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SOME REMARKS ON COMPLEX POWERS OF $(-\Delta)$ AND UMD SPACES

BY

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Introduction and notations

If X is a Banach space, $(\Omega, \mathscr{A}, \mu)$ a measure space and $1 \leq p < +\infty$, we will denote by $L_p(\Omega, X)$ ($L_p(\Omega)$ if $X = \mathbf{R}$), the Banach space of classes of Bochner measurable functions f from Ω to X such that

$$\int_\Omega \|f(t)\|_X^p \, d\mu(t) < +\infty,$$

equipped with the norm

$$\|f\|_p = \int_\Omega \|f(t)\|_X^p \, d\mu(t)^{1/p}.$$

We will also denote by $C_0^\infty(\mathbf{R}, X)$ ($C_0^\infty(\mathbf{R})$ if $X = \mathbf{R}$) the space of C^∞-functions from $\mathbf{R}$ to X such that $\lim_{t \to \pm\infty} \|f(t)\| = 0$, equipped with the norm

$$\|f\|_\infty = \sup\{\|f(t)\|_X, t \in \mathbf{R}\}.$$

We recall that X is UMD if martingale differences with values in X converge unconditionally in $L_2(\Omega, X)$ where Ω is any probability space, that is: there exists a constant $C > 0$, such that whenever $(M_k)_{k \in \mathbf{N}}$ is a bounded martingale in $L_2(\Omega, X)$ and $(\varepsilon_k)_{k \in \mathbf{N}}$ is a choice of signs,

$$\left\| \sum_{k=1}^{\infty} \varepsilon_k d_k \right\|_2 \leq C \left\| \sum_{k=1}^{\infty} d_k \right\|_2 \quad \text{where } d_{k+1} = M_{k+1} - M_k.$$

By a martingale, we mean that there exists an increasing sequence of σ-subalgebras $(\mathscr{A}_k)_{k \in \mathbf{N}}$ of $\mathscr{A}$ such that $E^{\mathscr{A}_k}[M_{k+1}] = M_k$, where $E^{\mathscr{A}_k}$ is the conditional expectation with respect to $\mathscr{A}_k$. It is well known that this

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condition is equivalent to the ζ-convexity of X and also to the fact that the X-valued Hilbert transform $\mathscr{H} \otimes \mathrm{Id}_X$ is a bounded operator on $L_2(\mathbf{R}, X)$. These results were proved by D. Burkholder [Bu] and J. Bourgain [B_1].

We will denote by Δ the Laplace operator on $C_0^\infty(\mathbf{R})$. We will use the well known fact that Δ is a convolution operator and the Fourier transform of its distribution kernel K is $K(x) = -x^2$ on $\mathbf{R}$.

Here we are interested in the operator $(-\Delta)^{is}$ where $s \in \mathbf{R}$: in agreement with theory of complex powers of operator [K], we will define this operator as the convolution by the kernel K_s, such that $\hat{K}_s(x) = (x^2)^{is}$ on $\mathbf{R}$. We know by results of E. Stein [S_1], [S_2] or of R. Edwards and G. Gaudry [EG], that this operator is bounded on $L_p(\mathbf{R})$ for all $p \in (1, +\infty)$. As a consequence of T. McConnel in [B_2] or J. Bourgain [M], it is easy to see that if X is UMD, then $(-\Delta)^{is} \otimes \mathrm{Id}_X$ is a bounded operator on $L_p(\mathbf{R}, X)$ for all $p \in (1, +\infty)$ and $s \in \mathbf{R}$.

Using techniques introduced in [B_1], we are going to prove an inverse property.

Main result

THEOREM. *Let $1 < p < \infty$ and X be a Banach space. If $(-\Delta)^{is} \otimes \mathrm{Id}_X$ is a bounded operator on $L_p(\mathbf{R}, X)$ for all $s \in \mathbf{R}$, then X is a UMD space.*

Proof. First of all, we can suppose that $p = 2$ (by using the results of T. Coulhon and D. Lamberton [CL]).

Then, it is shown in [V] that, under the hypothesis of the theorem,

$$s \to (-\Delta)^{is} \otimes \mathrm{Id}_X$$

is a strongly continuous group and thus the norm of $(-\Delta)^{is} \otimes \mathrm{Id}_X$ is uniformly bounded for s in compact subsets of $\mathbf{R}$.

