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CLOSED-FORM SOLUTIONS OF SOME PARTIAL DIFFERENTIAL EQUATIONS VIA QUASI-SOLUTIONS II

BY

LEE A. RUBEL[1]

Dedicated to my teacher, R. Creighton Buck, on the occasion of his retirement

Introduction

In this paper, we continue the work of Part I by studying a kind of separation of variables for some of the PDE's of mathematical physics. For example, in Section 1, we find all solutions of the form $u = \varphi(A(x) + B(y))$ of Laplace's equation in two variables, $u_{xx} + u_{yy} = 0$. In the terminology of Part I, we study which $A(x) + B(y)$ are quasi-solutions of Laplace's equation. (However, Part II can be read independently of Part I.) Surprisingly, the Jacobi elliptic functions appear naturally in this context. At present, their appearance seems to be an "accident" of computation. We have no conceptual explanation for their occurrence, or for why doubly-periodic functions should play a role in this kind of separation of variables.

Questions to be addressed at a future time involve expansions in series whose terms are constants times these Jacobi elliptic functions. How does one choose these constants? Are there any orthogonality relations? How fast is the convergence? These are natural questions especially for solving the Dirichlet problem in a rectangle, for which this form of separation of variables seems quite appropriate. Note that by using logarithms or exponentials, it is equivalent to the form $u = \psi(C(x)D(y))$.

Willard Miller, Jr., has shown the author how to derive many of the results of this paper by the method of differential-Stäckel matrices in [KAM]. This organized method has some advantages over the ad hoc methods of the present paper, but the calculations are still lengthy. The author thanks Professor Miller for his helpful communications.

Throughout this paper, all functions are supposed to be real-analytic on a domain in the appropriate Euclidean space. Alternatively, one could suppose

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that certain expressions that appear are zero-free. In this paper, the work is purely formal. Questions like boundary-value problems will be held for a possible future paper. Some of the calculations in this paper were done by Georg Reinhart at a Sun workstation, while he was supported in this work by a grant from the Research Board of the University of Illinois.

1. Laplace's equation in two variables

THEOREM 1. *Suppose* $u = \varphi(A(x) + B(y))$ *is a non-constant solution of* $u_{xx} + u_{yy} = 0$. *Then we must be in one of the following six cases, up to scaling and other normalizations*:

$$u = \cos x \cosh y, \tag{1.1}$$

$$u = \alpha x^2 + \beta x - \alpha y^2 + \gamma y, \tag{1.2}$$

$$u = \log(\cosh y + \cos x), \tag{1.3}$$

$$u = \log(\cosh y - \cos x), \tag{1.4}$$

$$\begin{aligned} A(x) &= \log[\operatorname{dn}(x\colon k) - k \operatorname{cn}(x\colon k)], \\ B(y) &= -\log[\operatorname{dn}(iy\colon k) - k \operatorname{cn}(iy\colon k)], \\ \varphi(t) &= \log \frac{\cosh t - 1}{\cosh t + 1}, \end{aligned} \tag{1.5}$$

$$\begin{aligned} A(x) &= \log[\operatorname{dn}(x\colon k) - k \operatorname{cn}(x\colon k)], \\ B(y) &= \log[\operatorname{dn}(iy\colon k) - k \operatorname{cn}(iy\colon k)], \\ \varphi(t) &= \log \frac{\cosh t - 1}{\cosh t + 1}. \end{aligned} \tag{1.6}$$

Furthermore, each of these six cases gives a solution of the form $\varphi(A(x) + B(y))$.

A word about "scaling and other normalizations" is in order. If $u(x, y)$ satisfies Laplace's equation, then so does

$$\gamma u(\alpha(x - x_0), \alpha(y - y_0)) + \beta$$

for any constants, γ, α, β, and any real constants x_0 and y_0. We also implicitly accept the change of variables $x \to y$, $y \to x$. Throughout this paper, where convenient, we have chosen the values of the parameters $\gamma, \alpha, \beta, (x_0, y_0)$ that give the simplest formulas for u.

Proof of Theorem 1. We suppose

$$u = \varphi(A(x) + B(y)). \tag{1.7}$$

In what follows, the argument of $\varphi, \varphi', \varphi''$ is always $A + B$. The argument of A, A', A'', A''' is always x and the argument of B, B', B'', B''' is always y. We have

$$u_x = \varphi' A', u_{xx} = \varphi'' A'^2 + \varphi' A'', \text{etc.} \tag{1.8}$$

so that

$$u_{xx} + u_{yy} = 0 \tag{1.9}$$

becomes

$$\varphi''(A'^2 + B'^2) + \varphi'(A'' + B'') = 0, \tag{1.10}$$

which we write as

$$\frac{A'^2 + B'^2}{A'' + B''} = -\frac{\varphi'(A + B)}{\varphi''(A + B)}. \tag{1.11}$$

Here we suppose we are *not* in Case (2′).

