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ON THE SIZES OF THE SETS OF INVARIANT MEANS

BY

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1. Introduction and notation

Let G be a locally compact group with a fixed Haar measure λ. If G is compact, we assume $\lambda(G) = 1$. Let $L^p(G)$ be the associated real Lebesgue spaces $(1 \leq p \leq \infty)$. For each $f \in L^\infty(G)$ and $x \in G$, ${}_xf \in L^\infty(G)$ is defined by ${}_xf(y) = f(xy)$, $y \in G$. Let P denote the set of all $f \in L^1(G)$ with $f \geq 0$ and $\|f\|_1 = \int_G |f(x)|\, dx = 1$. A functional $m \in L^\infty(G)^*$ is called a mean if $m(1) = 1$ and $m(f) \geq 0$ for each $f \in L^\infty(G)$ with $f \geq 0$. We denote the set of all left invariant means on $L^\infty(G)$ by LIM, i.e. all the mean m with $m({}_xf) = m(f), (x \in G, f \in L^\infty(G))$. For $\varphi \in P$ and $f \in L^\infty(G)$, $\varphi * f \in L^\infty(G)$ is defined by

$$\varphi * f(x) = \int_G \varphi(t) f(t^{-1}x)\, dt, \quad x \in G,$$

and the set of all topologically left invariant means, i.e. the mean m on $L^\infty(G)$ with $m(\varphi * f) = m(f)$ $(\varphi \in P,\ f \in L^\infty(G))$, is denoted by $TLIM$. For any set A, the cardinality of A is denoted by $|A|$.

Let $CB(G)$ be the Banach space of continuous bounded functions on G in the supremum norm. We can define a left invariant mean on $CB(G)$ as in the case of $L^\infty(G)$. We denote all left invariant means and all topologically left invariant means on $CB(G)$ by $LIM(CB(G))$ and $TLIM(CB(G))$, respectively. When $LIM \neq \phi$, we say that G is amenable. It is well known that any topologically left invariant mean is left invariant and G is amenable if and only if one of the following conditions is true: (a) $TLIM \neq \phi$. (b) $LIM(CB(G)) \neq \phi$. (c) $TLIM(CB(G)) \neq \phi$. Also, if G is amenable as a discrete group, then G is amenable (see [9]).

The size of $LIM \sim TLIM$ was first studied by Granirer [7] and Rudin [18]. They showed independently that $LIM \sim TLIM \neq \phi$ if G is nondiscrete and

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amenable as a discrete group. Rosenblatt in [16] proved the following:

THEOREM (Rosenblatt). *Let G be a σ-compact locally compact group. If G is nondiscrete and amenable as a discrete group, then there are at least 2^c mutually singular elements of LIM each of which is singular to any element of TLIM. In particular,* $|LIM \sim TLIM| \geq 2^c$.

In Section 2, we use the axiom of choice and Proposition 3.4 of Rosenblatt in [15] to divide a "small" open dense subset of G into infinitely many pairwise disjoint P.P. sets, i.e. the set E such that $\bigcap_{i=1}^{n} x_i E$ is not locally null for any $x_1, x_2, \ldots, x_n \in G$. Then we apply the technique used in Chou [3] to embed a large set $\mathscr{F}_1$ into $LIM \sim TLIM$. This result removes the condition of σ-compact for the above theorem of Rosenblatt (also see [13] Chapter 7).

The study of the size of LIM was initiated in Banach [1], Day [6] and Granirer [8]. Chou in [4] and [5] proved that $|LIM(CB(G))| \geq 2^c$ for any amenable noncompact locally compact group and $|LIM| = 2^{2^{|G|}}$ for a discrete infinite amenable group. Lau and Paterson in [11] showed that $|TLIM| = 2^{2^{d(G)}}$, where $d(G)$ is the smallest cardinality of a cover of G by compact sets. Since any topologically left invariant mean is left invariant. $|LIM| \geq 2^{2^{d(G)}}$. The general problem remains open (also see Paterson [13] Chapter 7 and Yang [20]).

