., then X has*

$$\left(\frac{1}{1 - 1/m} \left(K^{1+m} + \frac{1}{m} \right), m^{2/m} k(n)^{2+2/m} \right)\text{-U.A.P.}$$

for all integers $m > 1$.

Proof. Fix $m > 1$, let E be an n -dimensional subspace of X and let $T \in \mathcal{L}(X)$ satisfy $\|T\| \leq K$, $\pi_2(T) \leq k(n)$ and $T|_E = I_E$. By the Pietsch

Factorization Theorem [8, 17.3.7], there are a probability measure μ on a compact space S and operators $A \in \mathcal{L}(X, C(S))$, $B \in \mathcal{L}(C(S), L_2(\mu, S))$, $C \in \mathcal{L}(L_2(\mu, S), X)$ such that $T = CBA$ and $\|A\| = \|B\| = 1$, $\|C\| = \pi_2(T)$. Since the approximation numbers form a nonincreasing sequence and satisfy the inequality $a_k(PQR) \leq \|P\|a_k(Q)\|R\|$ for all operators P, Q, R , we get

$$\begin{aligned} \sup_i i^{m/2} a_i(T^{m+1}) &\leq \left(\sum_i a_i(T^{m+1})^{2/m} \right)^{m/2} \\ &\leq \|C\| \left\| (a_i((BAC)^m))_i \right\|_{l_{2/m}} \|BA\| \\ &= \pi_2(T) \left\| (a_i((BAC)^m))_i \right\|_{l_{2/m}}. \end{aligned} \tag{2}$$

Since BAC is in $\mathcal{L}(L_2(\mu, S))$ and since the second factor of the right hand side of the above inequality is nothing but the norm of $(BAC)^m$ in the Schatten class $C_{2/m}$, the Hölder inequality gives

$$\begin{aligned} \left\| (a_i((BAC)^m))_i \right\|_{l_{2/m}} &= c_{2/m}((BAC)^m) \\ &\leq c_2(BAC)^m \\ &= \pi_2(BAC)^m \\ &\leq \pi_2(B)^m \|A\|^m \|C\|^m \\ &\leq \pi_2(T)^m \\ &\leq k(n)^m. \end{aligned} \tag{3}$$

Putting (2) and (3) together we see that, if we take i_0 to be the smallest integer strictly greater than $m^{2/m}k(n)^{2+2/m}$, we have $a_{i_0}(T^{m+1}) < 1/m$, which means that we can find an operator $L \in \mathcal{L}(X)$ such that $\text{rank } L < i_0$ and

$$\|T^{m+1} - L\| < 1/m.$$

The operator $I_X - T^{m+1} + L$ is invertible with inverse V , $\|V\| < (1 - 1/m)^{-1}$. Consequently, if we define $T_0 = VL$, we have

$$\begin{aligned} \|T_0\| &\leq \|V\|(\|T^{m+1}\| + \|T^{m+1} - L\|) \leq (1 - 1/m)^{-1}(K^{m+1} + 1/m), \\ \text{rank } T_0 &\leq m^{2/m}k(n)^{2+2/m} \end{aligned}$$

and

$$T_0|_E = I_E. \tag{Q.E.D.}$$

Since $(K, k(n))$ -convex U.A.P. implies $(K, Kk(n)^{1/2})$ - π_2 -U.A.P. (recall that $\pi_2(I_E) = n^{1/2}$ if E is n -dimensional [8,28.2.4]), we have the following immediate.

COROLLARY 3. *If X has $(K, k(n))$ -convex U.A.P., then X has*

$$\left(\frac{1}{1 - 1/m} \left(K^{1+m} + \frac{1}{m} \right), m^{2/m} K^{2+2/m} k(n)^{1+1/m} \right)\text{-U.A.P.}$$

for all integers $m > 1$.

Now we are able to state and prove the main theorem:

THEOREM 4. *If X has $(K, k(n))$ -U.A.P., then X^* has U.A.P. with uniformity function $c_\delta k(n)^{1+\delta}$ for any $\delta > 0$. More precisely, X^* has*

$$\left(\frac{1}{1 - 1/m} \left([(1 + \varepsilon)K]^{1+m} + \frac{1}{m} \right), m^{2/m} [(1 + \varepsilon)K]^{2+2/m} k(n)^{1+1/m} \right)\text{-U.A.P.}$$

for all $\varepsilon > 0$ and all integers $m > 1$.

Proof. Let X have $(K, k(n))$ -U.A.P. Then, by [4], X^{**} has $(K, k(n))$ -U.A.P., too, and, in particular, X^{**} has $(K, Kk(n)^{1/2})$ - π_2^d -U.A.P. Since π_2 satisfies property (*) (see the remark above Lemma 1), Lemma 1 implies that X^* has

$$\left((1 + \varepsilon)K, (1 + \varepsilon)Kk(n)^{1/2} \right)\text{-}\pi_2\text{-U.A.P.}$$

for all $\varepsilon > 0$ and, finally, Proposition 2 concludes the proof. Q.E.D.

The nice dualization properties of convex U.A.P. and of α -U.A.P. together with Theorem 4 suggest the following conjecture:

CONJECTURE. *If X has U.A.P. with u.f. $k(n)$, then X^* has U.A.P. with u.f. $ck(n)$ for some constant c .*

It is interesting to note that the conjecture holds in the case when $k(n)$ has linear growth. In fact, in [3] we can find a refinement of an argument in [6] yielding that linear U.A.P. for X implies that X is a weak Hilbert space (as a matter of fact, in [3] it is also proved that, conversely, all weak Hilbert spaces have linear U.A.P., but we won't need this in the sequel). Given this result, we proceed as follows: X has linear U.A.P., and thus linear convex U.A.P.

Hence, X^* has linear convex U.A.P. and, a fortiori, (K, cn) - ν -U.A.P. for some c . Since X^* is a weak Hilbert space, we have $\sup_k ka_k(T) \leq c'\nu(T)$ for all $T \in \mathcal{L}(X^*)$ and some c' , and so an easy approximation and perturbation argument produces linear U.A.P. in X^* , as desired. At the moment, the above idea does not seem to generalize to higher growth rates of $k(n)$.

Let us conclude with an easy application of Theorem 4, which answers a question asked in [1]:

COROLLARY 5. *For any $K > 1$ let $k_1(n, K)$ be the smallest $k(n)$ such that L^1 has $(K, k(n))$ -U.A.P. Then there exist constants c_1, c_2 (depending on K) such that*

$$\exp(c_1 n) \leq k_1(n, K) \leq \exp(c_2 n).$$

Proof. By [2, Lemma 17], any $C(K)$ -space has the

$$\left((1 - \varepsilon)^{-1}, \frac{1}{2} \left(\frac{2}{\varepsilon} + 1 \right)^n \right)\text{-U.A.P.}$$

for all $\varepsilon > 0$. By Theorem 4, this shows that any $C(K)$ -predual has $(K, \exp cK)$ -U.A.P. for some constant c depending on K and for all $K > 1$. On the other hand, an exponential lower bound for $k_1(n, K)$ has been obtained in [1]. Q.E.D.

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