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QUADRATIC FORMS INVARIANT UNDER GROUP ACTIONS

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Introduction

Let K be a field and let G be a finite group. A K-bilinear form β: $V \times V \to K$ on a $K[G]$-module V is said to be *G-invariant* if $\beta(gv, gw) = \beta(v, w)$ for v, w in V and g in G. For simplicity, a symmetric nondegenerate G-invariant bilinear form will be called throughout a *G-form*.

In this paper we consider two equivalence relations on the set of G-forms on a given $K[G]$-module V, namely *isometry* and *projective isometry*. Two forms β_1 and β_2 are said to be *isometric* if there exists a K-automorphism f: $V \to V$ such that $\beta_1(f(v), f(w)) = \beta_2(v, w)$ for all v, w in V (notice that we *do not* require f to commute with the action of G). The forms β_1 and β_2 are said to be *projectively isometric* if there exists a non-zero constant k in K such that β_1 and $k\beta_2$ are isometric in the previous sense.

W. Feit proved in [2] for the cyclic group C_p of prime order p with $p \equiv 3 \pmod 4$, that all positive-definite C_p-forms on the irreducible $\mathbf{Q}[C_p]$-module of dimension $p - 1$ are projectively isometric. He also proved by giving an explicit counterexample that this is false for $p \equiv 1 \pmod 4$.

Our work originates in an attempt to generalize Feit's result. The question whether all positive-definite G-forms on a given irreducible $K[G]$-module V are projectively isometric is closely connected with two other problems, interesting for themselves. The first is the classification of all G-forms on V up to (projective) isometry. The second problem is to study the behavior of invariant forms under induction. More precisely, assuming that V is induced from a subgroup H of G, we wish to know which G-forms are obtained, up to isometry, by inducing H-forms (induction of forms is explained in Section 3).

We shall assume throughout this paper that the ground field K is a totally real number field, even though this hypothesis may not be essential for some of our statements.

Here is a summary of the contents of this article:

Section 1 explains the correspondence between symmetric G-invariant bilinear forms on V and G-invariant hermitian forms over the center of the

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endomorphism ring of V. This correspondence is applied repeatedly throughout the paper.

In Section 2 we calculate, under suitable hypotheses, the Hasse-Witt invariant of the difference of two G-forms (Proposition 2.1). We apply this result to obtain explicit criteria for (projective) isometry of G-forms (Theorem 2.2 and Theorem 2.4). We also generalize Feit's theorem [2] to arbitrary p-groups (Corollary 2.6).

Section 3 deals with induction of forms. We prove, under some assumptions, that a positive-definite G-form on an irreducible induced $K[G]$-module is isometric to a induced form (Theorem 3.1). In particular, for a nilpotent group G of odd order, all positive-definite G-forms on an irreducible $K[G]$ = module are obtained, up to isometry, by inducing forms invariant by a cyclic subgroup (Corollary 3.2).

1. Lifting forms to the endomorphism ring

Let K be a field and let G be a finite group. Let V be an irreducible $k[G]$-module endowed with a G-form β. Since V is irreducible, the endomorphism ring $\mathrm{End}_{K[G]}(V)$ is a (skew-)field. The form β induces an involution $e \mapsto \bar{e}$ on $\mathrm{End}_{K[G]}(V)$ defined by

$$\beta(ev, w) = \beta(v, \bar{e}w) \quad \text{for all } v, w \in V.$$

The restriction of this involution to the center E of $\mathrm{End}_{K[G]}(V)$ is independent of the choice of β: Let β' be another G-form on V. The form β' can be written $\beta'(v, w) = \beta(av, w)$ for some $K[G]$-automorphism a of V. For any z in the center E we have

$$\begin{aligned} \beta'(zv, w) &= \beta(azv, w) \\ &= \beta(zav, w) \\ &= \beta(av, \bar{z}w) \\ &= \beta'(v, \bar{z}w). \end{aligned}$$

The above computation shows that β' induces the same involution as β on E as claimed. This involution will be called the *canonical involution* on E.

