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PERTURBATION THEORY IN DIFFERENTIAL HOMOLOGICAL ALGEBRA II

BY

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1. Introduction

Perturbation theory is a particularly useful way to obtain relatively small differential complexes representing a given chain homotopy type. An important part of the theory is "the basic perturbation lemma" [RB], [G1], [LS] which is stated in terms of *modules* M and N of the same homotopy type. It has been known for some time that it would be useful to have a perturbation method which would respect extra structure. For example, if the initial modules M and N and the rest of the data have coalgebra, algebra, Lie, Lie-co, or Hopf structures, then there should be a corresponding "homological perturbation machine" that produces new data in the same category. A special case of this for algebra or coalgebra data has appeared in [GS]. The present paper is a continuation of part I [GL], but we have tried to make it relatively self-contained. Unless stated otherwise all modules will be over a commutative ring R with 1.

The use of "perturbation methods" in differential homological algebra has had a long history, much of which was indicated in part I and [GLS].

We wish to express our gratitude to the referee for his careful reading of an earlier version of this paper and his comments which have resulted in a major restructuring of the exposition.

In part I we showed that, in a special case, the basic perturbation lemma accepts algebra or coalgebra data *without a change* in the formulas, provided that we begin with an initiator (defined below) that itself satisfies an algebraic condition. The purpose of this paper is to cover the general case promised in part I [GL, §3.2].

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2. Basic perturbation theory

Unless otherwise stated, we work over an arbitrary commutative ring R with 1. We will use the term "module" to mean a chain complex over R. We also suppose that M and N are modules, ∇ and f are chain maps, and ϕ is a chain homotopy; $\nabla: M \to N$, $f: N \to M$, and $\phi: N \to N$, satisfying $f\nabla = 1_M$, $\nabla f = 1_N + D(\phi)$, where $D(\phi) = d_N\phi + \phi d_N$. We will refer to ∇f often enough to denote it by π. This information will be denoted succinctly by the diagram

$$\left(M \underset{f}{\overset{\nabla}{\rightleftarrows}} N, \phi\right).$$

Such objects and maps are called SDR-data [LS], [GS], [GL]. This notion of SDR-data first appears in §12 of Eilenberg-MacLane [EM1], where it is referred to as a *contraction*, and crucially (in terms of the history of homological perturbation theory) in [RB] and [WS]; in [GMu] SDR-data with some more structure is called *Eilenberg-Zilber data*, whereas in [HM] SDR-data with some more structure is called a *trivialized extension*. We call ∇ the inclusion and f the projection, while ϕ is simply called the homotopy. It was pointed out in [LS] that given f, ∇, ϕ, the homotopy ϕ could be changed to ϕ' so that $f\phi' = 0$, $\phi'\nabla = 0$, and $\phi'\phi' = 0$. This can be achieved, if necessary, by first setting $\phi'' = D(\phi)\phi D(\phi)$ and then $\phi' = \phi'' d\phi''$. We call these properties "side conditions". We point out that these conditions are crucial in the proof of the "algebra perturbation" lemma below.

2.1 The basic perturbation lemma

Consider SDR-data

$$\left(M \underset{f}{\overset{\nabla}{\rightleftarrows}} N, \phi\right)$$

and a new differential $\mathscr{D}$ on N. Let $t = \mathscr{D} - d$. Thus t satisfies

$$dt + td + t^2 = 0 \tag{2.1.1}$$

We now introduce a natural decreasing filtration. Let $\mathscr{A}$ be the algebra of non-commuting polynomials in ϕ, t, and d filtered by the weight in t. The filtration induced by the image of the obvious representation on End(N) will be denoted by I^n so that

$$\mathrm{End}(N) = I^0 \supset I^1 \supset \cdots \supset I^n \supset \cdots$$

For the purposes of (2.1.2) below, we also define filtrations for $\mathrm{Hom}(M, N)$ and $\mathrm{Hom}(N, M)$ using $I^n = f \circ I^n$ and $I^n = I^n \circ \nabla$ respectively.