We are going to work with the scalar multiplier $(x^2)^{is}$, $x \in \mathbf{R}$ on $L_2(\mathbf{R}, X)$. By the usual transference techniques developed by R. Coifman and G. Weiss in [CW] which are applicable in the vector valued setting as well by results of J. Bourgain [B_2], if $\mathbf{T}$ denotes the torus, we know that the discrete multiplier $((n^2)^{is})_{n \in \mathbf{Z}}$ is bounded on $L_2(\mathbf{T}, X)$.

By changing s to $s/2$ to simplify the notation we can work with the multiplier $m_s(n) = |n|^{is}$ and suppose that its norm is less than A for all $s \in [-1, +1]$. That means that if

$$f(\theta) = \sum_{j \in \mathbf{Z}} \lambda_j e^{ij\theta} \in L_2(\mathbf{T}, X)$$

and

$$m_s f(\theta) = \sum_{j \in \mathbf{Z}} m_s(j) \lambda_j e^{ij\theta},$$

then

$$\|m_s f\|_{L_2(\mathbf{T}, X)} \le A \|f\|_{L_2(\mathbf{T}, X)} \quad \text{for } s \in [-1, +1].$$

To prove that X is UMD, we are going to consider, as in [B_1], bounded X-valued martingales $(M_k)_{k \in \mathbf{N}}$ on $\mathbf{T}^{\mathbf{N}}$, associated with the filtration induced by the coordinates, defined by the inductive rule

$$M_{k+1}(\theta_1, \dots, \theta_{k+1}) = M_k(\theta_1 \cdots \theta_k) + \phi_k(\theta_1, \dots, \theta_k) \varphi_k(\theta_{k+1})$$

(so that $d_{k+1}(\theta_1, \dots, \theta_{k+1}) = \phi_k(\theta_1, \dots, \theta_k) \varphi_k(\theta_{k+1})$) with

$$(\theta_1, \dots, \theta_{k+1}) \in \mathbf{T}^{k+1},$$
$$\phi_k \in L_2(\mathbf{T}^k, X),$$
$$\varphi_k \in L_\infty(\mathbf{T}), \quad \int_{\mathbf{T}} \varphi_k(t)\, dt = 0,$$
$$\left\| \sum_{k=1}^{\infty} d_k \right\|_{L_1(\mathbf{T}^{\mathbf{N}}, X)} < +\infty.$$

By an approximation argument, we can assume first that $d_k = 0$ for $k > k_0$ and second that the ϕ_k- and φ_k-functions are respectively X-valued and $\mathbf{R}$-valued trigonometrical polynomials, namely, for $k \le k_0$,

$$\begin{cases} \phi_k(\theta_1, \dots, \theta_k) = \displaystyle\sum_{|j_1| \le L_{k,1}} \cdots \sum_{|j_k| \le L_{k,k}} a_{j_1 \cdots j_k} e^{ij_1\theta_1} \cdots e^{ij_k\theta_k} \\ \varphi_k(\theta) = \displaystyle\sum_{|j| \le K_k} b_j e^{ij\theta}, \quad b_0 = 0 \end{cases}$$

where

$$K_k, L_{k,j} \in \mathbf{N} \quad \text{for } 1 \le j \le k, 1 \le k \le k_0,$$
$$a_{j_1 \cdots j_k} \in X \quad \text{for } |j_1| \le L_{k,1}, \dots, |j_k| \le L_{k,k},$$
$$b_j \in \mathbf{R} \quad \text{for } |j| \le K_k, \quad b_0 = 0.$$

Then, with this notation we have to show that there exists a constant C

(independent of k_0) such that for all choices of signs $(\varepsilon_k)_{k\in\mathbf{N}}$, we have

$$\left\| \sum_{k=1}^{k_0} \varepsilon_k d_{k+1}(\theta_1 \cdots \theta_{k+1}) \right\|_{L_2(\mathbf{T}^{\mathbf{N}}, X)} \le C \left\| \sum_{k=1}^{k_0} d_{k+1}(\theta_1 \cdots \theta_{k+1}) \right\|_{L_2(\mathbf{T}^{\mathbf{N}}, X)}$$

If f is a trigonometric polynomial on $\mathbf{T}$, defined by

$$f(\theta) = \sum_{|j|\le L} \lambda_j e^{ij\theta}$$

we will denote by $\mathrm{sp}(f)$, the set of integers j such that $\lambda_j \neq 0$.