If K is a totally real number field, then the canonical involution on E is either trivial or it coincides with complex conjugation (see e.g. [1, (50.37)]). The dual vector space $V^* = \mathrm{Hom}_K(V, K)$ can be made into an E-vector space by setting $(e\phi)(v) = \phi(\bar{e}v)$ for e in E and ϕ in V^*. Similarly, the vector space $\mathrm{Hom}_E(V, E)$ has the E-vector space structure given by $(e\Phi)(v) = \Phi(v)\bar{e}$. We leave to the reader to see that the map

$$\begin{aligned} \mathrm{Hom}_E(V, E) &\to \mathrm{Hom}_K(V, K), \\ \Phi &\mapsto \mathrm{Tr}_{E/K}(\Phi) \end{aligned}$$

is an E-isomorphism. This isomorphism induces a bijection

$$\begin{aligned} \mathrm{Herm}_{E,G}(V) &\tilde{\rightarrow} \mathrm{Symm}_{K,G}(V), \\ h &\mapsto \mathrm{Tr}_{E/K}(h) \end{aligned} \tag{1}$$

between the set $\mathrm{Herm}_{E,G}(V)$ of G-invariant hermitian forms on V (with respect to the canonical involution) and the set $\mathrm{Symm}_{K,G}(V)$ of symmetric G-invariant bilinear forms on V.

We conclude this section by an example. Let C_n be the cyclic group of order n. The irreducible $\mathbf{Q}[C_n]$-module V of dimension $\varphi(n)$ can be identified with $\mathbf{Q}(\zeta)$, where ζ is a primitive n^{th} root of unity, and a fixed generator of C_n acts on $\mathbf{Q}(\zeta)$ by multiplication by ζ. The canonical involution on $E = \mathrm{End}_{\mathbf{Q}[C_n]}(V) = \mathbf{Q}(\zeta)$ is complex conjugation. Hence the bijection (1) can be written in this case

$$\begin{aligned} \mathbf{Q}(\zeta + \zeta^{-1}) &\rightarrow \mathrm{Symm}_{\mathbf{Q},G}(V), \\ a &\mapsto \beta_a, \end{aligned}$$

where β_a is the form give by $\beta_a(v, w) = \mathrm{Tr}_{\mathbf{Q}(\zeta)/\mathbf{Q}}(av\bar{w})$.

2. The classification of G-forms

We list here for convenience the notation that will be in force from now on:

G	:	a finite group
K	:	a totally real number field
V	:	a non-trivial irreducible $K[G]$-module
E	:	the center of $\mathrm{End}_{K[G]}(V)$
F	:	the subfield of E fixed by the canonical involution
$d_{E/F}$	:	the determinant of trace form $(x, y) \mapsto \mathrm{Tr}_{E/F}(xy)$
$Br_2(L)$	:	the subgroup of elements of order at most 2 in the Brauer group of L
N_a	:	the norm of the quaternion algebra $\left(\frac{a, d_{E/F}}{F}\right)$
$(\,,)_{\mathfrak{p}}$	:	the Hilbert symbol at the prime $\mathfrak{p}$
$W(L)$	:	the Witt ring of L
$I(L)$	:	the fundamental ideal of $W(L)$
ϕ_L	:	the Hasse-Witt homomorphism ϕ_L: $I^2(L) \rightarrow Br_2(L)$

We shall assume henceforth the condition

(*) $\operatorname{End}_{K[G]}(V)$ is a commutative field and the canonical involution is non-trivial.

Instances of representations satisfying condition (*) include faithful irreducible representations over K of the following types of groups: abelian groups of order ≥ 3, nilpotent groups of odd order (see [4], Satz 3).

We shall now calculate explicitly under assumption (*) the difference of two G-forms in the Witt group $W(K)$. The class of a bilinear form β in the Witt ring will be denoted by $[\beta]$.

(2.1) PROPOSITION. *Assume condition (*). Let β_1 and β_2 be two G-forms on V and let a be the unique element in F such that $\beta_1(x, y) = \beta_2(ax, y)$ for all x, y in V. Then the difference $[\beta_1] - [\beta_2]$ lies in $I^2(K)$ and its Hasse-Witt invariant is given by*

$$\phi_K([\beta_1] - [\beta_2]) = \dim_E(V)\operatorname{Cor}_{F/K}\left(\frac{a, d_{E/F}}{F}\right). \tag{2}$$