Case (2′). $A'' + B'' = 0$.

It is easy to see that in Case (2′) we have Case (2) of Theorem 1. Note now that the right-hand side of (1.11) is a function of $A + B$. Letting

$$J(R(x,y), S(x,y)) = \begin{vmatrix} \dfrac{\partial R}{\partial x} & \dfrac{\partial R}{\partial y} \\ \dfrac{\partial S}{\partial x} & \dfrac{\partial S}{\partial y} \end{vmatrix} \tag{1.12}$$

be the Jacobian determinant of the two function $R(x, y)$ and $S(x, y)$, we have

$$J\left(\frac{A'^2 + B'^2}{A'' + B''}, A + B\right) = 0. \tag{1.13}$$

Expanding (1.13) and simplifying, we get

$$\begin{aligned} &\left[(A'' + B'')2A'A'' - (A'^2 + B'^2)A'''\right]B' \\ &\quad = \left[(A'' + B'')2B'B'' - (A'^2 + B'^2)B'''\right]A' \end{aligned} \tag{1.14}$$

which becomes

$$2A''^2 - (A'^2 + B'^2)\frac{A'''}{A'} = 2B''^2 - (A'^2 + B'^2)\frac{B'''}{B'}. \tag{1.15}$$

Now (1.14) (and also (1.15)) is what we call a differential interlacing. Our purpose is to use some elimination procedures to get from it a differential interlacing in which the variables are separated. In (1.15), take $\partial/\partial x$ to get

$$4A''A''' - (A'^2 + B'^2)\left(\frac{A'''}{A'}\right)' - \frac{A'''}{A'}2A'A'' = -2A'A''\frac{B'''}{B'}. \tag{1.16}$$

Now simplify to get

$$2A''A''' - \left(\frac{A'''}{A'}\right)'(A'^2 + B'^2) = -2A'A''\frac{B'''}{B'}. \tag{1.17}$$

Now take $\partial/\partial y$ to get

$$-\left(\frac{A'''}{A'}\right)'' B'B'' = -A'A''\left(\frac{B'''}{B'}\right)'. \tag{1.18}$$

Dividing, we get

$$\left(\frac{A'''}{A'}\right)'/A'A'' = \left(\frac{B'''}{B'}\right)'/B'B'', \tag{1.19}$$

so that the variables are separated. Consequently, for some constant λ,

$$\left(\frac{A'''}{A'}\right)'/A'A'' = \lambda = \left(\frac{B'''}{B'}\right)'/B'B''. \tag{1.20}$$

For the time being we suppose we are not in Cases (3′)–(4′) where $\lambda = 0$. So $\lambda \neq 0$. We have

$$(A'''/A')' = \lambda A'A'', \tag{1.21}$$

$$A'''/A' = \frac{\lambda}{2}A'^2 + a \quad (a = \text{const}), \tag{1.22}$$

$$A''' = \frac{\lambda}{2}A'^3 + aA', \tag{1.23}$$

$$A''A''' = \frac{\lambda}{2}A'^3A'' + aA'A''. \tag{1.24}$$

$$\frac{A''^2}{2} = \frac{\lambda}{8}A'^4 + \frac{a}{2}A'^2 + \frac{b}{2} \quad (b = \text{const}), \tag{1.25}$$

$$A''^2 = \frac{\lambda}{4}A'^4 + aA'^2 + b. \tag{1.26}$$

Let

$$\mathfrak{A} = A', \quad \mathfrak{B} = B'. \tag{1.27}$$

Hence

$$\mathfrak{A}'^2 = \frac{\lambda}{4}\mathfrak{A}^4 + a\mathfrak{A}^2 + b. \tag{1.28}$$

$$\mathfrak{A}' = \pm\sqrt{\frac{\lambda}{4}\mathfrak{A}^4 + a\mathfrak{A}^2 + b} \tag{1.29}$$

$$\int_0^{\mathfrak{A}(x)} \frac{d\sigma}{\sqrt{\frac{\lambda}{4}\sigma^4 + a\sigma^2 + b}} = \pm x + c. \tag{1.30}$$

Taking a translate of $A(x)$ if necessary, we may suppose

$$c = 0. \tag{1.31}$$

Notice that the integral in (1.30) is an elliptic integral. Working with $\mathfrak{B}$ now, we similarly get

$$\int_0^{\mathfrak{B}(y)} \frac{d\sigma}{\sqrt{\frac{\lambda}{4}\sigma^4 + \bar{a}\sigma^2 + \bar{b}}} = \pm y, \tag{1.32}$$

where $\bar{a}$ and $\bar{b}$ are other constants. Normalizing and scaling, we have

$$\mathfrak{A}(x) = \alpha\,\mathrm{sn}(\beta x : k), \tag{1.33}$$

$$\mathfrak{B}(y) = \bar{\alpha}\,\mathrm{sn}(\bar{\beta} x : k) \tag{1.34}$$

where

$$\int_0^{\mathrm{sn}(x:k)} \frac{ds}{\sqrt{(1-s^2)(1-k^2s^2)}} = x \tag{1.35}$$

defines the Jacobi elliptic function (sinus amplitudinus) $\mathrm{sn}(x:k)$.

