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QUASIDIAGONAL OPERATOR ALGEBRAS

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Quasitriangular and quasidiagonal operators on Hilbert space were first introduced by Halmos in [Hal1], [Hal2] and quasitriangular operators in particular have played an especially important role in operator theory since that time (e.g., [DP], [AFV]). The natural extension of these concepts to quasitriangular and quasidiagonal algebras first appeared in [Ar] and then in [FAM]. Again, quasitriangular algebras have proved to be very important, especially in the area of similarity of nest algebras [An], [L], [D2]. On the other hand, very little is known about quasidiagonal algebras. Several interesting results appeared in [Pl] for the one particular case in which the quasidiagonal algebra is defined by a nest of finite-dimensional projections $\{P_n\}$ which increase to the identity operator. One result was that such a quasidiagonal algebra is always larger than the commutant of the nest modulo the compact operators. Thus, while a quasitriangular operator is the sum of a triangular operator and a compact operator, a quasidiagonal operator is not always the sum of a diagonal operator and a compact. In Theorem 10, we extend this result to all quasidiagonal algebras: if $\mathcal{QD}(\mathcal{N})$ is the quasidiagonal algebra determined by the infinite nest of projections $\mathcal{N}$, then $\mathcal{QD}(\mathcal{N})$ is always larger than $\mathcal{N}' + \mathcal{K}$, where $\mathcal{N}'$ is the commutant of the nest and $\mathcal{K}$ is the set of compact operators.

Even though a quasidiagonal operator is not necessarily diagonal plus compact, Halmos showed that it is nevertheless block diagonal plus compact. We also extend this result to arbitrary quasidiagonal algebras (Proposition 13), and this implies that there is an index obstruction for membership in quasidiagonal algebras defined by certain purely atomic nests, and it is the same as the obstruction for $\mathcal{N}' + \mathcal{K}$. In addition, we give an index obstruction for $\mathcal{N}' + \mathcal{K}$ if $\mathcal{N}$ is a continuous nest (Theorem 14), but it is not clear if the same is true for the corresponding quasidiagonal algebra. It does lead, however, to an alternate characterization of $\mathcal{N}' + \mathcal{K}$ in the continuous case (Theorem 15).

Another result in [Pl] is that the essential commutant of the quasidiagonal algebra considered there is $\mathbf{C}I + \mathcal{K}$. This provides an example of a nonseparable unital C^*-subalgebra of the Calkin algebra which does not equal its

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double commutant (by Voiculescu's reflexivity theorem [V, Theorem 1.8], this is not true if such an algebra is separable). We show, however, that the essential commutant is $\mathbf{C}I + \mathscr{K}$ for essentially only this one case. We also show that for a large class of nests, the essential commutant of the corresponding quasidiagonal algebra $\mathscr{Q}\mathscr{D}(\mathscr{N})$ is $C^*(\mathscr{N}) + \mathscr{K}$ (Corollary 23).

In this paper, $\mathscr{L}(\mathscr{H})$ will denote the set of bounded operators on a complex, separable, infinite dimensional Hilbert space $\mathscr{H}$, and the set of compact operators will be given by $\mathscr{K}(\mathscr{H})$, or just $\mathscr{K}$ if the Hilbert space is clear from the context. If $\mathscr{S}$ is a subset of $\mathscr{L}(\mathscr{H})$, then

$$\mathscr{S}' = \{T \in \mathscr{L}(\mathscr{H}) : TS = ST \text{ for all } S \in \mathscr{S}\}$$

is the *commutant* of $\mathscr{S}$, and $\mathscr{S}''$ denotes the *double commutant* $(\mathscr{S}')'$ of $\mathscr{S}$. The *essential commutant* of $\mathscr{S}$ is

$$\mathscr{S}_e' = \{T \in \mathscr{L}(\mathscr{H}) : TS - ST \in \mathscr{K} \text{ for all } S \in \mathscr{S}\}.$$

All projections on Hilbert space will be self-adjoint in this paper.

A *nest* $\mathscr{N}$ is a set of projections on $\mathscr{H}$ which is linearly ordered with respect to range inclusion, contains 0 and I, and is closed in the strong operator topology. A nest equipped with the strong operator topology is a compact separable complete metric space.

$$\operatorname{Alg} \mathscr{N} = \{T \in \mathscr{L}(\mathscr{H}) : P^{\perp}TP = 0 \text{ for all } P \in \mathscr{N}\}$$

is the *nest algebra* of $\mathscr{N}$, i.e., the set of all operators for which each projection in $\mathscr{N}$ is invariant. $\operatorname{Alg} \mathscr{N} + \mathscr{K}$ is the *quasitriangular algebra* of $\mathscr{N}$, denoted $\mathscr{Q}\mathscr{T}(\mathscr{N})$. Note that $(\operatorname{Alg} \mathscr{N}) \cap (\operatorname{Alg} \mathscr{N})^* = \mathscr{N}'$. On the other hand, $\mathscr{Q}\mathscr{T}(\mathscr{N}) \cap \mathscr{Q}\mathscr{T}(\mathscr{N})^*$ is the *quasidiagonal algebra* of $\mathscr{N}$, denoted $\mathscr{Q}\mathscr{D}(\mathscr{N})$. $\mathscr{Q}\mathscr{T}(\mathscr{N})$ is norm-closed [FAM, §1, Corollary 2], so $\mathscr{Q}\mathscr{D}(\mathscr{N})$ is a C^*-algebra which clearly contains $\mathscr{N}' + \mathscr{K}$. One objective of this paper is to show that if $\mathscr{N}$ is infinite, then $\mathscr{Q}\mathscr{D}(\mathscr{N}) \supsetneq \mathscr{N}' + \mathscr{K}$.

A projection E is an *atom* of $\mathscr{N}$ if $E = P_2 - P_1$ for some $P_1, P_2 \in \mathscr{N}$, $P_1 < P_2$, and there is no $P \in \mathscr{N}$ with $P_1 < P < P_2$. If $\mathscr{N}$ has no atoms, then $\mathscr{N}$ is *nonatomic*, or *continuous*. $\mathscr{N}$ is *purely atomic* if $\sum E_i = I$, where the sum is taken over all atoms E_i of $\mathscr{N}$ and convergence is in the strong operator topology. The operator $\delta_{\mathscr{N}}$ is defined by $\delta_{\mathscr{N}}(T) = \sum E_i T E_i$, again summing over all atoms and converging strongly. Note that if $\mathscr{N}$ is purely atomic, then $T \in \mathscr{N}'$ if and only if $T = \delta_{\mathscr{N}}(T)$. If $\mathscr{N}$ is continuous, then $\mathscr{N}$ can be parametrized by the unit interval, $\mathscr{N} = \{N_t : 0 \leq t \leq 1\}$, by defining $N_t = (\rho|_{\mathscr{N}})^{-1}(t)$ for some faithful normal state ρ on $\mathscr{L}(\mathscr{H})$.

Finally, $\mathscr{N}^{\perp}$ denotes the nest $\{P^{\perp} : P \in \mathscr{N}\}$. Note that $\mathscr{Q}\mathscr{T}(\mathscr{N}^{\perp}) = \mathscr{Q}\mathscr{T}(\mathscr{N})^*$, $\mathscr{Q}\mathscr{D}(\mathscr{N}^{\perp}) = \mathscr{Q}\mathscr{D}(\mathscr{N})$, and $\mathscr{N}' = (\mathscr{N}^{\perp})'$.

THEOREM 1 [FAM, Theorem 2.3]. *$T \in \mathcal{QT}(\mathcal{N})$ if and only if* (a) *$P^{\perp}TP \in \mathcal{K}$ for all $P \in \mathcal{N}$ and* (b) *the function $P \in \mathcal{N} \to P^{\perp}TP \in \mathcal{K}$ is continuous with respect to the strong operator topology on $\mathcal{N}$ and the norm topology on $\mathcal{K}$.*

The following characterization of $\mathcal{QD}(\mathcal{N})$ is an easy consequence.