Proof. Let $h\colon V \times V \to E$ be the hermitian form over E such that $\beta_1 = \operatorname{Tr}_{E/K}(h)$ (see (1)). Obviously we have $\beta_2 = \operatorname{Tr}_{E/K}(ah)$. We first compute the class of the form

$$X_a := \operatorname{Tr}_{E/F}(h) \perp \left(-\operatorname{Tr}_{E/F}(ah)\right)$$

in $W(F)$. Choosing a diagonalization we write $h = \langle c_1, \ldots, c_n \rangle$, where the coefficients c_i are in the fixed field F and n is the dimension of V over E. On the one hand we have

$$\begin{aligned} X_a &= \langle 1, -a \rangle \otimes \operatorname{Tr}_{E/F}(h) \\ &= \langle 1, -a \rangle \otimes \langle 1, -d_{E/F} \rangle \otimes \langle 2c_1, \ldots, 2c_n \rangle \\ &= \langle 1, -a, -d_{E/F}, ad_{E/F} \rangle \otimes \langle 2c_1, \ldots, 2c_n \rangle \\ &= N_a \otimes \langle 2c_1, \ldots, 2c_n \rangle. \end{aligned}$$

On the other hand, the forms N_a and cN_a are isometric over F for any c in F^*. Thus $[X_a] = n[N_a]$ in $W(F)$. In particular $[X_a]$ belongs to $I^2(F)$. Using the commutativity of the diagram

$$\begin{array}{ccc} I^2(F) & \xrightarrow{\phi_F} & Br_2(F) \\ {\scriptstyle \operatorname{Tr}_{F/K}}\downarrow & & \downarrow{\scriptstyle \operatorname{Cor}_{F/K}} \\ I^2(K) & \xrightarrow[\phi_K]{} & Br_2(K) \end{array}$$

(see e.g. [3, Section 6]) we obtain

$$\begin{aligned}\phi_K([\beta_1] - [\beta_2]) &= \phi_k \operatorname{Tr}_{F/K}([X_a]) \\ &= n\phi_K \operatorname{Tr}_{F/K}([N_a]) \\ &= n \operatorname{Cor}_{F/K}\phi_F([N_a]) \\ &= n \operatorname{Cor}_{F/K}\left(\frac{a, d_{E/F}}{F}\right). \quad \square\end{aligned}$$

We are now able to formulate the main result of this section.

(2.2) THEOREM. *Let V be a $K[G]$-module satisfying condition (*). Let β_1 and β_2 be two G-forms on V. Let $a \in F$ be such that $\beta_1(x, y) = \beta_2(ax, y)$.*

(I) *Suppose that $\dim_E(V)$ is even. Then β_1 and β_2 are isometric if and only if they have the same signature.*

(II) *Suppose that $\dim_E(V)$ is odd. Then β_1 and β_2 are isometric if and only if they have the same signature and*

$$\prod_{\mathfrak{P}|\mathfrak{p}} (a, d_{E/F})_{\mathfrak{P}} = 1 \tag{3}$$

for all primes $\mathfrak{p}$ of K.

Proof. Recall that forms over number fields are classified by rank, discriminant, Hasse invariant, and signature. Evidently β_1 and β_2 have the same rank and discriminant, and by hypothesis they have the same signature. Hence we need only to test the vanishing of the Hasse-Witt homomorphism ϕ_K on the difference $[\beta_1] - [\beta_2]$.

(I) If $\dim_E(V)$ is even then $\phi_K([\beta_1] - [\beta_2]) = 0$ by Proposition 2.1.

(II) If $\dim_E(V)$ is odd then identity (2) becomes

$$\phi_K([\beta_1] - [\beta_2]) = \operatorname{Cor}_{F/K}\left(\frac{a, d_{E/F}}{F}\right).$$

Let now $\mathfrak{p}$ be a prime of K and let $\mathfrak{P}$ be a prime of F above $\mathfrak{p}$. Using the commutativity of the diagram

$$\begin{array}{ccc} Br(F_{\mathfrak{P}}) & \xrightarrow{\operatorname{Cor}_{F_{\mathfrak{P}}/K_{\mathfrak{p}}}} & Br(K_{\mathfrak{p}}) \\ {\scriptstyle \operatorname{inv}_{F_{\mathfrak{P}}}}\downarrow & & \downarrow{\scriptstyle \operatorname{inv}_{K_{\mathfrak{p}}}} \\ \mathbf{Q}/\mathbf{Z} & = & \mathbf{Q}/\mathbf{Z} \end{array}$$