We note that, in the example concerning the Serre filtration in [G1], there is an error which was pointed out to us by Don Barnes [DB]. The point is that if we allow *all* operators, then the statement that every operator decreases the Serre filtration by one is false. The difficulty is overcome by restricting to the subring of operators generated by ∇, f, ϕ, and t and using the above filtration.

Remark. Note that (2.1.1) implies that $dt + td \in I^2$

Following [G1], we define

$$t_1 = t, \qquad t_{n+1} = (t\phi)^n t$$
$$\Sigma_n = t_1 + \cdots + t_n.$$

We call t the *initiator*. For each n define new maps as follows.
On M,

(i) $$\partial_{n+1} = \partial_n + ft_n\nabla = d + f\Sigma_n\nabla$$

(ii) $$\nabla_{n+1} = \nabla_n + \phi t_n \nabla = \nabla + \phi\Sigma_n\nabla,$$

and on N,

(iii) $$f_{n+1} = f_n + ft_n\phi = f + f\Sigma_n\phi.$$

(rv) $$\phi_{n+1} = \phi_n + \phi t_n \phi = \phi + \phi\Sigma_n\phi.$$

Then we have:

(2.1.2) Lemma [G1] (also see [RB], [WS]).

$$\begin{array}{ll} \partial_n\partial_n \in I^n, & \mathscr{D}\nabla_n - \nabla_n\partial_n \in I^n, \\ \partial_n f_n - f\mathscr{D} \in I^n, & f_n\nabla_n = 1_M, \\ \nabla_n f_n - 1_N - \mathscr{D}\phi_n - \phi_n\mathscr{D} \in I^n, & \phi_n\nabla_n = 0, \\ f_n\phi_n = 0, & \phi_n\phi_n = 0. \end{array}$$

Remark. The convergence of these maps occurs in many situations beyond [G1] and [RB]. For example, the basic perturbation lemma is applied to certain *iterated fibrations* in [LS] and specifically to *nilpotent groups* in [L3] to obtain complexes over the integers. Furthermore, the analogous construction works (converges) for some *non-nilpotent groups* such as the fundamental group of the Klein bottle as pointed out in [BL].

In the following, we will often use the notation $f \equiv g \bmod I^n$ if $f - g \in I^n$.

2.2 Perturbation of objects with structure

We will state perturbation theorems for coalgebras and algebras and give the details for coalgebras. We note however that no use of (co)associativity or (co)unit is used in the proofs. The reader should therefore think of "algebras" and "coalgebras" in the sense of universal algebra. A further remark will be made about this after the proofs of the algebra lemma. We will come back to the issue in another paper which will deal with the other algebraic structure theorems mentioned in the introduction.

We write our structure maps as $c\colon C \to C \otimes C$ for coalgebras C and $m\colon A \otimes A \to A$ for algebras A. As above, consider a given set of SDR-data

$$\left(M \underset{f}{\overset{\nabla}{\rightleftarrows}} N, \phi\right).$$

If M and N are coalgebras, then we say that ϕ is a "coalgebra homotopy" if (see [AH], [GMu])

$$c\phi = (1 \otimes \phi + \phi \otimes \pi)c.$$

Recall that the initiator t is a coderivation means

$$ct = (1 \otimes t + t \otimes 1)c.$$

When M and N are algebras, the corresponding notion of "algebra homotopy" is

$$\phi m = m(1 \otimes \phi + \phi \otimes \pi)$$

and t is a derivation means that

$$tm = m(1 \otimes t + t \otimes 1).$$

Briefly, the theorems and their duals say that if M and N are (co)algebras and ϕ is a (co)algebra homotopy and t is a (co)derivation, then for all $n \geq 1$:

(1) *If* ∇ *is a (co)algebra map, then so is* ∇_n *modulo* I^{n-1}.
(2) *If* f *is a (co)algebra map, then so is* f_n *modulo* I^{n-1}.
(3) *If* ∇ *and* f *are (co)algebra maps, then* ∂_n *is a (co)derivation.*
(4) ϕ_n *is a (co)algebra homotopy modulo* I^{n-1}.