PROPOSITION 2. *$T \in \mathcal{QD}(\mathcal{N})$ if and only if* (a) *$TP - PT \in \mathcal{K}$ for all $P \in \mathcal{N}$ and* (b) *the function $P \in \mathcal{N} \to TP - PT \in \mathcal{K}$ is continuous with respect to the strong operator topology on $\mathcal{N}$ and the norm topology on $\mathcal{K}$.*

Proof. If $T \in \mathcal{QD}(\mathcal{N})$, then $T \in \mathcal{QT}(\mathcal{N})$ and $T \in \mathcal{QT}(\mathcal{N})^* = \mathcal{QT}(\mathcal{N}^{\perp})$. Therefore, Theorem 1 implies that $P^{\perp}TP$ and $PTP^{\perp}$ are in $\mathcal{K}$ for all $P \in \mathcal{N}$ and the maps $P \to P^{\perp}TP$ and $P \to PTP^{\perp}$ are strong-norm continuous. Now (a) and (b) follow since $TP - PT = P^{\perp}TP - PTP^{\perp}$.

If (a) and (b) hold, then $P^{\perp}TP = P^{\perp}(TP - PT)P \in \mathcal{K}$ for all $P \in \mathcal{N}$. Now suppose $P_n \to P$ strongly. Then

$$\begin{aligned}\|P^{\perp}TP - P_n^{\perp}TP_n\| \le\ & \|P^{\perp}(TP - PT)(P - P_n)\| \\ & + \|(P^{\perp} - P_n^{\perp})(TP - PT)P_n\| \\ & + \|P_n^{\perp}((TP - PT) - (TP_n - P_nT))P_n\|.\end{aligned}$$

The first two terms converge to zero since $TP - PT$ is compact and $P_n \to P$ strongly [W, Lemma 1.8], and the last term converges to zero by property (b). Therefore, $T \in \mathcal{QT}(\mathcal{N})$ by Theorem 1. A similar estimate with $\|PTP^{\perp} - P_nTP_n^{\perp}\|$ shows that $T \in \mathcal{QT}(\mathcal{N}^{\perp}) = \mathcal{QT}(\mathcal{N})^*$. $\square$

A consequence of the last proposition is that $\mathcal{N}' + \mathcal{K} \subseteq \mathcal{QD}(\mathcal{N}) \subseteq \mathcal{N}_e' = C^*(\mathcal{N})_e'$. Now by [JP, Theorem 2.1], $\mathcal{N}' + \mathcal{K}$ is the essential commutant of $\mathcal{N}''$, and if $\mathcal{N}$ is a finite nest, then $\mathcal{N}'' = C^*(\mathcal{N})$, so the following result is immediate.

PROPOSITION 3. *If $\mathcal{N}$ is a finite nest, then $\mathcal{QD}(\mathcal{N}) = \mathcal{N}' + \mathcal{K}$.*

If $\mathcal{N}$ is a nest and Q is a projection in $\mathcal{N}'$, then $Q\mathcal{N}|_{Q\mathcal{H}}$ is also a nest, denoted $\mathcal{N}_Q$. $\mathcal{N}_Q$ is called an *induced nest* of $\mathcal{N}$.

LEMMA 4. *Suppose $\mathcal{N}$ is a nest and Q is a projection in $\mathcal{N}'$. Let T be an operator in $\mathcal{L}(Q\mathcal{H})$, and define $\tilde{T} \in \mathcal{L}(\mathcal{H})$ by $\tilde{T} = T \oplus 0$ with respect to the decomposition $\mathcal{H} = Q\mathcal{H} \oplus Q^{\perp}\mathcal{H}$. Then $\tilde{T} \in \mathcal{QD}(\mathcal{N})$ if and only if $T \in \mathcal{QD}(\mathcal{N}_Q)$, and $\tilde{T} \in \mathcal{N}' + \mathcal{K}$ if and only if $T \in (\mathcal{N}_Q)' + \mathcal{K}(Q\mathcal{H})$.*

Proof. If $\tilde{T} \in \mathcal{QD}(\mathcal{N})$, then $\tilde{T} - L \in \operatorname{Alg} \mathcal{N}$ for some compact operator L. Then $QLQ \in \mathcal{K}(Q\mathcal{H})$, and

$$QP^{\perp}(T - QLQ)QP = QP^{\perp}(Q\tilde{T}Q - QLQ)QP = QP^{\perp}(\tilde{T} - L)PQ = 0$$

for all $P \in \mathcal{N}$, so $T \in \mathcal{QT}(\mathcal{N}_Q)$. By interchanging P and $P^{\perp}$, we also have $T \in \mathcal{QT}(\mathcal{N}_Q^{\perp}) = \mathcal{QT}(\mathcal{N}_Q)^*$, so $T \in \mathcal{QD}(\mathcal{N}_Q)$. Conversely, if $T \in \mathcal{QD}(\mathcal{N}_Q)$, then $T - K \in \operatorname{Alg}(\mathcal{N}_Q) \subseteq \mathcal{L}(Q\mathcal{H})$ for some $K \in \mathcal{K}(Q\mathcal{H})$. Let $\tilde{K} = K \oplus 0$ with respect to the decomposition $Q\mathcal{H} \oplus Q^{\perp}\mathcal{H}$. Then

$$P^{\perp}(\tilde{T} - \tilde{K})P = P^{\perp}(Q\tilde{T}Q - Q\tilde{K}Q)P = QP^{\perp}(T - K)QP = 0$$

for all $P \in \mathcal{N}$, so $\tilde{T} \in \mathcal{QT}(\mathcal{N})$. Again, interchanging P and $P^{\perp}$ yields $\tilde{T} \in \mathcal{QT}(\mathcal{N})^*$, so $T \in \mathcal{QD}(\mathcal{N})$. The proof for $\mathcal{N}' + \mathcal{K}$ is similar. □

The following lemma is a generalization of [Pl, Proposition 4].

LEMMA 5. *Suppose $\mathcal{N}$ is purely atomic. Then $T \in \mathcal{N}' + \mathcal{K}$ if and only if $T - \delta_{\mathcal{N}}(T) \in \mathcal{K}$.*

Proof. Suppose $T = S + K$ where $S \in \mathcal{N}'$ and $K \in \mathcal{K}$. Then

$$\delta_{\mathcal{N}}(T) = \delta_{\mathcal{N}}(S) + \delta_{\mathcal{N}}(K) = S + \delta_{\mathcal{N}}(K) = T - K + \delta_{\mathcal{N}}(K),$$

so $T - \delta_{\mathcal{N}}(T) = K - \delta_{\mathcal{N}}(K)$. But $\delta_{\mathcal{N}}(K) \in \mathcal{K}$ by [FAM, Lemma 2], so $T - \delta_{\mathcal{N}}(T) \in \mathcal{K}$. The converse is clear since $\delta_{\mathcal{N}}(T) \in \mathcal{N}'$. □

We can now use this lemma to generalize [Pl, Lemma 17]. First, we say that $\mathcal{N}$ is an *order*-N *nest* if it has the form $\mathcal{N} = \{0, P_n, I: 1 \le n < \infty\}$ with $P_n < P_{n+1}$ for all n and $P_n \to I$ strongly. An operator S is *strictly upper triangular* with respect to $\mathcal{N}$ if $P_{n-1}^{\perp} SP_n = 0$ for all n. Similarly, S is *strictly lower triangular* if $P_n SP_{n-1}^{\perp} = 0$ for all n.