In Section 3, we prove that for any noncompact locally compact metrizable group G, $|LIM| = |TLIM|$. Since we have already known $|TLIM|$, this will give us the cardinality of LIM in this case. We also give examples which show that without the condition of metrizability this is not the case. Actually, $|LIM|$ can be as large as we want without changing $|TLIM|$ for some locally compact group. In this section, we also answer two problems raised by Rosenblatt in [15] and [17] on whether any $\theta \in LIM$ which is singular to every $\psi \in TLIM$ is concentrated on a "small" set of G and whether the discrete amenability assumption is necessary for a group G where there is an $f \in CB(G)$ and $\theta \in LIM(CB(G))$ with $\theta(f) = 1$ and $\psi(f) = 0$ for every $\psi \in TLIM(CB(G))$. We show that there are such $f \in CB(G)$ and $\theta \in LIM(CB(G))$ if $G = G_1 \times G_2$, where G_1 is any noncompact σ-compact nondiscrete group which is amenable as a discrete group and G_2 is any amenable group. This confirms Chou's conjecture $LIM(CB(G)) \neq TLIM(CB(G))$ for any noncompact amenable group in this case.

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2. The size of $LIM \sim TLIM$

Let $\mathscr{D}$ be the maximal ideal space of $L^\infty(G)$. With the Gelfand topology, $\mathscr{D}$ is a compact Hausdorff space. The Gelfand transform Λ is an isometry of

$L^\infty(G)$ onto $C(\mathscr{D})$, the algebra of real-valued continuous functions on $\mathscr{D}$ with the supremum norm. Note that if $\theta \in \mathscr{D}$, $L_x\theta \in \mathscr{D}$ is defined by $L_x\theta(f) = \theta({}_xf)$ for $f \in L^\infty(G)$ and $x \in G$. Each $\mu \in LIM$ can be identified with a G-invariant probability measure $\hat{\mu}$ on $\mathscr{D}: \hat{\mu}(\hat{f}) = \mu(f)$, $f \in L^\infty(G)$. We say that two $\mu_1, \mu_2 \in LIM$ are mutually singular if $\hat{\mu}_1$ and $\hat{\mu}_2$ are mutually singular as measures on $\mathscr{D}$. A λ-measurable set E of G is called permanently positive (P.P.) if for any $x_1, x_2, \ldots, x_n \in G$, $\bigcap_{i=1}^n x_iE$ is not locally null. E is called strictly positive (S.P.) if $V \cap \bigcap_{i=1}^n x_iE$ is not locally null for all open set V and $x_1, x_2, \ldots, x_n \in G$. Note that if G is σ-compact, a set E is P.P. if and only if

$$\lambda\left(\bigcap_{i=1}^n x_iE\right) > 0 \quad \text{for any } x_1, \ldots, x_n \in G$$

and E is S.P. if and only if

$$\lambda\left(\left(\bigcap_{i=1}^n x_iE\right) \cap V\right) > 0 \quad \text{for any open set } V \text{ in } G \text{ and } x_1, x_2, \ldots, x_n \in G$$

(see [15] p. 40 for more details).

Throughout this section G will denote a locally compact, noncompact and nondiscrete group. Let G_0 be a noncompact σ-compact open and closed subgroup of G (see [14], Proposition 22.24) and let $\{x_\alpha G_0 : \alpha \in \Lambda\}$ be all the left cosets of G_0 in G. Then

$$G = \bigcup_{\alpha \in \Lambda} x_\alpha G_0$$

is a disjoint union.

Definition 2.1. Let $\{A_\gamma : \gamma \in \Omega\}$ be a family of λ-measurable subsets of G_0. If $\gamma_1, \gamma_2, \ldots, \gamma_n \in \Omega$, V is an open subset in G_0 and $g_1^{(i)}, g_2^{(i)}, \ldots, g_{m_i}^{(i)} \in G_0$ $(i = 1, 2, \ldots, n)$, the set

$$F = \bigcap_{i=1}^n \bigcap_{k=1}^{m_i} g_k^{(i)} A_{\gamma_i} \cap V,$$

the intersection of finite elements of $\{A_\gamma : \gamma \in \Omega\}$, is called an *(FI)-form set relative to* $\{A_\gamma : \gamma \in \Omega\}$. *If for any (FI)*-form set F relative to $\{A_\gamma : \gamma \in \Omega\}$ we have $\lambda(F) > 0$, we call $\{A_\gamma : \gamma \in \Omega\}$ a *strictly positive (S.P.) family in* G_0.