For the precise statement we need some additional notation. Let

$$u = t\phi\colon N \to N, \quad v = \phi t\colon N \to N,$$
$$s_n = 1 + \Sigma_n\phi = 1 + u + \cdots u^n, \quad r_n = 1 + \phi\Sigma_n = 1 + v + \cdots v^n.$$

Note that

$$\pi^2 = \pi, \quad f\pi = f, \quad u\phi = 0, \quad u\pi = 0,$$
$$\phi v = 0, \quad \pi v = 0, \quad \phi\pi = 0, \quad \pi\phi = 0.$$

We will freely make use of these identities in the discussion that follows.

As we have stated, there are eight theorems, four for coalgebras and four for algebras which are completely dual. We will give proofs of the four coalgebra versions. Only in (2.2.3) do we need d to be a coderivation and then only on N. These proofs dualize in a straightforward manner to yield the algebra versions of the statements.

Define a filtration of $\mathrm{Hom}(N, M \otimes M)$ by $I^n = (f \otimes f)J^n c$ where $J^n \subset \mathrm{End}(N \otimes N)$ is the ideal generated by products of $t \otimes 1$, $1 \otimes t$, $\phi \otimes 1$, $1 \otimes \phi$, $d \otimes 1$, and $1 \otimes d$ with at least n factors equal to $1 \otimes t$ or $t \otimes 1$.

(2.2.1) PROPOSITION. *If $cf = (f \otimes f)c$, then for all $n \geq 0$,*

$$cf_{n+1} = (f_{n+1} \otimes f_{n+1})c \bmod I^n.$$

Proof. We have

$$f_{n+1} = f \sum_{i=0}^{n} u_i$$

so

$$(f_{n+1} \otimes f_{n+1})c = \sum_{0 \leq i,\, j \leq n} (f \otimes f)(u^i \otimes u^j)c.$$

On the other hand,

$$cf_{n+1} = cfs_n = (f \otimes f) \sum_{i=0}^{n} cu^i.$$

Put

$$\begin{aligned} \mathscr{P} &= (1 \otimes t + t \otimes 1)(1 \otimes \phi + \phi \otimes \pi) \\ &= 1 \otimes u + u \otimes \pi + t \otimes \phi - \phi \otimes t\pi. \end{aligned}$$

Then $cu = ct\phi = \mathscr{P}c$ and so $cu^i = \mathscr{P}^i c$. On the other hand, an easy induction using the conditions $u\pi = 0$, $f\pi = f$, $u\phi = 0$ and $f\phi = 0$ gives

$$(f \otimes f)\mathscr{P}^i = \sum_{k+l=i} (f \otimes f)(u^k \otimes u^l),$$

and so

$$
\begin{aligned}
&(f_{n+1} \otimes f_{n+1})c - cf_{n+1} \\
&\qquad = (f \otimes f)\Bigg(\sum_{0 \le i,\, j \le n} (u^i \otimes u^j) - \sum_{i+j \le n} (u^i \otimes u^j)\Bigg)c \\
&\qquad = \sum_{i+j>n} (f \otimes f)(u^i \otimes u^j)c \in I^n.
\end{aligned}
$$

The proof is now complete.

Remark. Note that in the proof above we have not assumed that either differential is a coderivation.

A similar proof using $r_n = 1 + v + \cdots v^n \colon N \to N$ (so that $\nabla_{n+1} = r_n \nabla$) gives:

(2.2.2) Proposition. *If $c\nabla = (\nabla \otimes \nabla)c$, then for all $n \geq 0$,*

$$c\nabla_{n+1} = (\nabla_{n+1} \otimes \nabla_{n+1})c \mod I^n.$$

We now consider the status of the sequence of "partial differentials" $\{\partial_n\}$. For this we use the combined hypotheses of (2.2.1) and (2.2.2).