LEMMA 6. *If $\mathcal{N}$ is an order-*N *nest, then there is a strictly upper triangular noncompact operator R^{++}, a strictly lower triangular operator R^{--}, and a compact operator D such that $R = R^{++} + D + R^{--}$ is compact.*

Proof. With a small modification, the argument is the same as in the proof of [Pl, Lemma 17]. We will include the details for completeness. For each n, choose a unit vector $e_n \in (P_n - P_{n-1})\mathcal{H}$. Let

$$\mathcal{H}_i = \operatorname{span}\{e_n: i^2 < n \le (i+1)^2\} \quad \text{for } i = 1, 2, \ldots.$$

If $S \in \mathscr{L}(\mathscr{H}_i)$, written as a matrix with respect to the basis $\{e_{i^2+1}, \ldots, e_{(i+1)^2}\}$, let S^+ denote the matrix obtained by replacing the entries below the main diagonal with zeros, and denote the map $S \to S^+$ by $+_i$. $\| +_i \| \to \infty$ since $\dim \mathscr{H}_i \to \infty$ [D1, Example 4.1], so there are operators $S_i \in \mathscr{L}(\mathscr{H}_i)$ such that $\|S_i\| = \| +_i \|^{-1}$ and $1 - 1/i < \|S_i^+\| \leq 1$. Define $R_i, R_i^+ \in \mathscr{L}((P_{(i+1)^2} - P_{i^2})\mathscr{H})$ by $R_i = S_i \oplus 0$ and $R_i^+ = S_i^+ \oplus 0$. Then $R = R_1 \oplus R_2 \oplus \cdots$ is compact, and thus so is $\delta_{\mathscr{N}}(R) = \delta_{\mathscr{N}}(R^+)$ by [FAM, Lemma 2], but $R^+ = R_1^+ \oplus R_2^+ \oplus \cdots$ is not compact. Now just let $D = \delta_{\mathscr{N}}(R)$, $R^{++} = R^+ - D$, and $R^{--} = R - R^+$. $\square$

Corollary 7. *If $\mathscr{N}$ is an order-**N** nest, then $\mathscr{Q}\mathscr{D}(\mathscr{N}) \supsetneq \mathscr{N}' + \mathscr{K}$.*

Proof. Let R, R^{++}, R^{--}, and D be the operators given by Lemma 6. Then

$$R^{++} \in \operatorname{Alg} \mathscr{N} \subseteq \mathscr{Q}\mathscr{T}(\mathscr{N})$$

and

$$R^{++} = -R^{--} + (R - D) \in \operatorname{Alg} \mathscr{N}^{\perp} + \mathscr{K} = \mathscr{Q}\mathscr{T}(\mathscr{N})^*,$$

so $R^{++} \in \mathscr{Q}\mathscr{D}(\mathscr{N})$. However, $R^{++} - \delta_{\mathscr{N}}(R^{++}) = R^{++}$ is not compact, so $R^{++} \notin \mathscr{N}' + \mathscr{K}$ by Lemma 5. $\square$

Remark. It would be interesting to have a concrete example of an operator in $\mathscr{Q}\mathscr{D}(\mathscr{N})$ which is not in $\mathscr{N}' + \mathscr{K}$. If we let $T_i = A_i^+$ be the operators in example 4.1 of [D1], then $A_i / \|T_i\|$ can be used in place of S_i in the proof of Lemma 6 to construct R^{++}. This would be a concrete example except that we don't know the norms $\|T_i\|$.

To show that $\mathscr{Q}\mathscr{D}(\mathscr{N}) \supsetneq \mathscr{N}' + \mathscr{K}$ in general, we first consider the case in which $\mathscr{N}$ has an infinite number of atoms. The set of atoms of $\mathscr{N}$ is linearly ordered by $E \prec F$ if $E = P_2 - P_1$ and $F = P_4 - P_3$ with $P_2 \leq P_3$ ($P_i \in \mathscr{N}$), so there must be either an increasing or decreasing sequence of atoms. By replacing $\mathscr{N}$ with $\mathscr{N}^{\perp}$ if necessary, we can assume that there is an increasing sequence of atoms $\{E_N\colon 1 \leq n < \infty\}$.

Theorem 8. *If $\mathscr{N}$ has an infinite number of atoms, then $\mathscr{Q}\mathscr{D}(\mathscr{N}) \supsetneq \mathscr{N}' + \mathscr{K}$.*

Proof. As indicated above, we can assume that $\mathscr{N}$ has an increasing sequence $\{E_n\colon 1 \leq n < \infty\}$ of atoms, i.e., $E_n = P_{2n} - P_{2n-1}$ with P_n in $\mathscr{N}$

and $P_{2n-1} < P_{2n} \leq P_{2n+1}$ for all n. Let

$$Q = \sum_{n=1}^{\infty} E_n \in \mathcal{N}''.$$

Then $\mathcal{N}_Q$ is an order-**N** nest consisting of the set of distinct projections

$$\{0, P'_n = QP_{2n}|_{Q\mathcal{H}}, I_{Q\mathcal{H}}: 1 \leq n < \infty\}.$$

Now Corollary 7 shows that there is an operator T in $\mathcal{Q}\mathcal{D}(\mathcal{N}_Q)$ but not in $(\mathcal{N}_Q)' + \mathcal{K}(Q\mathcal{H})$, and the result follows by Lemma 4. □

THEOREM 9. *If $\mathcal{N}$ is a continuous nest, then $\mathcal{Q}\mathcal{D}(\mathcal{N}) \supsetneq \mathcal{N}' + \mathcal{K}$.*

Proof. Choose an increasing sequence of projections $\{P_n\} \subseteq \mathcal{N}$ such that $P_n \to I$ strongly, and let $\mathcal{M}$ be the subnest $\{0 = P_0, P_n, I: 1 \leq n < \infty\}$. Let R, R^{++}, R^{--}, and D be the operators given by Lemma 6 for the nest $\mathcal{M}$. Now if $P \in \mathcal{N}$, $P \neq I$, then $P_{n-1} \leq P \leq P_n$ for some n, and therefore

$$P^{\perp}R^{++}P = P^{\perp}P^{\perp}{}_{n-1}R^{++}P_nP = 0,$$

so $R^{++} \in \operatorname{Alg} \mathcal{N}$. Similarly, $R^{--} \in \operatorname{Alg} \mathcal{N}^{\perp}$, so

$$R^{++} = -R^{--} + (R - D) \in \operatorname{Alg} \mathcal{N}^{\perp} + \mathcal{K} = \mathcal{Q}\mathcal{T}(\mathcal{N})^*.$$

Therefore, $R^{++} \in \mathcal{Q}\mathcal{D}(\mathcal{N})$. Also, $\mathcal{N}' + \mathcal{K} \subseteq \mathcal{M}' + \mathcal{K}$, and $R^{++} \notin \mathcal{M}' + \mathcal{K}$ by Lemma 5 since $R^{++} - \delta_{\mathcal{M}}(R^{++}) = R^{++}$ is not compact. □

THEOREM 10. *For any infinite nest $\mathcal{N}$, $\mathcal{Q}\mathcal{D}(\mathcal{N}) \supsetneq \mathcal{N}' + \mathcal{K}$.*

Proof. Theorems 8 and 9 cover the cases in which $\mathcal{N}$ either has an infinite number of atoms or is continuous. The only other possibility is that $\mathcal{N}$ has a finite positive number of atoms. Then since $\mathcal{N}$ is infinite, $\mathcal{N}$ must have a continuous interval. In other words, there are projections $P_1, P_2 \in \mathcal{N}$ with $P_1 < P_2$ such that $E(P_2 - P_1) = 0$ for every atom E of $\mathcal{N}$. Letting $Q = P_2 - P_1$, it follows that $\mathcal{N}_Q$ is a continuous nest on $Q\mathcal{H}$. Theorem 9 then implies that there is some operator $T \in \mathcal{Q}\mathcal{D}(\mathcal{N}_Q)$ which is not in $(\mathcal{N}_Q)' + \mathcal{K}(Q\mathcal{H})$, and the result follows from Lemma 4. □

PROPOSITION 11. *For any infinite nest $\mathcal{N}$, $(C^*(\mathcal{N}))'_e \supsetneq \mathcal{Q}\mathcal{D}(\mathcal{N})$.*