Lemma 2.2. *If* $\{A_\alpha : \alpha \in \Lambda\}$ *is an S.P. family in* G_0 *and the set* $A = \bigcup_{\alpha \in \Lambda} x_\alpha A_\alpha$, *then* A *is an S.P. subset of* G.

Proof. For any compact set K of G, there are $\alpha_1, \alpha_2, \ldots, \alpha_n \in \Lambda$ such that

$$K \subset \bigcup_{i=1}^{n} x_{\alpha_i} G_0$$

since G_0 is open. Hence $K \cap A = \bigcup_{i=1}^{n}(x_{\alpha_i} A_{\alpha_i}) \cap K$ is λ-measurable. By (11.31) of [10], A is λ-measurable. Given an open set V in G and $x_1, x_2, \ldots, x_n \in G$, there is an $\alpha_0 \in \Lambda$ with $V \cap x_{\alpha_0} G_0 \neq \phi$. Let V_0 be an open set in G_0 such that $V \supset x_{\alpha_0} V_0$. For each $1 \leq i \leq n$, there is a $\alpha_i \in \Lambda$ such that $y_i = x_{\alpha_0}^{-1} x_i x_{\alpha_i} \in G_0$. Hence

$$\begin{aligned} V \cap \bigcap_{i=1}^{n} x_i A &= V \cap \bigcap_{i=1}^{n} \bigcup_{\alpha \in \Lambda} x_i x_\alpha A_\alpha \\ &\supset x_{\alpha_0} V_0 \cap \bigcap_{i=1}^{n} x_i x_{\alpha_i} A_{\alpha_i} \\ &= x_{\alpha_0}\left(V_0 \cap \bigcap_{i=1}^{n} y_i A_{\alpha_i} \right). \end{aligned}$$

Also, $\lambda(x_{\alpha_0}(V_0 \cap \bigcap_{i=1}^{n} y_i A_{\alpha_i})) = \lambda(V_0 \cap \bigcap_{i=1}^{n} y_i A_{\alpha_i}) > 0$ since $\{A_\alpha : \alpha \in \Lambda\}$ is an S.P. family in G_0. Note that $V_0 \cap \bigcap_{i=1}^{n} y_i A_{\alpha_i}$ is a subset of G_0 which is σ-compact. $V_0 \cap \bigcap_{i=1}^{n} y_i A_{\alpha_i}$ is not locally null and so $V \cap \bigcap_{i=1}^{n} x_i A$ is not a locally null set. □

Let V_0 be an open dense subset of G_0 with $\lambda(V_0) < 1$. Then

$$V = \bigcup_{\alpha \in \Lambda} V_0^{-1} x_\alpha^{-1} = \left(\bigcup_{\alpha \in \Lambda} x_\alpha V_0 \right)^{-1}$$

is also an open and dense subset in G. Suppose that $V = \bigcup_{\alpha \in \Lambda} x_\alpha A_\alpha$, then each A_α is an open dense subset in G_0. We shall use Proposition 3.4 of [15] and the axiom of choice to divide V into infinitely many disjoint S.P. subsets as the following.

LEMMA 2.3. *For each $\alpha \in \Lambda$, there are subsets $A_\alpha^{(i)}$, $i = 1, 2, \ldots$ in G_0 such that*

$$A_\alpha = \bigcup_{i=1}^{\infty} A_\alpha^{(i)}$$

is a disjoint union and for each $i \geq 1$, $\{A_\alpha^{(i)} : \alpha \in \Lambda\}$ is an S.P. family in G_0.