(2.2.3) Proposition. *If f, ∇ are coalgebra maps, then for all $n \geq 0$,*

$$c\partial_{n+1} = (\partial_{n+1} \otimes 1 + 1 \otimes \partial_{n+1})c \quad (\textit{no congruence necessary}).$$

Proof. Given the formula for $(f \otimes f)\mathscr{P}^i$ in the previous proposition and the fact that $u\nabla = 0$, it is now routine to verify that

$$c\partial_{n+1} = cd + \sum_{i=0}^{n-1} (f \otimes f)(1 \otimes u^i t + u^i t \otimes 1)(\nabla \otimes \nabla)c.$$

On the other hand,

$$
\begin{aligned}
&(\partial_{n+1} \otimes 1 + 1 \otimes \partial_{n+1})c \\
&\qquad = cd + (f \otimes f)\Bigg(\sum_{i=1}^{n} u^{i-1}t \otimes 1 + 1 \otimes u^{i-1}t \Bigg)(\nabla \otimes \nabla)c
\end{aligned}
$$

so we are done.

We now consider the effect of the perturbation on the homotopy ϕ. For this we may drop the hypotheses on f and ∇.

(2.2.4) PROPOSITION. *For all $n \geq 0$*

$$c\phi_{n+1} = (1 \otimes \phi_{n+1} + \phi_{n+1} \otimes \pi)c \bmod I^n.$$

Proof. We introduce the notation $\tilde{\phi} = 1 \otimes \phi + \phi \otimes \pi$. We then have

$$c\phi_n = \sum_{i=0}^{n} \tilde{\phi} c u^i = \sum_{i=0}^{n} \tilde{\phi} \mathscr{P}^i c$$

while it is easily seen that

$$\begin{aligned} 1 \otimes \phi_{n+1} + \phi_{n+1} \otimes \pi &= 1 \otimes r_n \phi + \phi s_n \otimes r_n \pi s_n \\ &= \sum_{i=0}^{n} 1 \otimes v^i \phi + \sum_{0 \leq i,j,k \leq n} \phi u^i \otimes v^j \pi u^k. \end{aligned}$$

An induction that is now routine and left to the reader shows that

$$\tilde{\phi} \mathscr{P}^n = 1 \otimes v^n \phi + \sum_{i+j+k=n} \phi u^i \otimes v^j \pi u^k.$$

We thus have

$$(1 \otimes \phi_{n+1} + \phi_{n+1} \otimes \pi)c - c\phi_{n+1} = \sum_{i+j+k>n} \phi u^i \otimes v^j \pi u^k \in I^n.$$

Remark. Note that, as in (2.2.1) and (2.2.2), we did not assume that either differential was a coderivation in the proof of (2.2.4).

We will state the algebra version of the above theorems for future reference.

(2.2.5) PROPOSITION. *Suppose that*

$$\left(M \underset{\beta}{\overset{\alpha}{\rightleftarrows}} A, \nu\right)$$

is SDR-data where M and A are algebras, t: $A \to A$ is an initiator such that t is *a derivation and ν is an algebra homotopy*:

$$tm = m(t \otimes 1 + 1 \otimes t), \quad \nu m = m(1 \otimes \nu + \nu \otimes \alpha\beta).$$

(1) *If $\alpha m = m(\alpha \otimes \alpha)$ then for all $n \geq 0$,*

$$m(\alpha_{n+1} \otimes \alpha_{n+1}) = \alpha_{n+1} m \bmod I^n.$$

(2) *If $\beta m = m(\beta \otimes \beta)$ then for all $n \geq 0$,*

$$m(\beta_{n+1} \otimes \beta_{n+1}) = \beta_{n+1} m \bmod I^n.$$

(3) *If α and β are algebra maps, then for all $n \geq 1$, ∂_n is a derivation.*
(4) *For all $n \geq 0$, $m(1 \otimes \nu_{n+1} + \nu_{n+1} \otimes \alpha\beta) = \nu_{n+1} m \bmod I^n$.*