Proof. Let $\{P_n\}$ be an increasing or decreasing sequence of projections in $\mathcal{N}$. By replacing $\mathcal{N}$ with $\mathcal{N}^{\perp}$ if necessary, we can assume the sequence increases. Let $P_0 = 0$ and let $P = \bigvee P_n \in \mathcal{N}$, so $P_n \to P$ strongly. For each

n, let e_n be a unit vector in $(P_n - P_{n-1})\mathscr{H}$, and let V be the partial isometry defined by $Ve_n = e_{n+1}$ for all n. Then $\|P_n^\perp VP_n\| = 1$ for all n, but $P^\perp VP = 0$, so $V \notin \mathscr{QD}(\mathscr{N})$ by Proposition 2. Now if $Q \in \mathscr{N}$, then either $Q \geq P$, or $P_{n-1} \leq Q < P_n$ for some n. In the first case $QV - VQ = 0$, and in the second case

$$(QV - VQ)h = -\langle h, e_{n-1}\rangle Q^\perp e_n - \langle h, Qe_n\rangle e_{n+1} \qquad \text{for } h \in \mathscr{H},$$

so $QV - VQ$ is compact. Therefore, $V \in \mathscr{N}_e' = (C^*(\mathscr{N}))_e'$. □

One of the first results on quasidiagonal operators was given in [Hal2, p. 903], where it was shown that every quasidiagonal operator is the sum of a block diagonal operator and a compact. In terms of nests, if $\{0, P_n, I: 1 \leq n < \infty\}$ is an order-**N** nest with $\dim P_n < \infty$ for all n, then each operator $T \in \mathscr{QD}(\mathscr{N})$ has a compact perturbation which is *block diagonal* with respect to $\mathscr{N}$, i.e., there is a compact operator K and a subsequence P_{n_k} such that

$$(T - K)P_{n_k} = P_{n_k}(T - K)$$

for all k. Equivalently, an operator S is block diagonal if there is an infinite subnest $\mathscr{M}$ such that $S \in \mathscr{M}'$. The set of all such block diagonal operators is denoted by $\mathscr{BD}(\mathscr{N})$. Now if we remove the finite dimensional restriction, so $\mathscr{N}$ is a general order-**N** nest, then the same argument given in [Hal2] works in this case also because $TP_n - P_nT \in \mathscr{K}$ for all $T \in \mathscr{QD}(\mathscr{N})$ (by Proposition 2).

Proposition 12. *If $\mathscr{N}$ or $\mathscr{N}^\perp$ is an order-**N** nest, then $\mathscr{QD}(\mathscr{N}) \subsetneq \mathscr{BD}(\mathscr{N}) + \mathscr{K}$.*

Proof. Since $\mathscr{BD}(\mathscr{N}^\perp) = \mathscr{BD}(\mathscr{N})$, it is enough to prove the result if $\mathscr{N}$ is an order-**N** nest. From the above remarks, it only remains to show that the containment is strict. For each n, let V_n be a partial isometry with initial space in $(P_{2n-1} - P_{2n-2})\mathscr{H}$ (where $P_0 = 0$) and final space in $(P_{2n} - P_{2n-1})\mathscr{H}$, and let $V = V_1 \oplus V_2 \oplus \cdots$. Then $\|P_{2n-1}^\perp VP_{2n-1}\| = 1$ for all n, so $V \notin \mathscr{QD}(\mathscr{N})$ by Proposition 2, but V commutes with the subnest $\{0, P_{2n}, I\}$. □

The notion of a block diagonal operator does not generalize easily to arbitrary nests. After all, if $T \in \mathscr{QD}(\mathscr{N})$, then Proposition 3 shows that for any *finite* subnest $\mathscr{M}$ of $\mathscr{N}$, there is a compact perturbation of T which commutes with $\mathscr{M}$. The content of the last proposition is that there is an *infinite* subnest with this property. Now if $\mathscr{N}$ or $\mathscr{N}^\perp$ is order-**N**, then $\mathscr{N}$ has just one strong accumulation point, namely I or 0, respectively.

Because a nest is compact, every infinite nest must have at least one accumulation point, but there may be many, and they need not include 0 or I. However, at each accumulation point P, there must be either an increasing or decreasing sequence of projections in the nest which converges to P. With this in mind, we will generalize the notion of a block diagonal operator as follows.

DEFINITION. If $\mathcal{N}$ is an infinite nest and P is a strong accumulation point of $\mathcal{N}$, then an operator T is *block diagonal at P with respect to $\mathcal{N}$* if there is an increasing or decreasing sequence $\{P_n: 1 \leq n < \infty\} \subseteq \mathcal{N}$ such that $P_n \to P$ strongly and $TP_n = P_nT$ for all n. $\mathcal{BD}_P(\mathcal{N})$ denotes the set of all such block diagonal operators.

PROPOSITION 13. *If $\mathcal{N}$ is an infinite nest and P is an accumulation point of $\mathcal{N}$, then $\mathcal{QD}(\mathcal{N}) \subsetneq \mathcal{BD}_P(\mathcal{N}) + \mathcal{K}$. In fact, if $\mathcal{M}$ is any infinite subnest with accumulation point P, then $\mathcal{QD}(\mathcal{N}) \subsetneq \mathcal{BD}_P(\mathcal{M}) + \mathcal{K}$.*

Proof. Suppose that $\{P_n\}$ is a sequence of increasing projections in $\mathcal{M}$ such that $P_n \to P$ strongly. The argument for a decreasing sequence is similar. Let T be an operator in $\mathcal{QD}(\mathcal{N})$. Then $TP - PT \in \mathcal{K}$, so $P^\perp TP, PTP^\perp \in \mathcal{K}$ also. Now $PTP \in \mathcal{QD}(\mathcal{P})$, where $\mathcal{P}$ is the order-N nest $\{0, P_n, P = I|_{P\mathcal{H}}\}$ on $P\mathcal{H}$. It follows from Proposition 12 that $PTP = S + K$, where $S \in \mathcal{L}(P\mathcal{H})$ is block diagonal with respect to some subsequence $\{P_{n_k}\}$ and $K \in \mathcal{K}(P\mathcal{H})$. Considering S and K as operators on $\mathcal{H}$, we have

$$\begin{aligned} P_{n_k}(S + P^\perp TP^\perp) &= P_{n_k}P(S + P^\perp TP^\perp) = P_{n_k}PS = SPP_{n_k} \\ &= (S + P^\perp TP^\perp)PP_{n_k} = (S + P^\perp TP^\perp)P_{n_k} \end{aligned}$$

for all k. It follows that

$$T = (S + P^\perp TP^\perp) + (K + P^\perp TP + PTP^\perp) \in \mathcal{BD}_P(\mathcal{M}) + \mathcal{K}.$$

To show that the inclusion is strict, define an isometry V just as in the proof of Proposition 12, using the sequence $\{P_n\}$. □

If T is a semi-Fredholm operator, let $i(T)$ denote the Fredholm index of T. One of the uses of these results on block diagonal operators is the following: if $\mathcal{N}$ is an order-N nest with all atoms finite-dimensional, and T in $\mathcal{QD}(\mathcal{N})$ is semi-Fredholm, then $i(T) = 0$. This follows from Proposition 12 since each subnest $\mathcal{M}$ has finite-dimensional atoms and therefore $i(S) = 0$ for every semi-Fredholm operator S in $\mathcal{M}'$. Proposition 13 implies the same result if $\mathcal{N}$ is purely atomic with all atoms finite-dimensional and a finite number of accumulation points. Of course, if $\mathcal{N}$ has just one infinite-dimensional atom, then $\mathcal{N}'$, and therefore $\mathcal{QD}(\mathcal{N})$ also, has semi-Fredholm operators of arbitrary index. On the other hand, if $\mathcal{N}$ is continuous, then it turns out that there is again an index obstruction for membership in $\mathcal{N}' + \mathcal{K}$. We will use the notation $T \sim R$ to mean that T is unitarily equivalent to a compact perturbation of R, i.e., if $T \in \mathcal{L}(\mathcal{H}_T)$ and $R \in \mathcal{L}(\mathcal{H}_R)$, then $T \sim R$ if $T = U^*RU + K$ for some $K \in \mathcal{K}(\mathcal{H}_T)$ and unitary operator $U: \mathcal{H}_T \to \mathcal{H}_R$.