We are going to use Bourgain's transform [B1]: For ψ in $\mathbf{T}$, and for a monotone increasing $\mathbf{N}$-valued sequence $(N_k)_{k\in\mathbf{N}}$, we can define

$$F(\psi) = \sum_{k=1}^{k_0} \phi_k(\theta_1 + N_1\psi, \dots, \theta_k + N_k\psi)\varphi_k(\theta_{k+1} + N_{k+1}\psi) \in L_2(\mathbf{T}, X),$$

$$f_k(\psi) = \phi_k(\theta_1 + N_1\psi, \dots, \theta_k + N_k\psi)\varphi_k(\theta_{k+1} + N_{k+1}\psi) \in L_2(\mathbf{T}, X),$$

$$S_k = \sum_{j=1}^{k} N_j L_{k,j} \in \mathbf{N}.$$

Since $b_0 = 0$, with this notation, we get

$$\mathrm{sp}(f_k) \subset [-S_k - N_{k+1}K_k, S_k - N_{k+1}] \cup [-S_k + N_{k+1}, S_k + N_{k+1}K_k].$$

The aim is to prove that we can choose $s \in \mathbf{R}$ and an increasing sequence $(N_k)_{k\in\mathbf{N}}$ of integers such that the multiplier $m_s(n) = |n|^{is}$, $n \in \mathbf{Z}$, acts almost like a given choice of sign ε_k on each f_k.

LEMMA 1. *Let $\delta_k > 0$, $s > 0$, $\varepsilon_k = \pm 1$ and choose $\varepsilon_k = e^{ip_k\pi}$ with $p_k \in \mathbf{N}$. Then*

$$e^{(p_k\pi - \delta_k)/s} \le |n| \le e^{(p_k\pi + \delta_k)/s} \Rightarrow \left| |n|^{is} - \varepsilon_k \right| \le \delta_k.$$

Proof of Lemma 1. It is an easy application of the inequalities

$$\left| |n|^{is} - \varepsilon_k \right| = |e^{is\log|n|} - e^{ip_k\pi}| \le |p_k\pi - s\log|n||$$

and

$$|p_k\pi - s\log|n|| \le \delta_k \Leftrightarrow e^{(p_k\pi - \delta_k)/s} \le |n| \le e^{(p_k\pi + \delta_k)/s}.$$

LEMMA 2. *There exist* $s \in (0,1]$, *two increasing* **N**-*valued sequences* $(N_k)_{k \in \mathbf{N}}$ *and* $(p_k)_{k \in \mathbf{N}}$ *and a decreasing non-negative sequence* $(\delta_k)_{k \in \mathbf{N}}$, *converging to* 0 *such that*

$$\delta_{k+1} \leq \frac{\varepsilon}{2^{k+1}} \frac{1}{\|f_k\|_{L_2(\mathbf{T},X)}}, \tag{1}$$

$$e^{ip_{k+1}\pi} = \varepsilon_{k+1}, \tag{2}$$

$$[N_{k+1} - S_k, N_{k+1}K_k + S_k] \subset [e^{(p_{k+1}\pi - \delta_{k+1})/s}, e^{(p_{k+1}\pi + \delta_{k+1})/s}], \tag{3}$$

$$N_{k+1} - S_k \geq N_k K_{k-1} - S_{k-1}. \tag{4}$$

Proof of Lemma 2. First of all, note that (1), (2), (4) can be verified with $N_k \to +\infty$, $p_k \to +\infty$, $\delta_k \to 0$ sufficiently fast.

The main problem is to deal with (3).