The formulas in the proofs lend themselves to convergence lemmas. For example, suppose $\{f_n\}$ is such that there is an n such that $f_{n+1} = f_n$ then $f t_n \phi = f u^n = 0$ hence $f u^i = 0$, for all $i \geq n$, i.e. $f_{n+k} = f_n$ for all $k \geq 0$, i.e. $\{f_i\}$ is naive convergent. Now let $m = 2n$, then

$$(f_{m+1} \otimes f_{m+1})c - cf_{n+1} = \sum_{i+j>2n} (fu^i \otimes fu^j)c$$

hence at least one of i, j must be $\geq n$ so the expression is zero and

$$(f_\infty \otimes f_\infty)c = cf_\infty.$$

Therefore:

(2.2.6) PROPOSITION. *If $f_{n+1} = f_n$, then naive convergence takes place and $(f_\infty \otimes f_\infty)c = cf_\infty$.*

Similar statements hold for ∇_∞ and ϕ_∞:

(2.2.7) PROPOSITION. *If $\nabla_{n+1} = \nabla_n$, then $(\nabla_\infty \otimes \nabla_\infty)c = c\nabla_\infty$.*

(2.2.8) PROPOSITION. *If $\phi_{n+1} = \phi_n$, then $(1 \otimes \phi_\infty + \phi_\infty \otimes \pi)c = c\phi_\infty$.*

We also have, of course, the three corresponding dual algebra statements.

Remark. The proofs just given do not give results directly for Lie, Lie-co, or commutative algebras. In these cases, the (anti) symmetry must be taken into account. For example, note that the definition of an algebra homotopy we have given is not a good definition for commutative algebras. If m is a commutative operation, then the left hand side of the equation $\nu m = m(1 \otimes$

$\nu + \nu \otimes \pi)$ is symmetric, but the right hand side is not. We will come back to this issue in another paper.

3. The tensor trick

It may seem a bit difficult to make sure that the (co)algebra conditions above are satisfied. This difficulty is resolved in the so-called *tensor trick* (see [GL, (3.2)], [GLS] [HK]). Consider now a connected differential graded module Z (i.e., Z is non-negatively graded and $Z_0 = R$), denote by $T^c s\overline{Z}$ the tensor coalgebra on the suspension $s\overline{Z}$ of the submodule $\overline{Z}$ of positive elements with the tensor product differential. Now consider SDR-data satisfying the side conditions with no other assumptions on the maps:

$$\left(M \underset{f}{\overset{\nabla}{\rightleftarrows}} N, \phi\right)$$

but with M and N connected. We form the corresponding tensor coalgebra of the given data to obtain new SDR-data

$$\left(T^c s\overline{M} \underset{Tf}{\overset{T\nabla}{\rightleftarrows}} T^c s\overline{N}, T\phi\right).$$

Here $T\nabla$ and Tf are an abuse of notation for the obvious maps and

$$T\phi = T_1\phi \oplus T_2\phi \oplus \cdots \oplus T_n\phi \oplus \cdots$$

where, with s implicit,

$$\begin{aligned} T_n\phi = {} & \phi \otimes \pi \otimes \cdots \otimes \pi + 1 \otimes \phi \otimes \pi \otimes \cdots \otimes \pi \\ & + \cdots + 1 \otimes \cdots \otimes 1 \otimes \phi \otimes \pi \otimes \cdots \otimes \pi + \cdots \\ & + 1 \otimes \cdots \otimes 1 \otimes \phi, \quad n \text{ factors}. \end{aligned}$$

This construction appears in [EM, I]. The differentials involved are just the tensor product differentials. The maps $T\nabla$, Tf are maps of coalgebras, and $T\phi$ is a coalgebra homotopy. Now let N be an algebra and take the initiator t on $T^c s\overline{N}$ to be the "algebra part" of the bar construction differential on N, i.e.,

$$t(a_1 \otimes \cdots a_n) = \sum \pm a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n.$$

As introduced in [GL], the tensor trick consists of applying propositions (2.2.1)–(2.2.4) to the tensor coalgebra data above. The formulas of the basic perturbation lemma converge because the correcting terms decrease the number of tensors by one and so will be zero after a finite number of steps.