We now see that in order for $\mathfrak{A} = A'$ and $\mathfrak{B} = B'$ to fit (1.14), some restrictions must be placed on $a, b, \bar{a}, \bar{b}$ (equivalently on $\alpha, \beta, \bar{\alpha}, \bar{\beta}$). Rewrite (1.15) as

$$\frac{A'''}{A'}\left[\frac{2A'A''^2 - A'^2A'''}{A'''} - B'^2\right] = \frac{B'''}{B'}\left[\frac{2B'B''^2 - B'^2B'''}{B'''} - A'^2\right]. \tag{1.36}$$

But from (1.23)

$$A''' = \frac{\lambda}{2}A'^3 + aA' \tag{1.23}$$

and similarly

$$B''' = \frac{\lambda}{2}B'^3 + \bar{a}B', \tag{1.23'}$$

the left-hand side of (1.36) becomes

$$-\frac{\lambda}{2}A'^2B'^2 - aB'^2 + 2A''^2 - \left[\frac{\lambda}{2}A'^4 + aA'^2\right]. \tag{1.37}$$

Now use (1.26) to see that the left-hand side of (1.36) becomes

$$-\frac{\lambda}{2}A'^2B'^2 - aB'^2 + \left(\frac{\lambda}{2}A'^4 + 2aA'^2 + 2b\right) - \left(\frac{\lambda}{2}A'^4 + aA'^2\right), \tag{1.38}$$

which simplifies to

$$-\frac{\lambda}{2}A'^2B'^2 - aB'^2 + aA'^2 + 2b. \tag{1.39}$$

Similarly, the right-hand side of (1.36) becomes

$$-\frac{\lambda}{2}A'^2B'^2 - \bar{a}A'^2 + \bar{a}B'^2 + 2\bar{b}. \tag{1.40}$$

They are equal exactly when

$$-aB'^2 + aA'^2 + 2b = -\bar{a}A'^2 + \bar{a}B'^2 + 2\bar{b}, \tag{1.41}$$

or

$$(a + \bar{a})A'^2 = (a + \bar{a})B'^2 + 2(\bar{b} - b). \tag{1.42}$$

Thus, we have the necessary restriction

$$a + \bar{a} = 0, \quad \bar{b} = b. \tag{1.43}$$

Now, if we consider

$$\int_0^{\mathfrak{A}(x)} \frac{ds}{\sqrt{(1 - s^2)(1 - k^2s^2)}} \tag{1.44}$$

then the quartic polynomial under the radical is

$$(1 - s^2)(1 - k^2s^2) = 1 - (1 + k^2)s^2 + k^2s^4. \tag{1.45}$$

When we pass to $\mathfrak{B}$ (with $\bar{b} = b$ but $\bar{a} = -a$), we must get

$$1 + (1 + k^2)s^2 + k^2 s^4 = (1 + s^2)(1 + k^2 s^2) \tag{1.46}$$

so that

$$\int_0^{\mathfrak{B}(y)} \frac{ds}{\sqrt{(1 + s^2)(1 + k^2 s^2)}} = \pm y \tag{1.47}$$

Thus,

$$\mathfrak{B}(y) = \mp i \operatorname{sn}(iy : k). \tag{1.48}$$

Note here that, because $\lambda \neq 0$, we have $k \neq 0$.

At this point, we give a brief summary of some basic definitions and facts about the Jacobi elliptic functions (see [WHW])

Let

$$u = \int_0^y (1 - t^2)^{-1/2} (1 - k^2 t^2)^{-1/2} \, dt. \tag{1.49}$$

Then

$$y = \operatorname{sn}(u : k), \tag{1.50}$$

$$\operatorname{sn}^2(u : k) + \operatorname{cn}^2(u : k) = 1, \tag{1.51}$$

$$k^2 \operatorname{sn}^2(u : k) + \operatorname{dn}^2(u : k) = 1, \tag{1.52}$$

$$\operatorname{cn}(0 : k) = \operatorname{dn}(0 : k) = 1,$$

$$\frac{d}{du} \operatorname{sn}(u : k) = \operatorname{cn}(u : k) \operatorname{dn}(u : k), \tag{1.53}$$

$$\frac{d}{du} \operatorname{cn}(u : k) = \operatorname{sn}(u : k) \operatorname{dn}(u : k), \tag{1.54}$$