THEOREM 14. *Suppose $\mathcal{N}$ is continuous and T is a semi-Fredholm operator in $\mathcal{N}' + \mathcal{K}$. Then $i(T) = 0$, ∞, or $-\infty$.*

Proof. $T = S + K$, where $S \in \mathcal{N}'$ and $K \in \mathcal{K}$. It is well-known that since $\mathcal{N}$ is continuous, then $C^*(S) \cap \mathcal{K} = \{0\}$. To see this, first parametrize $\mathcal{N}$ by $\mathcal{N} = \{N_t\colon t \in [0,1]\}$, and note that if $C^*(S) \cap \mathcal{K} \neq \{0\}$, then $C^*(S)$ contains some nonzero finite rank projection E. But then the map $t \to EN_t$ is strongly continuous, so the map $t \to \operatorname{trace}(EN_t)$ is continuous also, a contradiction since $\operatorname{trace}(EN_0) = 0$ and $\operatorname{trace}(EN_1) = \operatorname{rank}(E)$. It now follows from Voiculescu's Theorem [V, Theorem 1.3] that

$$S \sim S \oplus S \oplus S \oplus \cdots \in \mathcal{L}(\mathcal{H} \oplus \mathcal{H} \oplus \mathcal{H} \oplus \cdots).$$

Thus, S and $S \oplus S \oplus \cdots$ must be semi-Fredholm and $i(T) = i(S) = i(S \oplus S \oplus \cdots)$. But this implies that either $\ker(S) = \{0\}$ or $\ker(S^*) = \{0\}$. If both are $\{0\}$, then $i(T) = i(S) = 0$. If $\dim(\ker(S)) > 0$, then $\dim(\ker(S \oplus S \oplus \cdots)) = \infty$, so $i(T) = i(S \oplus S \oplus \cdots) = \infty$. Similarly, if $\dim(\ker(S^*)) > 0$, then $i(T) = -\infty$. □

QUESTION. If $\mathcal{N}$ is a continuous nest, does $\mathcal{QD}(\mathcal{N})$ contain Fredholm operators with nonzero index? If so, this would give another proof that $\mathcal{QD}(\mathcal{N}) \supsetneq \mathcal{N}' + \mathcal{K}$ in the continuous case, by virtue of Theorem 14. We have been unable to answer this question, but the following example comes tantalizingly close, and is interesting in its own right.

Example. Let $\mathcal{N} = \{N_t\colon t \in [0,1]\}$ be a continuous nest on $\mathcal{H}$, and define $\mathcal{N}^\varepsilon$ for $0 \le \varepsilon \le 1$ to be the nest $\{0, N_t\colon t \ge \varepsilon\}$ (thus, $\mathcal{N}^0 = \mathcal{N}$). We will show that there is a Fredholm operator X with $i(X) = -1$ such that $X \in \mathcal{QD}(\mathcal{N}^\varepsilon)$ for all ε, $0 < \varepsilon \le 1$, but $X \notin \mathcal{QD}(\mathcal{N})$. First, let $\mathcal{H}_1 = L^2[0,1]$ with Lebesgue measure m, and define a projection P_t on $\mathcal{H}_1$ by $P_t f = \chi_{[0,t]} f$ for each t in $[0,1]$. Let $\mathcal{P}$ be the continuous nest $\{P_t\colon 0 \le t \le 1\}$ and $\mathcal{P}^\varepsilon = \{0, P_t\colon t \ge \varepsilon\}$, $0 \le \varepsilon \le 1$. By Andersen's Theorem [An, Theorem 3.5.5], there is a unitary operator $U\colon \mathcal{H} \to \mathcal{H}_1$ such that $U^*P_tU - N_t \in \mathcal{K}$ for all t and the map $t \to U^*P_tU - N_t$ is strong-norm continuous. Since the map $[0,1] \to (\mathcal{N}, \text{strong})$ by $t \to N_t$ is a homeomorphism, and the same is true for the map $t \to P_t$, it follows easily from Proposition 2 that

$$U^*\mathscr{D}\mathscr{D}(\mathscr{P})U = \mathscr{D}\mathscr{D}(\mathscr{N}) \quad \text{and} \quad U^*\mathscr{D}\mathscr{D}(\mathscr{P}^\varepsilon)U = \mathscr{D}\mathscr{D}(\mathscr{N}^\varepsilon).$$

Let $j \in \mathscr{H}_1$ be the constant function $j(x) = 1$, let Q be the projection onto span(j), and let $\mathscr{H}_2 = Q^\perp \mathscr{H}_1 \subseteq \mathscr{H}_1$. Then $\mathscr{Q} = \{Q_t = Q^\perp \wedge P_t \colon 0 \leq t \leq 1\}$ is a continuous nest on $\mathscr{H}_2$. By applying Andersen's Theorem again, there is unitary operator $V\colon \mathscr{H}_1 \to \mathscr{H}_2$ (i.e., an isometry in $\mathscr{L}(\mathscr{H}_1)$ whose range has codimension 1) such that the map $t \to V^*Q_tV - P_t \in \mathscr{K}(\mathscr{H}_1)$ is strong-norm continuous, and it follows that the map $P_t \to Q_tV - VP_t \in \mathscr{K}(\mathscr{H}_1)$ is strong-norm continuous. Note that V is a Fredholm operator with index -1. Now Proposition 2 shows that $V \in \mathscr{D}\mathscr{D}(\mathscr{P}^\varepsilon)$ if and only if the map $P_t \to P_tV - VP_t \in \mathscr{K}(\mathscr{H}_1)$ is strong-norm continuous for $\varepsilon \leq t \leq 1$, and in that case it would follow that $X = U^*VU$ is a Fredholm operator of index -1 in $\mathscr{D}\mathscr{D}(\mathscr{N}^\varepsilon)$.

Since $P_t \to Q_tV - VP_t \in \mathscr{K}(\mathscr{H}_1)$ is strong-norm continuous, the map $P_t \to P_tV - VP_t \in \mathscr{K}(\mathscr{H}_1)$ is strong-norm continuous if and only if the map $P_t \to P_tV - Q_tV \in \mathscr{K}(\mathscr{H}_1)$ is strong-norm continuous. But $P_tV - Q_tV$ can be computed directly. If $0 < t < 1$, let $\{e_n\colon 0 \leq n < \infty\}$ and $\{f_n\colon 0 \leq n < \infty\}$ be orthonormal bases for $P_t\mathscr{H}_1$ and $P_t^\perp\mathscr{H}_1$, respectively, with

$$e_0 = \frac{1}{\sqrt{t}}\chi_{[0,t]} \quad \text{and} \quad f_0 = \frac{1}{\sqrt{1-t}}\chi_{[t,1]}.$$