Convergence in the dual situation involving the "coalgebra part" of the cobar construction is much more subtle. It is clear that, with the tensor trick, we obtain information not about the original data, but about the corresponding "classifying spaces". We immediately obtain the result of [GS]. Applications to K-T. Chen's work [KC] are given in [GLS] and applications to E. Brown [EB] and T. Kadeishvili's work [TK] are given below. Related results are discussed in Heubschmann-Kadeishvili [HK].

4. Consolidating earlier theorems

The algebra perturbation lemma can be used to consolidate several theorems appearing in [EB], [TK], [GMu], [G2], and [GS]. This observation began in [GL] where we showed a relationship between the work in [GMu], [G2] and [GS] in a special case. We now lift the restriction of that special case. The theorems we have in mind will be referred to using the notation of this paper.

4.1 Cobar and bar construction results

Recall the cobar construction. For a connected differential graded module Z, denote by $T^a s^{-1}\overline{Z}$, the tensor algebra on the desuspension of $\overline{Z}$ with the tensor product differential. The cobar construction $\Omega\overline{C}$ [FA] on a (differential graded coalgebra) (C, Δ) is $T^a s^{-1}\overline{C}$ with total differential $d + t$ where the "coalgebra part" t is the graded derivation determined by

$$tc = \Sigma \pm c_i \otimes c_i'$$

where $\Delta c = \Sigma c_i \otimes c_i'$.

Gugenheim's theorem in [G2] is the following:

(4.1) THEOREM [G2, (2.1)]. *Given SDR-data*

$$\left(H \underset{f}{\overset{\nabla}{\rightleftarrows}} C, \phi\right)$$

where C is a simply connected coalgebra, H is a module with trivial differential ($d = 0$), and ∇ and f are merely assumed to be module maps, there is a derivation

$$\delta\colon T^a s^{-1}\overline{H} \to T^a s^{-1}\overline{H}$$

and a twisting cochain

$$\tau\colon C \to (T^a s^{-1}\overline{H}, \delta)$$

such that the induced map of differential graded algebras

$$\hat{\tau}\colon \overline{\Omega}C \to T^a s^{-1}\overline{H}$$

is an isomorphism in homology.

Gugenheim's formulation of this perturbation result was inspired directly by K-T. Chen's work [KC] as explained in [GLS]. The result itself had already been discovered by T. Kadeishvili [TK], although, at the time, we were unfortunately not aware of this fact. R. Hain has worked out an analogous result in the Lie setting [RH2]. The result above was generalized to the case where the differential in H above is not assumed to be zero by Gugenheim and Stasheff in an unpublished manuscript that had circulated for some years before the published (dual) version appeared in [GS].

(4.2) THEOREM [GS (§II)]. *Suppose that*

$$\left(M \underset{f}{\overset{\nabla}{\rightleftarrows}} A, \phi\right)$$

is SDR-data where A is an algebra, M is a differential module (both connected), and f and ∇ are merely assumed to be module maps. There exists a coderivation of the tensor coalgebra

$$\partial\colon T^c s\overline{M} \to T^c s\overline{M}$$

and a twisting cochain

$$\tau\colon T^c s\overline{M} \to A$$

such that the induced map of differential coalgebras

$$\hat{\tau}\colon T^c s\overline{M} \to \overline{B}A$$

induces an isomorphism in homology.

Even when $d = 0$, the result involving the cobar construction is more subtle because the initiator t raises the filtration given by the number of tensor factors. We will deal with this in another paper that addresses convergence issues in several general settings.