$$\frac{d}{du} \operatorname{dn}(u : k) = -k^2 \operatorname{sn}(u : k) \operatorname{cn}(u : k), \tag{1.55}$$

$$\int \operatorname{sn}(u : k) \, du = \frac{1}{2k} \log \frac{\operatorname{dn}(u : k) - k \operatorname{cn}(u : k)}{\operatorname{dn}(u : k) + k \operatorname{cn}(u : k)}, \tag{1.56}$$

$$\operatorname{sn}(-u : k) = -\operatorname{sn}(u : k), \tag{1.57}$$

$$\operatorname{cn}(-u : k) = \operatorname{cn}(u : k),$$

$$\operatorname{dn}(-u ; k) = \operatorname{dn}(u : k).$$

By simple operations

$$(1.58) \qquad \int \operatorname{sn}(u\colon k)\, du = \frac{1}{k} \log[\operatorname{dn}(u\colon k) - k \operatorname{cn}(u\colon k)].$$

We assume we are in Case 5′ where $k \neq \pm 1$. In this case ((5′)–(6′)) we wish to find φ. We seek to determine Φ in

$$(1.59) \qquad \frac{A''(x) + B''(y)}{A'(x)^2 + B'(y)^2} = \Phi(A(x) + B(y)).$$

Here, in (1.48), we have chosen the minus sign. Later we will do the plus sign. Write

$$(1.60) \qquad \operatorname{sn}(x\colon k) = s, \operatorname{sn}(iy\colon k) = \sigma.$$

Then

(1.61)

$$\frac{A'' + B''}{A'^2 + B'^2} = \frac{(1 - s^2)^{1/2}(1 - k^2 s^2)^{1/2} + (1 - \sigma^2)^{1/2}(1 - k^2\sigma^2)^{1/2}}{s^2 - \sigma^2},$$

$$(1.62) \qquad \frac{(1 - s^2)^{1/2}(1 - k^2 s^2)^{1/2} + (1 - \sigma^2)^{1/2}(1 - k^2\sigma^2)^{1/2}}{s^2 - \sigma^2}$$
$$= \Phi\left(\frac{1}{k} \log \frac{(1 - k^2 s^2)^{1/2} - k(1 - s^2)^{1/2}}{(1 - k^2\sigma^2)^{1/2} - k(1 - \sigma^2)^{1/2}}\right).$$

Write

$$(1.63) \quad R = \frac{(1 - s^2)^{1/2}(1 - k^2 s^2)^{1/2} + (1 - \sigma^2)^{1/2}(1 - k^2\sigma^2)^{1/2}}{s^2 - \sigma^2}$$

$$(1.64) \qquad T = \frac{(1 - k^2 s^2)^{1/2} - k(1 - s^2)^{1/2}}{(1 - k^2\sigma^2)^{1/2} - k(1 - \sigma^2)^{1/2}}.$$

Georg Reinhart has found and verified on a computer, using the Macsyma program, that

$$(1.65) \qquad R = W(T)$$

where

$$(1.66) \qquad W(z) = k\frac{z^2 + 1}{z^2 - 1}.$$

This was done as follows. Set $\sigma = 0$. Then R becomes

$$\tilde{R}^2(s) = \frac{(1 - s^2)^{1/2}(1 - k^2 s^2)^{1/2} + 1}{s^2} \tag{1.67}$$

and T becomes

$$\tilde{T} = \frac{1}{1-k}\left[(1 - k^2 s^2)^{1/2} - k(1 - s^2)^{1/2}\right] \tag{1.68}$$

and we have

$$\tilde{R} = W(\tilde{T}) \tag{1.69}$$

so that

$$W(z) = \tilde{R}(\tilde{T}^{-1}(z)), \tag{1.70}$$

and W can be found by straightforward computation. Hence

$$\Phi(t) = k\,\frac{\cosh kt}{\sinh kt} = -\frac{\varphi''(t)}{\varphi'(t)}, \tag{1.71}$$

which is a simple differential equation for φ, whose solution, up to a constant factor, is

$$\log \frac{\cosh kt - 1}{\sinh kt + 1}. \tag{1.72}$$

Absorbing factors of k into $A(x)$ and $B(y)$, we are led to the solution (1.5).

If we choose the plus sign in (1.48), the computations are very similar. We get

(1.63*)
$$R^* = \frac{(1 - s^2)^{1/2}(1 - k^2 s^2)^{1/2} - (1 - \sigma^2)^{1/2}(1 - k^2\sigma^2)^{1/2}}{s^2 - \sigma^2},$$

(1.64*)
$$T = \left[(1 - k^2 s^2)^{1/2} - k(1 - s^2)^{1/2}\right]\left[(1 - k^2\sigma^2)^{1/2} - k(1 - \sigma^2)^{1/2}\right],$$

$$R^* = W^*(T^*), \tag{1.65*}$$

where

$$W^*(z) = -k\frac{(1 - k^2)^2 + z^2}{(1 - k^2) - z^2}, \tag{1.66*}$$

and this leads to the solution (1.6).