Proof. Fix a $\alpha_0 \in \Lambda$. There are disjoint S.P. subsets $A_{\alpha_0}^{(0)}$ and $A_{\alpha_0}^{(1)}$ in G_0 such that $A_{\alpha_0} = A_{\alpha_0}^{(0)} \cup A_{\alpha_0}^{(1)}$ by Proposition 3.4 of [15] since G_0 is σ-compact. Suppose Λ_0 is a subset of Λ with $\alpha_0 \in \Lambda_0$. Set

$$(*)\quad \begin{aligned} \mathscr{A}_{\Lambda_0} = \Big\{ & A_\alpha^{(0)} : \alpha \in \Lambda_0,\ A_\alpha^{(0)} \subset A_\alpha,\ A_\alpha^{(1)} = A_\alpha \sim A_\alpha^{(0)} \text{ such that } \big\{A_\alpha^{(i)} : \alpha \in \Lambda_0\big\} \\ & \text{is an S.P. family } (i = 0, 1) \text{ and } A_{\alpha_0}^{(0)} \cap A_\alpha^{(1)} = \emptyset, \\ & A_{\alpha_0}^{(1)} \cap A_\alpha^{(0)} = \phi \text{ for all } \alpha \in \Lambda_0 \Big\}. \end{aligned}$$

Let $\Lambda_0 = \{\alpha_0\}$. We can see that such $\mathscr{A}_{\Lambda_0}$ exists and $\mathscr{A}_{\Lambda_0} \neq \phi$. Take a partial order in the family of all the nonempty $\mathscr{A}_{\Lambda_0}$ as the following. Put $\mathscr{A}_{\Lambda_0} \leq \mathscr{A}_{\Lambda'_0}$ if and only if $\Lambda_0 \subset \Lambda'_0$ and if $\alpha \in \Lambda_0$, then $A_\alpha^{(0)}$ in $\mathscr{A}_{\Lambda_0}$ is the same as in $\mathscr{A}_{\Lambda'_0}$ for $\Lambda_0 \subset \Lambda$ and $\Lambda'_0 \subset \Lambda$. Then it is clear that $\leq$ is a partial order. For each chain $\{\mathscr{A}_{\Lambda_0^{(p)}} : p \in \Sigma\}$, put $\Lambda_0 = \bigcup_{p \in \Sigma} \Lambda_0^{(p)}$; then $\Lambda_0 \subset \Lambda$ and $\alpha_0 \in \Lambda_0$. If $\alpha \in \Lambda_0$, then there is $p \in \Sigma$ such that $\alpha \in \Lambda_0^{(p)}$. Let $A_\alpha^{(0)}$ be the same as in $\mathscr{A}_{\Lambda_0^{(p)}}$. Then $A_\alpha^{(0)}$ is well-defined since $\{\mathscr{A}_{\Lambda_0^{(p)}} : p \in \Sigma\}$ is a chain. Also it is clear that $(*)$ is satisfied. Since for any $\alpha_1, \alpha_2, \ldots, \alpha_n \in \Lambda_0$, there is $p \in \Sigma$ such that $\alpha_1, \alpha_2, \ldots, \alpha_n \in \Lambda_0^{(p)}$, both

$$\big\{A_\alpha^{(0)} : \alpha \in \Lambda_0\big\} \quad \text{and} \quad \big\{A_\alpha^{(1)} : \alpha \in \Lambda_0\big\}$$

are S.P. families. Hence $\mathscr{A}_{\Lambda_0} = \{A_\alpha^{(0)} : \alpha \in \Lambda_0\}$ is an upper bound of $\{\mathscr{A}_{\Lambda_0^{(p)}} : p \in \Sigma\}$. By Zorn's Lemma, there is a maximal $\mathscr{A}_{\Lambda_0}$. Claim $\Lambda = \Lambda_0$. If not, let $\alpha \in \Lambda \sim \Lambda_0$, then there are disjoint S.P. subsets $V_\alpha^{(0)}$ and $V_\alpha^{(1)}$ in G_0 such that $A_\alpha = V_\alpha^{(0)} \cup V_\alpha^{(1)}$. Put