Thus, $j = \sqrt{t}\,e_0 + \sqrt{1-t}\,f_0$, and $\{g, e_n, f_n\colon 1 \leq n < \infty\}$ is an orthonormal basis for $\mathscr{H}_2 = V\mathscr{H}_1$, where $g = \sqrt{1-t}\,e_0 - \sqrt{t}\,f_0$. Let $h \in \mathscr{H}_2$. Then

$$\begin{aligned} P_th - Q_th = \langle h, g\rangle P_tg &= \left(\frac{1-t}{t}\left(\int_0^t h\,dm\right) - \left(\int_t^1 h\,dm\right)\right)\chi_{[0,t]} \\ &= \frac{1}{t}\left(\int_0^t h\,dm\right)\chi_{[0,t]} \end{aligned}$$

since $\int_0^1 h\,dm = 0$. Thus, $P_tV - Q_tV$ is a rank one operator, and an easy calculation shows that the map

$$t \to \frac{1}{t}\left(\int_0^t h\,dm\right)\chi_{[0,t]}$$

is continuous on the open interval $(0, 1)$ with respect to the norm topology on

$\mathscr{K}(\mathscr{H}_1)$. Also,

$$\begin{aligned}\|(P_1h - Q_1h) - (P_th - Q_th)\|_2 &= \left\|\frac{1}{t}\left(\int_0^t h\,dm\right)\chi_{[0,t]}\right\|_2 \\ &= \left\|\frac{1}{t}\left(\int_t^1 h\,dm\right)\chi_{[0,t]}\right\|_2 \\ &\le \frac{1}{t}\|h\|_2\sqrt{1-t}\,\sqrt{t} \to 0 \quad \text{as } t \to 1,\end{aligned}$$

so this map is continuous at 1 also. Thus, $P_t \to P_tV - Q_tV \in \mathscr{K}(\mathscr{H}_1)$ is strong-norm continuous on the interval $(0, 1]$. However, it is not continuous at 0. To see this, define a function h_t for each t in $(0, \frac{1}{2})$ by

$$h_t = \frac{1}{\sqrt{2t}}(\chi_{[0,t]} - \chi_{[1-t,1]}).$$

Then $h_t \in \mathscr{H}_2$ and $\|h_t\|_2 = 1$, but

$$\|(P_th_t - Q_th_t) - (P_0h_t - Q_0h_t)\|_2 = \left\|\frac{1}{t}\left(\int_0^t h_t\,dm\right)\chi_{[0,t]}\right\|_2 = \frac{1}{\sqrt{2}}. \qquad \square$$

In the proof of Theorem 14, it is noted that if $T \in \mathscr{N}' + \mathscr{K}$, then T has a compact perturbation S with the property that $C^*(S) \cap \mathscr{K} = \{0\}$. It is interesting to note that the converse is also true.

THEOREM 15. *If $C^*(S) \cap \mathscr{K} = \{0\}$, then there is a continuous nest $\mathscr{N}$ such that $S \in \mathscr{N}' + \mathscr{K}$.*

Proof. By Voiculescu's Theorem, $S \sim S \oplus S \oplus \cdots$. Let $\tilde{\mathscr{H}} = \int_X^\oplus \mathscr{H}\,dm$ be the direct integral of $\mathscr{H}$ over X, where $X = [0, 1]$ and m is Lebesgue measure. Then $S \oplus S \oplus S \oplus \cdots$ is unitarily equivalent to $\tilde{S} = \int_X^\oplus S(x)\,dm$, where $S(x) = S$ for all $x \in X$ [Had, Lemma D]. For each $t \in [0, 1]$, define a projection

$$P_t = \int_X^\oplus \chi_{[0,t]}(x) I(x)\,dm$$

on $\tilde{\mathscr{H}}$, where $I(x) = I|_{\mathscr{H}}$ for each x. Then $\mathscr{M} = \{P_t\colon 0 \le t \le 1\}$ is a continuous nest and clearly $P_t\tilde{S} = \tilde{S}P_t$ for all t. Writing $S = U^*\tilde{S}U + K$, with U unitary and K compact, it follows that $S \in (U^*\mathscr{M}U)' + \mathscr{K}$. Now just let $\mathscr{N} = U^*\mathscr{M}U$. $\square$

The Weyl–von Neumann–Berg Theorem states that if N is a normal operator, then there is an order-$\mathbf{N}$ nest $\mathscr{N}$, with each atom one-dimensional, such that $N = D + K$ for some $D \in \mathscr{N}'$ and some compact operator K [B]. The last theorem provides us with the following continuous analogue (and, since the theorem is a direct consequence of Voiculescu's Theorem, this gives another indication why the title of [V] is appropriate).

COROLLARY 16. *If N is a normal operator, then $N = D + K$, where K is compact and D is in $\mathscr{N}'$ for some continuous nest $\mathscr{N}$.*

Proof. We can assume that N is not compact. Let Ω be an at most countable dense subset of the essential spectrum of N, and let D be a diagonal operator with point spectrum equal to Ω and such that each eigenvalue has infinite multiplicity. Then $N \sim D$ since their essential spectra are the same [Pe, Corollary 2.13], and $C^*(D) \cap \mathscr{K} = \{0\}$. Now apply Theorem 15 to D. $\square$

We turn now to essential commutants of quasidiagonal algebras. If $\mathscr{N}$ is an order-$\mathbf{N}$ nest with all atoms finite-dimensional, then it was shown in [Pl, Theorem 20] that $\mathscr{Q}\mathscr{D}(\mathscr{N})'_e = \mathbf{C}I + \mathscr{K}$. As noted in that paper, this provides a nonseparable selfadjoint counterexample to Voiculescu's reflexivity theorem for separable subalgebras of the Calkin algebra [V, Theorem 1.8]. There are other nonselfadjoint examples; for instance, $\mathbf{C}I + \mathscr{K}$ is the essential commutant of both $\operatorname{Alg}\mathscr{N}$ and $\mathscr{Q}\mathscr{T}(\mathscr{N})$ for every nest [CP]. In the quasidiagonal case, however, it is easy to see that this property is the exception rather than the rule. Before proving this, note that since $\mathscr{N}' + \mathscr{K} \subseteq \mathscr{Q}\mathscr{D}(\mathscr{N}) \subseteq C^*(\mathscr{N})'_e$ and $\mathscr{N}'' + \mathscr{K} = (\mathscr{N}')'_e$ by [JP], it follows that

$$\mathscr{N}'' + \mathscr{K} = (\mathscr{N}' + \mathscr{K})'_e \supseteq \mathscr{Q}\mathscr{D}(\mathscr{N})'_e \supseteq (C^*(\mathscr{N})'_e)'_e \supseteq C^*(\mathscr{N}) + \mathscr{K}$$
$$= \overline{\operatorname{span}(\mathscr{N})} + \mathscr{K}.$$

Note that if $\mathscr{N}$ is countable, then $(C^*(\mathscr{N})'_e)'_e = C^*(\mathscr{N}) + \mathscr{K}$ by [V, Theorem 1.8]. In the following lemma, it is easy to see that $Q\mathscr{Q}\mathscr{D}(\mathscr{N})Q$ and $\mathscr{Q}\mathscr{D}(\mathscr{N}_Q)$ are the same sets except that $Q\mathscr{Q}\mathscr{D}(\mathscr{N})Q$ is technically a subset of $\mathscr{L}(\mathscr{H})$ and $\mathscr{Q}\mathscr{D}(\mathscr{N}_Q)$ is a subset of $\mathscr{L}(Q\mathscr{H})$. Similar remarks apply to $(Q\mathscr{Q}\mathscr{D}(\mathscr{N})Q)'_e$ and $\mathscr{Q}\mathscr{D}(\mathscr{N}_Q)'_e$.