Let us now do the case

$$k = \pm 1. \tag{1.73}$$

In this case, we have

$$\int_0^{\mathfrak{B}(y)} \frac{ds}{1+s^2} = \pm y, \tag{1.74}$$

so that

$$\mathfrak{B}(y) = B'(y) = \pm \tan y, \tag{1.75}$$

$$B(y) = \pm \log \cos y. \tag{1.76}$$

Similarly,

$$A(x) = \log \cosh x. \tag{1.77}$$

It is quickly verified that

$$\varphi(t) = \exp t, \tag{1.78}$$

so that we have the solution given by (1.1).

We now treat the only case remaining, namely $\lambda = 0$. This means

$$\mathfrak{A}'^2 = a\mathfrak{A}^2 + b. \tag{1.79}$$

Then up to a normalization,

$$A = \cos x. \tag{1.80}$$

We then find

$$B = \pm \cosh y. \tag{1.81}$$

From

$$\frac{A'' + B''}{A'^2 + B'^2} = -\frac{\varphi''(A+B)}{\varphi'(A+B)} = W(A+B) \tag{1.82}$$

say, we have

$$\frac{-\cos x \pm \cosh y}{\sin^2 x + \sinh^2 y} = W(\cos x \pm \cosh y), \tag{1.83}$$

so that

$$W(t) = \frac{1}{t} \tag{1.84}$$

and therefore

$$\varphi(t) = \log t. \tag{1.85}$$

In other words, we are in Case (3) or Case (4). Thus, the first (direct) part of the theorem is proved. For the second ("furthermore") part, we observe that all the steps in the first part are reversible, so that each of Cases (1)–(6) does indeed give a solution of the desired form. To be on the safe side, Georg Reinhart has run a successful direct computer check, using the Mathematica program on Cases (5) and (6). It was not possible to fully verify that these cases are solutions, but some "random" numerical values were plugged in, and the answer was indeed zero in these cases. Of course Cases (1)–(4) are easily checked by hand computation.

2. The wave equation in two variables

Since $u(x, y)$ is a solution of the wave equation in two variables, if and only if $u(x, iy)$ is a solution of the Laplace equation in two variables, and since Theorem 1 is purely formal, we have the following result, with no need for further proof.

THEOREM 2. *Suppose $u = \varphi(A(x) + B(y))$ is a non-constant solution of $u_{xx} - u_{yy} = 0$. Then we must be in one of the following six cases, up to scaling and other normalizations*:

$$u = \cos x \cos y \text{ (also } u = \cosh x \cosh y), \tag{2.1}$$

$$u = ax^2 + ay^2 + \alpha x + \beta y, \tag{2.2}$$

$$u = \log(\cos x + \cos y) \quad (\textit{also } u = \log(\cosh x + \cosh y)), \tag{2.3}$$

$$u = \log(\cos y - \cos x) \quad (\textit{also } u = \log(\cosh y - \cosh x)), \tag{2.4}$$

$$\begin{aligned} A(x) &= \log[\mathrm{dn}(x : k) - k\,\mathrm{cn}(x : k)], \\ B(y) &= -\log[\mathrm{dn}(y : k) - k\,\mathrm{cn}(y : k)], \\ \varphi(t) &= \log \frac{\cosh t - 1}{\cosh t + 1}, \end{aligned} \tag{2.5}$$

$$\begin{aligned} A(x) &= \log[\mathrm{dn}(x : k) - k\,\mathrm{cn}(x : k)], \\ B(y) &= \log[\mathrm{dn}(y : k) - k\,\mathrm{cn}(y : k)], \\ \varphi(t) &= \log \frac{\sinh t - 1}{\cosh t + 1}. \end{aligned} \tag{2.6}$$

Furthermore, each of these six cases gives a solution of the form $\varphi(A(x) + B(y))$.

We remark that Case (2.5) (or (2.6)) can be used to prove an addition formula involving the Jacobi elliptic functions. One example of an addition formula is (see [WHW])

$$\operatorname{sn}(x+y)=\frac{\operatorname{sn} x \operatorname{cn} y \operatorname{dn} y+\operatorname{sn} y \operatorname{cn} x \operatorname{dn} x}{1-k^2 \operatorname{sn}^2 x \operatorname{sn}^2 y}. \tag{2.7}$$