4.2 A_∞-structures

An A_∞-algebra means a connected module M along with a differential ∂ which is a coderivation of the tensor coalgebra $T^c s\overline{M}$ and a perturbation of the tensor product differential, i.e. differs from the tensor product differential by terms which increase the number of tensor factors. We write $(T^c s\overline{M}, \partial)$ simply as $\tilde{B}M$ [JS]. For later use, it may be helpful to recall another interpretation of A_∞-algebras in terms of maps. Since the differential ∂: $T^cM \to T^cM$ is a coderivation, we can look at the induced maps

$$m_i\colon \otimes^i M \to M \quad \text{for } i \geq 1.$$

These are the composites

$$\begin{tikzcd} \otimes^i M \arrow[r, "m_i"] \arrow[d] & M \\ T^c s\overline{M} \arrow[r, "\partial"] & T^c s\overline{M}. \arrow[u, "\pi"'] \end{tikzcd}$$

Because of the fact that the differential is a coderivation, the maps m_i satisfy interesting relations. For example, m_1 is the differential on M, and m_2 is an operation on A for which m_1 is a derivation. The map m_3 is a chain homotopy giving the obstruction to the associativity of m_2. Similarly, in the dual case (which follows by a proof that is completely dual to the one in [GS] but under a stronger assumption that guarantees convergence), we call the differential algebra $(T^a s^{-1}H, \delta)$ a A_∞-coalgebra and write it simply as $\tilde{\Omega}H$.

In light of our current work, we see that the more general theorem 4.2 is really only one third of the whole picture given by the tensor trick where we obtain new SDR-data

$$\left(\tilde{B}M \underset{f_\infty}{\overset{\nabla_\infty}{\rightleftarrows}} \overline{B}A, \phi_\infty\right)$$

with ∇_∞ and f_∞ maps of differential coalgebras (ϕ_∞ a coalgebra homotopy). Project ∇_∞ to A,

$$\begin{tikzcd} \tilde{B}M \arrow[r] \arrow[dr, "\tau"'] & \overline{B}A, \arrow[d, "\pi"] \\ & A \end{tikzcd}$$

using the obvious projection π onto $\overline{A} \subset A$, which is a universal twisting cochain. We obtain a twisting cochain τ as in the earlier theorems, but now we have a map f_∞ in the reverse direction as well as a homotopy for the new maps. Before pointing out the significance of the "reverse map", we review the related result from [GL], viz. in the case when the projection f in the original data is multiplicative (thus also assuming that M is a d.g. algebra). We recapture the recursive formula for the twisting cochain given by

Gugenheim and Munkholm [GMu]:

$$\tau_0 = 0$$
$$\tau_n = \sum_{i+j=n} \phi(\tau_i \cup \tau_j).$$

We furthermore recapture the homotopy in [GMu].

4.3 Homotopy twisting cochains and homotopy representations

For an application of the limit projection f_∞, recall the theorem of E. Brown:

(4.3) THEOREM [EB (9.1)]. *Let τ be a twisting cochain $\tau\colon C \to A$, where C is a coalgebra and A is an algebra, and L is a module over A. Let $E =$* End$(H(L))$ *denote the endomorphism algebra of the homology of L with trivial differential. There is a twisting cochain $\tau^*\colon C \to E$ such that the twisted tensor product complexes $C \otimes_\tau L$ and $C \otimes_{\tau^*} H(L)$ are homology equivalent.*

Recall also Kadeishvili's theorems, the first of which we have already mentioned:

(4.4) THEOREM [TK, p. 232, pp. 235–236]. (1) *Given an algebra A, there is an A_∞-structure on $H(A)$ and a twisting cochain $\tau\colon \tilde{B}H(A) \to A$.*

(2) *Given a twisting cochain $\mu\colon C \to A$, there is a "homotopy twisting cochain" (see the definition below) $\tilde{\mu}\colon C \to H(A)$ with respect to the A_∞-structure on $H(A)$ given in* (1) *above for which μ and $\tau\tilde{\mu}$ are equivalent, i.e., the twisting cochains are homotopic in the sense of twisting cochains* [TK, *p.* 235], [GMu].

Generally, if M is an A_∞-algebra, so that $\tilde{B}M$ exists as above, then a *homotopy twisting cochain* $\tau\colon C \to M$ (where C is a coalgebra) is an R-module map such that there exists a differential coalgebra map $\hat{\tau}$ making the following diagram commute:

$$\begin{tikzcd} C \arrow[rr, "\tau"] \arrow[dr, "\hat{\tau}"'] & & M \\ & \tilde{B}M \arrow[ur, "\pi"'] & \end{tikzcd}$$

where π is as above for $\bar{B}A$.