(see e.g. [5, Section 1]), and taking the sum over all primes $\mathfrak{P}$ lying above $\mathfrak{p}$, we obtain the commutative diagram

$$\begin{array}{ccc} \bigoplus_{\mathfrak{P}|\mathfrak{p}} Br_2(F_{\mathfrak{P}}) & \xrightarrow{\prod_{\mathfrak{P}|\mathfrak{p}} \mathrm{Cor}_{F_{\mathfrak{P}}/K_{\mathfrak{p}}}} & Br_2(K_{\mathfrak{p}}) \\ {\scriptstyle \oplus_{\mathfrak{P}|\mathfrak{p}} \mathrm{inv}_{F_{\mathfrak{P}}}} \downarrow & & \downarrow {\scriptstyle \mathrm{inv}_{K_{\mathfrak{p}}}} \\ \bigoplus_{\mathfrak{P}|\mathfrak{p}} \mathbf{Z}/2\mathbf{Z} & \xrightarrow{\text{sum}} & \mathbf{Z}/2\mathbf{Z}, \end{array}$$

which shows immediately that the $\mathfrak{p}$-component of $\mathrm{Cor}_{F/K}(a, d_{E/F}/F)$ is given by the product of Hilbert symbols

$$\prod_{\mathfrak{P}|\mathfrak{p}} (a, d_{E/F})_{\mathfrak{P}}.$$

This completes the proof of the theorem. □

(2.3) *Remark.* If $\mathfrak{P}$ is inert in E then $Br_2(F_{\mathfrak{P}})$ can be identified with $F_{\mathfrak{P}}^*/N_{E/F}(E_{\mathfrak{P}}^*)$. With this identification, the natural map $Br_2(F_{\mathfrak{P}}) \to \mathbf{Z}/2\mathbf{Z}$ is given by $x \mapsto \mathrm{ord}_{\mathfrak{P}}(x) \pmod 2$. Thus, in this case, condition (3) becomes

$$\sum_{\mathfrak{P}|\mathfrak{p}} \mathrm{ord}_{\mathfrak{P}}(a) \equiv 0 \pmod 2,$$

or equivalently,

$$\mathrm{ord}_{\mathfrak{p}}\left(N_{F/K}(a)\right) \equiv 0 \left(\mathrm{mod}\, 2f_{\mathfrak{p}}\right),$$

where $f_{\mathfrak{p}}$ is the inertial degree of $\mathfrak{p}$ in F.

With the same notation, we have:

(2.4) THEOREM. *Let V be a irreducible $K[G]$-module satisfying (*).*

(I) *If $\dim_E(V)$ is even, then all positive-definite invariant bilinear forms are isometric.*

(II) *If $\dim_E(V)$ is odd, then the following statements are equivalent:*

(a) *$[F : K]$ is odd;*

(b) *All positive-definite G-forms are projectively isometric.*

Proof. (I). Direct consequence of Theorem 2.2.

(II). Assume now that $\dim_E(V)$ is odd.

(a) ⇒ (b). Since $[F : K]$ is odd and E/K is normal, we can choose $d_{E/F}$ in K^*. Let β_1 and β_2 be positive-definite G-forms. Let a be in F such that $\beta_1(x, y) = \beta_2(ax, y)$ for all x, y in V. Let $\mathfrak{p}$ be a prime of K and fix a prime

$\mathfrak{P}_0$ of F above $\mathfrak{p}$. Let $\Gamma = \mathrm{Gal}(F/K)$. With this notation we have

$$\prod_{\mathfrak{P}|\mathfrak{p}} (a, d_{E/F})_{\mathfrak{P}} = \prod_{\sigma \in \Gamma/\Gamma_{\mathfrak{P}_0}} (\sigma(a), d_{E/F})_{\mathfrak{P}_0}$$
$$= (N_{F/K}(a), d_{E/F})_{\mathfrak{P}_0}$$
$$= \prod_{\mathfrak{P}|\mathfrak{p}} (N_{F/K}(a), d_{E/F})_{\mathfrak{P}}$$

(note that the order of $\Gamma_{\mathfrak{P}_0}$ is odd). Applying Theorem 2.2 we conclude that β_1 and $N_{F/K}(a)\beta_2$ are isometric.