(Here, we have suppressed the parameter k in sn, cn, dn.) We proceed here as follows. We have the solution of the wave equation $u(x, y) = \varphi(A(x) + B(y))$ of (2.5), where $A(x)$ and $B(y)$ are expressions in the Jacobi elliptic functions. Hence, for this u, perhaps adding a constant to make $u(0,0) = 0$,

$$u(x,y)=f(x+y)+g(x-y). \tag{2.8}$$

First setting $y = x$ and then $y = -x$, we get, on supposing that $f(0) = g(0) = 0$,

$$f(x)=u\left(\frac{x}{2},\frac{x}{2}\right), \tag{2.9}$$

$$g(y)=u\left(\frac{y}{2},-\frac{y}{2}\right). \tag{2.10}$$

Hence (2.8) becomes

$$u(x,y)=u\left(\frac{x+y}{2},\frac{x+y}{2}\right)+u\left(\frac{x-y}{2},-\frac{x-y}{2}\right), \tag{2.11}$$

which is the desired addition formula.

3. The heat equation

THEOREM 3. *Suppose $u(x,t) = \varphi(A(x) + B(t))$ is a nonconstant real solution of the heat equation $u_{xx} = u_t$. Then, apart from scalings and translations, u must have one of the following seven forms*:

$$u(x,t)=x, \tag{3.1}$$

$$u(x,t)=\frac{x^2}{2}+t, \tag{3.2}$$

$$u(x,t)=e^{x+t}, \tag{3.3}$$

$$u(x,t)=e^{-x+t} \tag{3.4}$$

$$u(x,t)=e^{-t}\sin x, \tag{3.5}$$

$$u(x,t)=e^{t}\sinh x, \tag{3.6}$$

$$u(x,t)=\operatorname{erf}\frac{x}{2\sqrt{t}}. \tag{3.7}$$

Moreover, each of (3.1)–(3.7) *is a solution of the form* $\varphi(A(x) + B(t))$.

Proof of Theorem 3. We start by supposing

$$u_{xx} - u_t = 0, \tag{3.8}$$

$$u = \varphi(A(x) + B(t)). \tag{3.9}$$

We then have

$$\varphi'' A'^2 + \varphi' A'' - \varphi' B' = 0, \tag{3.10}$$

$$\varphi''/\varphi' = \frac{B' - A''}{A'^2} = \Phi(A + B), \tag{3.11}$$

or, if we suppose, for a while, that

$$B' - A'' \neq 0, \tag{3.12}$$

then

$$-\frac{\varphi'}{\varphi''} = \frac{A'^2}{B' - A''} = \Psi(A + B). \tag{3.13}$$

Consequently

$$J\left(\frac{A'^2}{B' - A''}, A + B\right) = 0, \tag{3.14}$$

where J, as before, denotes the Jacobian determinant. Thus

$$\left[(B' - A'')2A'A'' + A'^2 A'''\right]B' + A'^3 B'' = 0, \tag{3.15}$$

or

$$\frac{(B' - A'')2A'A'' + A'^2 A'''}{A'^3} = -\frac{B''}{B'}. \tag{3.16}$$

Take $\partial/\partial t$ in (3.16) to get

$$B''\frac{2A''}{A'^2} = \left(\frac{B''}{B'}\right)' \tag{3.17}$$

Suppose also

$$B''(t) \neq 0. \tag{3.18}$$

Then

$$\frac{2A''}{A'^2} = -\left(\frac{B''}{B'}\right)'/B'', \tag{3.19}$$

and the variables are separated. We have, for some constant λ,

$$\frac{2A''}{A'^2} = \lambda, \tag{3.20}$$

$$\left(\frac{B''}{B'}\right)' = -\lambda B''. \tag{3.21}$$

Now suppose

$$\lambda \neq 0. \tag{3.22}$$

Then we have

$$A' = \frac{2}{1 - \lambda x}, \; A'' = \frac{2\lambda}{(a - \lambda x)^2}, \; A''' = \frac{4\lambda^2}{(a - \lambda x)^3} \tag{3.23}$$

for some constant a, and thus

$$A = -\frac{2}{\lambda}\log(a - \lambda x) + b, \tag{3.24}$$

for some constant b. Consequently

$$e^{\lambda A} = \frac{e^b}{(a - \lambda x)^2}. \tag{3.25}$$

From (3.21)

$$\frac{B''}{B'} = -\lambda B' + c. \tag{3.26}$$

for some constant c. We now show that

$$c = 0, \tag{3.27}$$

so that

$$\frac{B''}{B'} = -\lambda B'. \tag{3.28}$$

From (3.26), we have

$$\log B' = -\lambda B + ct + d, \tag{3.29}$$

for a constant d. Then we have

$$B' = ke^{-\lambda B}e^{ct}, \tag{3.30}$$

$$e^{\lambda B}B' = ke^{ct}, \tag{3.31}$$

$$\frac{1}{\lambda}e^{\lambda B} = \frac{k}{c}e^{ct} + \alpha, \tag{3.32}$$

if we suppose

$$c \neq 0. \tag{3.33}$$

We rewrite (3.32) as

$$e^{\lambda B} = \tilde{k}e^{ct} + \tilde{\alpha}, \tag{3.34}$$

$$B = \frac{1}{\lambda}\log(\tilde{k}e^{ct} + \tilde{\alpha}), \tag{3.35}$$

$$B' = \frac{c}{\lambda}\frac{\tilde{k}e^{ct}}{\tilde{k}e^{ct} + \tilde{\alpha}}, \tag{3.36}$$