$$\begin{aligned} A_\alpha^{(0)} &= \big(V_\alpha^{(0)} \cup A_\alpha \cap A_{\alpha_0}^{(0)}\big) \sim A_{\alpha_0}^{(1)}, \\ A_\alpha^{(1)} &= \big(V_\alpha^{(1)} \cup A_\alpha \cap A_{\alpha_0}^{(1)}\big) \sim A_{\alpha_0}^{(0)}; \end{aligned}$$

then $A_\alpha = A_\alpha^{(0)} \cup A_\alpha^{(1)}$ is a disjoint union and

$$A_\alpha^{(0)} \cap A_{\alpha_0}^{(1)} = \phi, \quad A_\alpha^{(1)} \cap A_{\alpha_0}^{(0)} = \phi.$$

Claim

$$\big\{A_\beta^{(0)} : \beta \in \Lambda_0 \cup \{\alpha\}\big\}$$

is an S.P. family. For any (FI)-form set F_0 relative to $\{A_\beta^{(0)} : \beta \in \Lambda_0\}$ and any $g_1, g_2, \ldots, g_{m_\alpha} \in G_0$, since $\{A_\beta^{(0)} : \beta \in \Lambda_0\}$ is an S.P. family and $V =$

$\bigcap_{i=1}^{m_\alpha} g_i A_\alpha$ is an open dense subset of G_0, we have

$$\begin{aligned} 0 &< \lambda\left(F_0 \cap \bigcap_{i=1}^{m_\alpha} g_i A_{\alpha_0}^{(0)} \cap V\right) \\ &= \lambda\left(F_0 \cap \bigcap_{i=1}^{m_\alpha} g_i A_{\alpha_0}^{(0)} \cap \bigcap_{l=1}^{m_\alpha} g_l A_\alpha\right) \\ &\le \lambda\left(F_0 \cap \bigcap_{i=1}^{m_\alpha} g_i\left(A_{\alpha_0}^{(0)} \cap A_\alpha\right)\right) \\ &= \lambda\left(F_0 \cap \bigcap_{i=1}^{m_\alpha} g_i\left(A_{\alpha_0}^{(0)} \cap A_\alpha^{(0)}\right)\right) \\ &\le \lambda\left(F_0 \cap \bigcap_{i=1}^{m_\alpha} g_i A_\alpha^{(0)}\right); \end{aligned}$$

i.e. for any (FI)-form set F relative to $\{A_\beta^{(0)} : \beta \in \Lambda_0 \cup \{\alpha\}\}$, $\lambda(F) > 0$. Therefore

$$\left\{A_\beta^{(0)} : \beta \in \Lambda_0 \cup \{\alpha\}\right\}$$

is an S.P. family. Similarly, $\{A_\beta^{(1)} : \beta \in \Lambda_0 \cup \{\alpha\}\}$ is an S.P. family in G_0. Therefore

$$\mathscr{A}_{\Lambda_0 \cup \{\alpha\}} \ge \mathscr{A}_{\Lambda_0} \quad \text{and} \quad \mathscr{A}_{\Lambda_0 \cup \{\alpha\}} \ne \mathscr{A}_{\Lambda_0}$$

which is a contradiction. Hence $\Lambda = \Lambda_0$.

Suppose for each $\alpha \in \Lambda$, $A_\alpha = A_\alpha^{(1)} \cup A_\alpha^{(2)} \cup \cdots \cup A_\alpha^{(n)}$ is a disjoint union and for each $1 \le i \le n$, $\{A_\alpha^{(i)} : \alpha \in \Lambda\}$ is an S.P. family. Also, if $i \ne j$,

$$(**) \qquad A_{\alpha_0}^{(i)} \cap A_\alpha^{(j)} = \phi \quad \text{for any } \alpha \in \Lambda.$$