An easy computation shows that (3) is equivalent to

$$\frac{p_{k+1}\pi - \delta_{k+1}}{\log(N_{k+1} - S_k)} \leq s \leq \frac{p_{k+1}\pi - \delta_{k+1}}{\log(N_{k+1}K_k + S_k)}$$

Choose

$$p_{k+1} \cong \frac{s}{\pi} \log N_{k+1}, \quad N_k \to +\infty.$$

Then, up to negligible terms, the inequalities become

$$s - \frac{\delta_{k+1}}{\log N_{k+1}} \lesssim s \lesssim \frac{s \log N_{k+1} + \delta_{k+1}}{\log(N_{k+1}K_k)}$$
$$= s - \frac{s \log K_k}{\log(N_{k+1}K_k)} + \frac{\delta_{k+1}}{\log(N_{k+1}K_k)}.$$

This condition can be realised if and only if

$$s \log K_k \leq \delta_{k+1},$$

that is,

$$s \leq \frac{\delta_{k+1}}{\log K_k}.$$

So, if we choose s less than

$$\inf_{k<k_0} \left\{ \frac{\delta_{k+1}}{\log K_k} \right\},$$

then if (N_k) tends to $+\infty$ sufficiently fast, (1)–(4) hold.

Back to the proof of the theorem. Let $\varepsilon > 0$ and $(\varepsilon_k)_{k \in \mathbf{N}}$ be any sequence of signs.

Let us suppose that s, $(N_k)_{k \in \mathbf{N}}$, $(p_k)_{k \in \mathbf{N}}$ and $(\delta_k)_{k \in \mathbf{N}}$ are given by Lemma 2. Then, assuming (1)–(4), we are going to describe the action of the multiplier $m_s(n) = |n|^{is}$, $n \in \mathbf{Z}$, on $F(\psi)$:

With (1)–(4) it is clear that for $k \leq k_0$,

$$\operatorname{sp}(f_k) \subset \left[-e^{(p_{k+1}\pi + \delta_{k+1})/s}, -e^{(p_{k+1}\pi - \delta_{k+1})/s}\right] \cup \left[e^{(p_{k+1}\pi - \delta_{k+1})/s}, e^{(p_{k+1}\pi - \delta_{k+1})/s}\right].$$

Then, we can write

$$\left\| m_s F(\psi) - \sum_{k=1}^{k_0} \varepsilon_k f_k(\psi) \right\|_{L_2(\mathbf{T}, X)} \leq \sum_{k=1}^{k_0} \delta_{k+1} \|f_k\|_{L_2(\mathbf{T}, X)} \leq \varepsilon.$$

And then, by hypothesis,

$$\left\| \sum_{k=1}^{k_0} \varepsilon_k f_k(\psi) \right\|_{L_2(\mathbf{T}, X)} \leq \|m_s F\|_{L_2(\mathbf{T}, X)} + \varepsilon$$
$$\leq A \|F\|_{L_2(\mathbf{T}, X)} + \varepsilon.$$

We can integrate this last inequality in $\theta_1, \cdot, \theta_k, \ldots$. Using the invariance of the measure on $\mathbf{T}$ by the transform $\theta_j \to \theta_j + N_j \psi$, it is easy to see that we obtain

$$\left\| \sum_{k=1}^{k_0} \varepsilon_k \phi_k \varphi_k \right\|_{L_2(\mathbf{T}^{\mathbf{N}}, X)} \leq A \left\| \sum_{k=1}^{k_0} \phi_k \varphi_k \right\|_{L_2(\mathbf{T}^{\mathbf{N}}, X)} + \varepsilon$$

or equivalently,

$$\left\| \sum_{k=1}^{k_0} \varepsilon_k d_{k+1} \right\|_{L_2(\mathbf{T}^{\mathbf{N}}, X)} \leq A \left\| \sum_{k=1}^{k_0} d_{k+1} \right\|_{L_2(\mathbf{T}^{\mathbf{N}}, X)} + \varepsilon.$$

Letting $\varepsilon \to 0$ proves the theorem with $C = A$.

Remark. This theorem is also true for some other operators on $L_2(\mathbf{R}, X)$ of type $T \otimes \mathrm{Id}_X$, where T is a convolution operator on $L_2(\mathbf{R})$ with "nice" associated multiplier.

The origin of my interest in complex powers of operators is the paper of G. Dore and A. Venni [DV]. See also [G] for an extension of their result.

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