$$B'' = \frac{c^2}{\lambda}\tilde{k}\frac{\tilde{\alpha}e^{ct}}{(\tilde{k}e^{ct} + \tilde{\alpha})^2}. \tag{3.37}$$

Now try this A and B in (3.15). The left-hand side of (3.15) becomes

$$(\lambda B'^2 + B'')\frac{8\lambda}{(a - \lambda x)^3}. \tag{3.38}$$

If this is to vanish, then by (3.26), we must have (3.27), contrary to (3.33). Hence (3.27) holds in any event. Hence

$$\frac{B''}{B'} = -\lambda B' \tag{3.39}$$

from which we get

$$e^{\lambda B} = lt + \alpha, \tag{3.40}$$

$$B = \frac{1}{\lambda}\log(\alpha + lt), \tag{3.41}$$

$$B' = \left(\frac{l}{\lambda}\right)e^{-\lambda B}. \tag{3.42}$$

Note that

$$A'' = \frac{2\lambda}{\kappa} e^{\lambda A}$$

for a suitable constant κ.

$$(3.43) \qquad \frac{B' - A''}{A'^2} = \frac{\left(\frac{l}{\lambda}\right) e^{-\lambda B} - \frac{2\lambda}{\kappa} e^{\lambda A}}{\frac{4}{\kappa} e^{\lambda A}}.$$

$$(3.44) \qquad \frac{B' - A''}{A'^2} = \frac{\left(\frac{l}{\lambda}\right) \frac{2\lambda}{\kappa} - e^{\lambda(A+B)}}{\frac{4}{\kappa} e^{\lambda(A+B)}},$$

which makes manifest the functional dependence of (3.11). We have

$$(3.45) \qquad \frac{\varphi''(s)}{\varphi'(s)} = \frac{\kappa l}{4\lambda} e^{-\lambda s} - \frac{\lambda}{2}$$

$$(3.46) \qquad \log \varphi'(s) = -\frac{\kappa l}{4\lambda^2} e^{-\lambda s} - \frac{\lambda}{2} s + \beta,$$

and for simplicity we choose

$$(3.47) \qquad \beta = 0.$$

This yields

$$(3.48) \qquad \varphi(s) = \int_0^s \exp\left(-\frac{kl}{4\lambda^2} e^{-\lambda\sigma}\right) e^{-\lambda\sigma/2}\, d\sigma.$$

Make the substitution

$$(3.49) \qquad \rho = e^{-\lambda\sigma/2},\ r = e^{-\lambda s/2}$$

to get

$$(3.50) \qquad \varphi(s) = -\frac{2}{\lambda} \int_0^r \exp\left(-\frac{kl}{4\lambda^2} \rho^2\right) d\rho,$$

which we change by an additive constant to get

$$(3.51) \qquad \varphi(s) = \frac{\sqrt{\pi}}{2\sqrt{\gamma}} \operatorname{erf}\left(r\sqrt{\gamma}\right)$$

where

$$\gamma = \frac{\kappa l}{4\lambda^2}. \tag{3.52}$$

Now putting $s = A + B$ and using (3.24) and (3.41), we get

$$u = \operatorname{erf}\left(\xi \frac{x}{\sqrt{t}}\right), \tag{3.53}$$

after taking simplifying normalizations, where ξ is a constant. But it is easy to see that the only value of ξ for which (3.53) is a solution of $u_{xx} = u_t$ is