LEMMA 17. *If $T \in \mathscr{Q}\mathscr{D}(\mathscr{N})'_e$ and Q is a projection in $\mathscr{N}'$, then T, $QTQ \in (Q\mathscr{Q}\mathscr{D}(\mathscr{N})Q)'_e$, and therefore QTQ, viewed as an operator on $Q\mathscr{H}$, is in $\mathscr{Q}\mathscr{D}(\mathscr{N}_Q)'_e$.*

Proof. $T \in (Q\mathscr{Q}\mathscr{D}(\mathscr{N})Q)'_e$ since $Q\mathscr{Q}\mathscr{D}(\mathscr{N})Q \subseteq \mathscr{Q}\mathscr{D}(\mathscr{N})$, so $TQCQ - QCQT \in \mathscr{K}$ for all $C \in \mathscr{Q}\mathscr{D}(\mathscr{N})$, and therefore $QTQCQ - QCQTQ \in \mathscr{K}$ for all $C \in \mathscr{Q}\mathscr{D}(\mathscr{N})$. □

PROPOSITION 18. *$\mathscr{Q}\mathscr{D}(\mathscr{N})'_e = \mathbf{C}I + \mathscr{K}$ if and only if either* (a) *$\mathscr{N}$ is finite with exactly one infinite-dimensional atom, or* (b) *$\mathscr{N}$ is countable with exactly one left or one right accumulation point and all atoms finite-dimensional.*

Proof. If $\mathscr{Q}\mathscr{D}(\mathscr{N})'_e = \mathbf{C}I + \mathscr{K}$, then $C^*(\mathscr{N}) + \mathscr{K} = \mathbf{C}I + \mathscr{K}$ also by the remarks preceding Lemma 17, and this implies that either (a) or (b) must hold. On the other hand, if (a) holds, then $C^*(\mathscr{N}) = \mathscr{N}''$ and $\mathbf{C}I + \mathscr{K} = C^*(\mathscr{N}) + \mathscr{K}$, which yields the desired result. If (b) holds, then either $\mathscr{N}$ or $\mathscr{N}^{\perp}$ has the form

$$0 = P_0 < P_1 < P_2 < \cdots < Q_0 < Q_1 < \cdots < Q_m = I$$

with $P_n \to Q_0$ strongly and $\dim(P_n - P_{n-1})\mathscr{H}$, $\dim(Q_n - Q_{n-1})\mathscr{H} < \infty$ for all possible n. Now if $T \in \mathscr{N}'' \cap \mathscr{Q}\mathscr{D}(\mathscr{N})'_e$, then $Q_0TQ_0 \in \mathscr{Q}\mathscr{D}(\mathscr{N}_{Q_0})'_e$ by Lemma 17. Thus, the problem is reduced to the order-$\mathbf{N}$ case with finite-dimensional atoms, the case proved in [Pl], and it follows from that result that

$$Q_0TQ_0 \in \mathbf{C}I|_{Q_0\mathscr{H}} + \mathscr{K} = \mathbf{C}Q_0 + \mathscr{K},$$

and therefore $T \in \mathbf{C}I + \mathscr{K}$. □

This proposition can also be obtained from Corollary 21 below.

Observe that if $T \in \mathscr{N}''$ and E is an atom of $\mathscr{N}$, then $TE = ET = ETE = \lambda E$ for some $\lambda \in \mathbf{C}$.

THEOREM 19. *Suppose $T \in \mathscr{N}'' \cap \mathscr{Q}\mathscr{D}(\mathscr{N})'_e$, $\{E_i\}$ is any increasing or decreasing* (*with respect to $\prec$*) *sequence of atoms of $\mathscr{N}$, and $\{\lambda_i\} \subseteq \mathbf{C}$ is the sequence defined by $TE_i = \lambda_i E_i$. Then $\{\lambda_i\}$ converges.*

Proof. Suppose there is such a sequence $\{E_i\}$ with $\{\lambda_i\}$ not convergent. $\{\lambda_i\}$ is bounded since T is a bounded operator, so there are at least two cluster points α and β of the sequence. Then there is a subsequence $\{\lambda_{i_j}\}$ such that $\lambda_{i_{2k-1}} \to \alpha$ and $\lambda_{i_{2k}} \to \beta$. Let $\alpha_k = \lambda_{i_{2k-1}}$, $F_k = E_{i_{2k-1}}$, $\beta_k = \lambda_{i_{2k}}$, $G_k = E_{i_{2k}}$, $A = \Sigma(\alpha_k - \alpha)F_k$, and $B = \Sigma(\beta_k - \beta)G_k$. Each F_k and G_k is in $C^*(\mathscr{N})$, so $A, B \in C^*(\mathscr{N})$. Let $S = T - A - B$. We will show that $S \notin \mathscr{Q}\mathscr{D}(\mathscr{N})'_e$, so neither is T, a contradiction.

Assume that $S \in \mathscr{Q}\mathscr{D}(\mathscr{N})'_e$, and let $Q = \Sigma F_k + \Sigma G_k$. By Lemma 17,

$$QSQ = \alpha \sum F_k + \beta \sum G_k \in (Q\mathscr{Q}\mathscr{D}(\mathscr{N})Q)'_e.$$

For each k, let f_k and g_k be unit vectors in $F_k\mathscr{H}$ and $G_k\mathscr{H}$, respectively, let P be the projection onto $\operatorname{span}\{g_k\}$, and let V be the partial isometry with initial space $P\mathscr{H}$ and final space $\operatorname{span}\{f_k\}$ defined by $Vg_k = f_k$. Let P_k be the projection onto $\operatorname{span}\{g_l\colon 1 \leq l \leq k\}$, and note that

$$\mathscr{N}_Q = \{0, Q_1 = F_1, Q_2 = F_1 + G_1, Q_3 = F_1 + G_1 + F_2, \ldots, Q\}$$

and

$$\mathscr{N}_P = \{0, P_k, P\colon 1 \leq k < \infty\}$$

are order-**N** nests. By Lemma 6, there is a strictly upper triangular operator $R^{++} \in \mathscr{Q}\mathscr{D}(\mathscr{N}_P)$ which is not compact, and an operator R^{--} which is strictly lower triangular with respect to $\mathscr{N}_P$ such that $R^{++} + R^{--} \in \mathscr{K}$. Then VR^{++}, viewed as an operator on $Q\mathscr{H}$, is not compact, and it lies in $\operatorname{Alg}(\mathscr{N}_Q)$ since it is strictly upper triangular with respect to $\mathscr{N}_Q$. On the other hand, $-VR^{--}$ is a compact perturbation of VR^{++} and is strictly lower triangular with respect to $\mathscr{N}_Q$, so it follows that $VR^{++} \in \mathscr{Q}\mathscr{D}(\mathscr{N}_Q)$. Now view VR^{++} as an operator on $\mathscr{H}$, so VR^{++} is in $\mathscr{Q}\mathscr{D}(\mathscr{N})$ by Lemma 4, and therefore is in $Q\mathscr{Q}\mathscr{D}(\mathscr{N})Q$. But then

$$QSQVR^{++} - VR^{++}QSQ = (\alpha - \beta)VR^{++}$$

is not compact, contradicting Lemma 17. □

COROLLARY 20. *If $\mathscr{N}$ or $\mathscr{N}^{\perp}$ is an order-*$\mathbf{N}$ *nest, then* $\mathscr{Q}\mathscr{D}(\mathscr{N})'_e = C^*(\mathscr{N}) + \mathscr{K}$.

Proof. By replacing $\mathscr{N}$ with $\mathscr{N}^{\perp}$ if necessary, we can assume $\mathscr{N}$ is order-**N**. Let $\{E_i\}$ be the atoms of $\mathscr{N}$, with $E_i \prec E_{i+1}$ for all i. Now if $T \in \mathscr{N}'' \cap \mathscr{Q}\mathscr{D}(\mathscr{N})'_e$, then $T = \Sigma\lambda_i E_i$ with $\lambda_i \to$ some λ by Theorem 19. But then $T - \lambda I = \Sigma(\lambda_i - \lambda)E_i \in C^*(\mathscr{N})$, so $T \in C^*(\mathscr{N})$ also. □

COROLLARY 21. *If $\mathscr{N}$ is a nest which has a finite number of accumulation points, then*

$$\mathscr{Q}\mathscr{D}(\mathscr{N})'_e = C^*(\mathscr{N}) + \mathscr{K}.$$

Proof. From the hypothesis, there are projections $0 < P_1 < P_2 < \cdots < P_n < I$ in $\mathscr{N}$ such that $\mathscr{N}_{P_k - P_{k-1}}$ or $\mathscr{N}^{\perp}_{P_k - P_{k-1}}$ is order-**N** for each k. Suppose $T \in \mathscr{N}'' \cap \mathscr{Q}\mathscr{D}(\mathscr{N})'_e$. Then Lemma 17 and Corollary 20 imply that

$$T(P_k - P_{k-1}) = (P_k - P_{k-1})T(P_k - P_{k-1}) \in C^*\left(\mathscr{N}_{P_k - P_{k-1}}\right)$$

for each k. The result follows since $C^*(\mathcal{N}_{P_k - P_{k-1}}) \subseteq C^*(\mathcal{N})$ and $T = \Sigma\, T(P_k - P_{k-1})$. $\square$