(b) ⇒ (a). Let β_1 be a positive-definite G-invariant form on V. Choose a prime $\mathfrak{p}$ of K such that $\mathfrak{p}O_F$ is the product of $[F:K]$ distinct primes of F which are inert in E (such a prime exists by Tchebotarev's Density Theorem). Let $\mathfrak{P}_1, \dots, \mathfrak{P}_{[F:K]}$ be the primes of F lying above $\mathfrak{p}$ and let a be a totally positive element in F satisfying

$$\mathrm{ord}_{\mathfrak{P}_i}(a) = \begin{cases} 1 & \text{if } i = 1, \\ 0 & \text{if } i > 1. \end{cases}$$

Let $\beta_2(v, w) = \beta_1(av, w)$. By hypothesis, there exists k in K^* such that $k\beta_1$ and β_2 are isometric. By Theorem 2.2 part II (and Remark 2.3), we must have

$$\sum_{\mathfrak{P}|\mathfrak{p}} \mathrm{ord}_{\mathfrak{P}}(a) \equiv \sum_{\mathfrak{P}|\mathfrak{p}} \mathrm{ord}_{\mathfrak{P}}(k) \pmod 2. \tag{4}$$

The left hand side of (4) is equal to 1 by the construction of a, and, since $\mathfrak{p}$ is totally decomposed in F, the right hand side of (4) is given by

$$\sum_{\mathfrak{P}|\mathfrak{p}} \mathrm{ord}_{\mathfrak{P}}(k) = [F:K]\mathrm{ord}_{\mathfrak{p}}(k).$$

Therefore $[F:K]$ must be odd. □

(2.5) Corollary. *Let $K = \mathbf{Q}$ and assume condition (*). Suppose that G acts faithfully on V and that $\dim_E(V)$ is odd. If all positive-definite G-forms on V are projectively isometric, then the center $Z(G)$ of G is cyclic and its order is either 2^ν with $0 \le \nu \le 2$, or of the form p^ν or $2p^\nu$ with p prime and $p \equiv 3 \pmod 4$.*

Proof. Let $n = |Z(G)|$. The statement being trivial for $n \le 2$ we may assume $n > 2$. Since G acts faithfully, the center $Z(G)$ is mapped injectively into E^*, therefore $Z(G)$ is cyclic and E contains the cyclotomic field $\mathbf{Q}(\zeta_n)$.

By Theorem 2.4 Part II the degree $[\mathbf{Q}(\zeta_n + \bar{\zeta}_n): \mathbf{Q}]$ must be odd, or equivalently, $\varphi(n)/2$ must be odd. This is true only for $n = 4$ or of the form p^α or $2p^\alpha$ with $p \equiv 3 \pmod 4$. □

We also have a generalization of Feit's result.

(2.6) COROLLARY. *Let G be a p-group ($p > 2$). Let V be a simple non-trivial $\mathbf{Q}[G]$-module. The following statements are equivalent*:

(a) $p \equiv 3 \pmod 4$;

(b) *All positive-definite G-forms on V are projectively isometric.*

Proof. Evident consequence of Theorem 2.4, since in this case E contains $\mathbf{Q}(\zeta_p)$ and is contained in $\mathbf{Q}(\zeta_{|G|})$. Notice also that condition (*) is automatically satisfied. □

3. Induction of forms

We keep the conventions and the notation from the previous section. Let H be a subgroup of G and let U be a $K[H]$-module. We write the induced $K[G]$-module $\mathrm{Ind}_H^G(U) = K[G] \otimes_{K[H]} U$ in the form

$$\mathrm{Ind}_H^G(U) = \bigoplus_{i=1}^{r} x_i \otimes U,$$

where $\{x_1, \ldots, x_r\}$ is a system of representatives of the left cosets of $G \pmod H$. Let β be an H-form on U. The induced module $\mathrm{Ind}_H^G(U)$ inherits naturally the G-form $\tilde{\beta}$ defined by

$$\tilde{\beta}(x_i \otimes v, x_j \otimes w) = \delta_{ij}\beta(v, w),$$

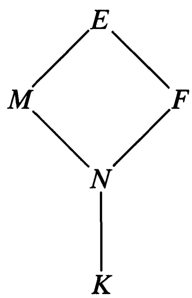
which will be called the form induced from β. We have the following result.