Now we will explain the relationship between these theorems and the tensor trick. To begin, we recall the left regular representation of an algebra M.

$$\rho: M \to \mathrm{Hom}(M, M) = \mathrm{End}(M)$$
$$a \mapsto (b \mapsto ab).$$

The map ρ is an algebra map precisely because the operation in M is *associative*. If M is an A_∞-algebra, the operation in M is not generally associative. However, the map ρ above is still defined for an A_∞-algebra M, but it may no longer be an algebra map because of the possible lack of associativity. It is however an algebra map up to homotopy in the strong (A_∞) sense, (see [GMu] for a discussion of such maps). This fact was shown in R. Nowlan's thesis [RN]. To be precise, we have that ρ extends to a map of differential graded coalgebras

$$\tilde{B}M \to \bar{B}\,\mathrm{End}(M)$$

or equivalently, there is a twisting cochain $\hat{\rho}$ which makes the following diagram commute:

$$\begin{tikzcd} \tilde{B}M \arrow[rr, "\hat{\rho}"] \arrow[dr, "\pi"'] & & \mathrm{End}(M) \\ & M \arrow[ur, "\rho"'] & \end{tikzcd}$$

Note that if the differential in M is trivial, then the differential in $\mathrm{Hom}(M, M) = \mathrm{End}(M)$ is trivial and $\bar{B}\,\mathrm{End}(M)$ is just the ordinary bar construction of the R-algebra $\mathrm{End}(M)$.

By this device, we may convert a homotopy twisting cochain $\tau: C \to M$ into an ordinary twisting cochain $\hat{\tau}: C \to \mathrm{End}(M)$; we take $\hat{\tau} = \hat{\rho}\tau$. We can then extend the theorems above for a given twisting cochain $\tau: C \to A$, assuming that we have SDR-data

$$\left(M \underset{f}{\overset{\nabla}{\rightleftarrows}} A, \phi\right)$$

where A is an algebra. We may use the homotopy twisting cochain obtained by projecting the composite

$$f_\infty\bar{\tau}: C \xrightarrow{\bar{\tau}} \bar{B}A \xrightarrow{f_\infty} \tilde{B}M$$

onto M via the universal homotopy twisting cochain π, and, using the discussion above, also obtain the associated twisting cochain

$$\widehat{\pi f_\infty \tilde{t}}: C \to \operatorname{End}(M).$$

(4.5) THEOREM. *Let $C \to A$ be a twisting cochain where C is a coalgebra and A is an algebra. Let*

$$\left(M \underset{f}{\overset{\nabla}{\rightleftarrows}} A, \phi\right)$$

be SDR-data for some connected module M (with possibly non-zero differential). There is an A_∞-structure on M, a homotopy twisting cochain

$$\tilde{\tau}: C \to M$$

and a twisting cochain

$$\tau^*: C \to \operatorname{End}(M)$$

such that we have factorizations

$$\begin{tikzcd} C \arrow[r, "\tau^*"] \arrow[d, "\hat{\tau}"'] \arrow[dr, "\tilde{\tau}"] & \operatorname{End}(M) \\ \bar{B}A \arrow[r, "\pi f_\infty"'] & M. \arrow[u, "\rho"'] \end{tikzcd}$$

with f_∞ a coalgebra homology equivalence (f_∞ is the map obtained from the tensor trick).

Note that π itself is a homotopy twisting cochain which induces the identity on $\tilde{B}M$ (i.e., π is the homotopy twisting cochain associated to the A_∞-structure obtained from the tensor trick).

As a consequence of this theorem, the twisted tensor products $C \otimes_\tau A$, and $C \otimes_{\tau^*} M$ are homology equivalent (compare [GL, (4.1)]). One can also define the "homotopy twisted tensor product" $C \otimes_{\tilde{\tau}} M$ [TK, p. 235] and this complex is also homology equivalent to $C \otimes_\tau A$.

We leave the details of the proof and the extension to the case of modules L over A as in Brown's theorem to the reader.

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