$$\xi = \frac{1}{2}. \tag{3.54}$$

Thus, we are in Case (3.7). We now consider the cases we excluded, namely

$$B' - A'' = 0,\ B'' = 0, \quad \text{and} \quad \lambda = 0.$$

The first of these is included in the second, so we now assume

$$B' - A'' = 0. \tag{3.55}$$

Then the variables separate, so that we easily have

$$A = x^2/2, \quad B = t, \tag{3.56}$$

or, in a degenerate case,

$$A = x, \quad B = 0. \tag{3.57}$$

Thus (3.57) accounts for Case (3.1) and (3.56) for Case 3.2. Now we consider

$$B'' = 0, \quad \text{but } B' - A'' \neq 0. \tag{3.58}$$

We then have

$$B' = \lambda = \text{const}. \tag{3.59}$$

We have

$$\varphi''/\varphi' = \frac{\lambda - A''}{A'^2} = \Phi(A + B) \tag{3.60}$$

and taking $\partial/\partial t$ we get

$$\Phi'(A+B)B' = 0 \tag{3.61}$$

so that

$$\Phi'(s) = 0, \tag{3.62}$$

$$\Phi = \text{const} = \mu \tag{3.63}$$

$$\varphi''/\varphi' = \mu, \tag{3.64}$$

so that

$$\varphi(s) = e^s, \tag{3.65}$$

up to scaling and translation. Hence

$$u = e^{A(x)+t} \tag{3.66}$$

or

$$u = e^{A(x)-t}. \tag{3.67}$$

From $u_{xx} = u_t$ in (3.66) we get

$$A'' + A'^2 = 1 \tag{3.68}$$

and, in (3.67),

$$A'' + A'^2 = -1 \tag{3.69}$$

which lead to Cases (3.3), (3.4), (3.5), and (3.6).

Finally, we have the case

$$\lambda = 0,\ B' - A'' \neq 0,\ B'' \neq 0, \tag{3.70}$$

where λ appears in (3.28). Hence $B'' = 0$, which is excluded by (3.70). This completes the proof of the first part of Theorem 3. The "moreover" part is easily verified by direct computation.

4. Laplace's equation in more than two variables

We state our theorem for three variables, but it will be obvious how to adapt the statement and proof for any number, exceeding two, of variables. The gist of the theorem is that for more than two variables, there is no new separation of variables possible.

THEOREM 4. *Suppose that $A'(x)B'(y)C'(z) \neq 0$ and that $u(x, y, z)$ is not a quadratic polynomial. Then there exists no non-constant function φ such that*

$$u = \varphi(A(x) + B(y) + C(z))$$

is a solution of

$$u_{xx} + u_{yy} + u_{zz} = 0.$$

Proof. Reasoning as in the proof of Theorem 1, we have

$$J_{x,y}\left(\frac{A'^2 + B'^2 + C'^2}{A'' + B'' + C''}, A + B + C\right) = 0, \tag{4.1}$$

where $J_{x,y}$ is the Jacobian determinant

$$J_{x,y}(F, G) = \begin{vmatrix} \dfrac{\partial F}{\partial x} & \dfrac{\partial F}{\partial y} \\ \dfrac{\partial G}{\partial x} & \dfrac{\partial G}{\partial y} \end{vmatrix}. \tag{4.2}$$

Expanding (4.1), we get

$$\begin{aligned} &\left[(A'' + B'' + C'')2A'A'' - (A'^2 + B'^2 + C'^2)A'''\right]B' \\ &\quad = \left[(A'' + B'' + C'')2B'B'' - (A'^2 + B'^2 + C'^2)B'''\right]A' \end{aligned} \tag{4.3}$$

unless

$$A''(x) + B''(y) + C''(z) = 0. \tag{4.4}$$

But in case of (4.4), the variables separate, and we see that A, B, C are each quadratic polynomials, and the result follows in this case. Otherwise, we see from (4.3) that

$$C''(2A'B'A'' - 2A'B'B'') - C'^2(A'''B' - A'B''') \tag{4.5}$$

is independent of the variable z. Now we do a case analysis.

Suppose first that

$$C'' \neq 0. \tag{4.6}$$

Then

$$A'B'A'' = A'B'B'' \tag{4.7}$$

and

$$A'''B' \equiv B'''A'. \tag{4.8}$$

If we now suppose that A and B are both non-constant then from (4.7), $A''(x) = \lambda$, $B''(y) = \lambda$ and so A and B must be quadratic polynomials. Then C must also be a quadratic polynomial, and the result follows in this case.

If $C'' = 0$, we may suppose $C' \neq 0$, since if $C' = 0$, the result follows.

So C' is a non-zero constant, and it follows from (4.5) that (4.8) must hold. By interchanging the variables, the arguing as above, we may assume that $A'' = 0$, $A' = \text{const} \neq 0$. Then from (4.7), it follows that B' is also a constant, and the result follows in this, the last case.

5. A remark

Remark 5. We note that if A, B, φ and ψ are non-constant and if

$$\varphi(A(x) + B(y)) = \psi(C(x) + D(y))$$

then

$$\varphi(s) = \psi(as + b), \quad A(x) = \alpha C(x) + d, \quad B(y) = \alpha D(y) + e,$$

so that there is no redundancy in the list of separated solutions that occurs in Theorem 1, say. For then, on letting $\psi^{-1}\varphi = \sigma$, we have

$$\sigma(A(x) + B(y)) = C(x) + D(y).$$

Taking $\partial^2/\partial x \partial y$, we get $\sigma''(A(x) + B(y))A'(x)B'(y) = 0$, so that $\sigma'' = 0$, and the result follows.

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University of Illinois
Urbana, Illinois