COROLLARY 22. *If $\mathcal{N}$ is a nest whose accumulation points form an increasing or decreasing sequence, then $\mathcal{QD}(\mathcal{N})'_e = C^*(\mathcal{N}) + \mathcal{K}$.*

Proof. By replacing $\mathcal{N}$ with $\mathcal{N}^\perp$ if necessary, we can assume that the accumulation points form an increasing sequence. Let $P_1 < P_2 < \cdots$ be the nonzero accumulation points, and let $P_0 = 0$. If $\mathcal{N}_{P_1 - P_0}$ is finite, define $A = P_1 - P_0$ and otherwise $A = 0$. Let $P = \bigvee P_k$ and $B = P - A$. Now if $T \in \mathcal{N}'' \cap \mathcal{QD}(\mathcal{N})'_e$, then $T = ATA + BTB + P^\perp\, TP^\perp$, and $ATA, P^\perp\, TP^\perp \in C^*(\mathcal{N})$ since $\mathcal{N}_A$ and $\mathcal{N}_{P^\perp}$ are finite. Thus, it is enough to show that $BTB \in C^*(\mathcal{N})$. Equivalently, we can assume that $P = I$ and $\mathcal{N}_{P_1 - P_0}$ is infinite.

Now for each k, either $\mathcal{N}_{P_k - P_{k-1}}$ is an order-**N** nest, $\mathcal{N}^\perp_{P_k - P_{k-1}}$ is order-**N**, or else $\mathcal{N}_{P_k - P_{k-1}}$ is order isomorphic to the extended integers. If the last case holds, choose any $Q_k \in \mathcal{N}$ with $P_{k-1} < Q_k < P_k$. Let $\mathcal{S} = \{Q_k\} \cup \{P_k\}$. $\mathcal{S}$ forms an increasing sequence, so its projections can be relabeled by $0 = R_0 < R_1 < R_2 < \cdots$, with $I = \bigvee R_k$. Now $\mathcal{N}_{R_k - R_{k-1}}$ or $\mathcal{N}^\perp_{R_k - R_{k-1}}$ is order-**N** for each $k \geq 1$. As in the proof of Corollary 21, it follows that if $T \in \mathcal{N}'' \cap \mathcal{QD}(\mathcal{N})'_e$, then $T(R_k - R_{k-1}) \in C^*(\mathcal{N}_{R_k - R_{k-1}})$ for each k.

For each k, let $F_k = R_k - R_{k-1}$ and let $\{E_{kl}\colon 1 \leq l < \infty\}$ be the atoms of $\mathcal{N}_{F_k}$. Define $\mu_k = \lim_{l\to\infty} \lambda_{kl}$, where $TE_{kl} = \lambda_{kl} E_{kl}$. The limit exists by Theorem 19. Now for each k, choose an atom $E_{kl_k} \in \{E_{kl}\}$. No matter how $\{E_{kl_k}\}$ is chosen, Theorem 19 implies that λ_{kl_k} converges as $k \to \infty$. It follows that there is a sequence $\{\varepsilon_k\}$ of positive real numbers such that $\varepsilon_k \to 0$ and $|\lambda_{kl} - \mu_k| < \varepsilon_k$ for all k and l. In addition, the sequence $\{\mu_k\}$ must converge to some μ. Now let $S = T - \mu I$. Then $SF_k \in C^*(\mathcal{N}_{F_k}) \subseteq C^*(\mathcal{N})$ for each k and $\|SF_k\| < \varepsilon_k$, so $S = \Sigma SF_k \in C^*(\mathcal{N})$ and thus $T \in C^*(\mathcal{N})$. $\square$

COROLLARY 23. *Suppose that $\mathcal{N}$ is a nest with subnests $\mathcal{N}_1 \subsetneq \mathcal{N}_2 \subsetneq \cdots \subsetneq \mathcal{N}_m = \mathcal{N}$ such that*

(a) *$\mathcal{N}_1$ is finite, and*

(b) *for each atom E of $\mathcal{N}_j$, $j < m$, either $(\mathcal{N}_{j+1})_E$ or $(\mathcal{N}^\perp_{j+1})_E$ is order-***N***. Then $\mathcal{QD}(\mathcal{N})'_e = C^*(\mathcal{N}) + \mathcal{K}$.*

Proof. Note that $\mathcal{N}_1$ may just be the trivial nest $\{0, I\}$. We will just sketch the proof, as it is simply a generalization of the previous corollary. Let $T \in \mathcal{N}'' \cap \mathcal{QD}(\mathcal{N})'_e$, and first suppose that F is an atom of $\mathcal{N}_{m-2}$. Then Corollary 22 implies that $TF \in C^*(\mathcal{N}_F)$. But now we can apply the same argument as in the proof of Corollary 22 to show that if A is an atom of $\mathcal{N}_{m-3}$, then $TA \in C^*(\mathcal{N}_A)$. More specifically, let $\{F_k\}$ be the atoms of $(\mathcal{N}_{m-2})_A$, and choose atoms E_k of $\mathcal{N}$ with $E_k < F_k$. Let $\{\lambda_k\}$ be defined by

$TE_k = \lambda_k E_k$. Then the sequence $\{\lambda_k\}$ converges to some μ, and μ is independent of the choice of the sequence $\{E_k\}$. Now let $S = T - \mu I$. It follows that $SF_k \in C^*(\mathcal{N}_{F_k}) \subseteq C^*(\mathcal{N}_A)$ for each k, and $\|SF_k\| \to 0$, so $S = \Sigma SF_k \in C^*(\mathcal{N}_A)$ and thus $T \in C^*(\mathcal{N}_A)$.

The remainder of the proof simply repeats this same argument: next show that $T \in C^*(\mathcal{N}_A)$ for each atom A of $\mathcal{N}_{m-4}$, then $T \in C^*(\mathcal{N}_A)$ for each atom A of $\mathcal{N}_{m-5}$, etc. Finally, we will have that $T \in C^*(\mathcal{N}_A)$ for each atom A of $\mathcal{N}_1$. But $\mathcal{N}_1$ is finite and $C^*(\mathcal{N}_A) \subseteq C^*(\mathcal{N})$, so $T \in C^*(\mathcal{N})$. □

Note that a nest which satisfies the hypotheses of Corollary 23 can be quite complicated, so this is a substantial generalization of [Pl, Theorem 20]. Also, if $\mathcal{N}$ is such a nest, then

$$(\mathcal{D}\mathcal{D}(\mathcal{N})'_e)'_e = C^*(\mathcal{N})'_e,$$

so $\mathcal{D}\mathcal{D}(\mathcal{N}) \neq (\mathcal{D}\mathcal{D}(\mathcal{N})'_e)'_e$ by Proposition 11. It follows that $\mathcal{D}\mathcal{D}(\mathcal{N})$ is a nonseparable selfadjoint counterexample of [V, Theorem 1.8] in this case as well.

Question. Is there a nest $\mathcal{N}$ for which $\mathcal{D}\mathcal{D}(\mathcal{N})'_e \neq C^*(\mathcal{N}) + \mathcal{K}$? As we have seen, for purely atomic nests Theorem 19 imposes a very strong restriction on membership in $\mathcal{D}\mathcal{D}(\mathcal{N})'_e$.

Addendum. Kenneth R. Davidson has shown that the question following Theorem 14 has a negative answer [D3]. Along with Theorem 14, this shows that $\mathcal{D}\mathcal{D}(\mathcal{N})$ and $\mathcal{N}' + \mathcal{K}$ have the same index properties if $\mathcal{N}$ is a continuous nest.

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