(3.1) THEOREM. *Let H be a normal subgroup of prime index p in G ($p > 2$) and let U be a $K[H]$-module. Suppose that $V = \mathrm{Ind}_H^G(U)$ is irreducible and satisfies condition (*) of the previous section. Then any positive-definite G-form on V is isometric to a G-form induced from a positive-definite H-form on U.*

Proof. The induction functor Ind_H^G provides an injection of $M := \mathrm{End}_{K[H]}(U)$ into $E = \mathrm{End}_{K[G]}(V)$. Two cases have to be distinguished.

(a) $M = E$. In this case $\mathrm{Res}_H^G(V)$ is the orthogonal sum of non-isomorphic $K[H]$-submodules $x_i \otimes U$, where $\{x_1 = 1, x_2, \ldots, x_p\}$ is a system of representatives of G/H. Any G-form β on V is in this case the orthogonal sum of p copies of $\bar{\beta}: U \times U \to K$, where $\bar{\beta}$ is given by $\bar{\beta}(u, v) := \beta(1 \otimes u, 1 \otimes v)$.

(b) $M \subsetneqq E$. In this case $\mathrm{Res}_H^G(V)$ is isomorphic to $U \oplus \cdots \oplus U$; therefore $E \cong \mathbf{M}_p(M)^{G/H}$. Comparing the dimensions over M, we see that E/M is an extension of degree p. Note that complex conjugation is non-trivial on M (since $[E : M]$ is odd, M cannot be contained in the subfield F of E fixed by complex conjugation). Let N be the intersection $M \cap F$. We sketch here for clarity the related tower of fields:



Let β_1 and β_2 be positive definite G-forms on V and assume that β_1 is a form induced from H, that is

$$\beta_1(x_i \otimes u, x_j \otimes v) = \delta_{ij}\bar{\beta}_1(u, v),$$

where $\bar{\beta}_1$ is a positive-definite H-form on U. Let B_i: $V \times V \to N$ be such that

$$\beta_i(x, y) = \mathrm{Tr}_{E/K}(B_i(x, y)) \quad \text{for } i = 1, 2.$$

Since $[F : N] = p$ is odd, by Theorem 2.4 Part II, the forms B_1 and B_2 are projectively isometric, that is there exists a in N such that aB_1 and B_2 are isometric. Applying the trace $\mathrm{Tr}_{N/K}$ we see that the forms $\beta_3 := \mathrm{Tr}_{N/K}(aB_1)$ and $\beta_2 = \mathrm{Tr}_{N/K}(B_2)$ are isometric. We finish the proof by showing that β_3 is an induced form as well

$$\begin{aligned} \beta_3(x_1 \otimes u, x_j \otimes v) &= \mathrm{Tr}_{N/K} B_3(x_i \otimes u, x_j \otimes v) \\ &= \mathrm{Tr}_{N/K} B_1(x_i \otimes au, x_j \otimes v) \\ &= \bar{\beta}_1(au, v)\delta_{ij}. \quad \square \end{aligned}$$

(3.2) Corollary. *Let G be a nilpotent group of odd order and let V be an irreducible $K[G]$-module. Let β be a positive-definite G-form on V. Then there exists a divisor n of $|G|$ and a totally positive element a in the cyclotomic field $K(\zeta_n)$ such that β is isometric to an orthogonal sum*

$$\beta_0 \perp \cdots \perp \beta_0,$$

where $\beta_0(x, y) = Tr_{K(\zeta_n)/K}(ax\bar{y})$.

Proof. We may assume that G acts faithfully on V. We prove the corollary by induction on the order of G. For $|G| = 1$ the statement is trivial. For $|G| > 1$ we have two possible cases.

(1) If $V = \mathrm{Ind}_H^G(U)$, where H is a subgroup of index p, then we apply Theorem 3.1 and the induction hypothesis.

(2) If V is not induced, then, by [4, Section 3], the group G must be cyclic. □

(3.3) *Remark.* Corollary 3.2 together with Feit's theorem [2] give an alternative proof for Corollary 2.6.

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