Note that for each $1 \le i \le n$, $A_\alpha^{(i)}$ is an S.P. set in G_0. By Proposition 3.4 of [15] again, there are S.P. sets $A_{\alpha_0}^{(n,0)}$ and $A_{\alpha_0}^{(n,1)}$ in G_0 such that $A_{\alpha_0}^{(n)} = A_{\alpha_0}^{(n,0)} \cup A_{\alpha_0}^{(n,1)}$ is a disjoint union. With the similar order and the argument as above, there is a maximal $\mathscr{A}_{\Lambda_0}$ for every subset Λ_0 of Λ with $\alpha_0 \in \Lambda_0$, where

$$\begin{aligned} \mathscr{A}_{\Lambda_0} = \Big\{ & A_\alpha^{(n,0)} : \alpha \in \Lambda_0, \mathscr{A}_\alpha^{(n,0)} \subset A_\alpha^{(n)}, A_\alpha^{(n,1)} = A_\alpha^{(n)} \sim A_\alpha^{(n,0)}, \\ & \left\{A_\alpha^{(n,i)} : \alpha \in \Lambda_0\right\} \text{ is an S.P. family } (i = 0, 1) \text{ and} \\ & A_{\alpha_0}^{(n,0)} \cap A_\alpha^{(n,1)} = \phi, A_{\alpha_0}^{(n,1)} \cap A_\alpha^{(n,0)} = \phi \Big\}. \end{aligned}$$

Then $\Lambda_0 = \Lambda$. Indeed, if $\alpha \in \Lambda \sim \Lambda_0$, by Proposition 3.4 of [15], there are disjoint S.P. sets $V_\alpha^{(n,0)}$ and $V_\alpha^{(n,1)}$ in G_0 such that $A_\alpha^{(n)} = V_\alpha^{(n,0)} \cup V_\alpha^{(n,1)}$. Put

$$A_\alpha^{(n,0)} = \left(V_\alpha^{(n,0)} \cup A_\alpha^{(n)} \cap A_{\alpha_0}^{(n,0)}\right) \sim A_{\alpha_0}^{(n,1)},$$
$$A_\alpha^{(n,1)} = \left(V_\alpha^{(n,1)} \cup A_\alpha^{(n)} \cap A_{\alpha_0}^{(n,1)}\right) \sim A_{\alpha_0}^{(n,0)};$$

then $A_\alpha^{(n)} = A_\alpha^{(n,0)} \cup A_\alpha^{(n,1)}$ is a disjoint union and

$$A_{\alpha_0}^{(n,0)} \cap A_\alpha^{(n,1)} = \phi, \quad A_{\alpha_0}^{(n,1)} \cap A_\alpha^{(n,0)} = \phi.$$

Claim

$$\left\{A_\beta^{(n,0)} : \beta \in \Lambda_0 \cup \{\alpha\}\right\}$$

is an S.P. family. Let F_0 be a (FI)-form set relative to $\{A_\beta^{(n,0)} : \beta \in \Lambda_0\}$ and $g_1, g_2, \ldots, g_{m_\alpha} \in G_0$. Note that $V = \bigcap_{i=1}^{m_\alpha} g_i A_\alpha$ is open and $F_0 \cap \bigcap_{i=1}^{m_\alpha} g_i A_{\alpha_0}^{(n,0)}$ is an (FI)-form set relative to $\{A_\beta^{(n,0)} : \beta \in \Lambda_0\}$. Hence

$$\begin{aligned} 0 &< \lambda\left(F_0 \cap \bigcap_{i=1}^{m_\alpha} g_i A_{\alpha_0}^{(n,0)} \cap \bigcap_{l=1}^{m_\alpha} g_l A_\alpha\right) \\ &\leq \lambda\left(F_0 \cap \bigcap_{i=1}^{m_\alpha} g_i\left(A_{\alpha_0}^{(n,0)} \cap A_\alpha\right)\right) \\ &= \lambda\left(F_0 \cap \bigcap_{i=1}^{m_\alpha} g_i\left(A_{\alpha_0}^{(n,0)} \cap A_\alpha^{(n,0)}\right)\right) \\ &\leq \lambda\left(F_0 \cap \bigcap_{i=1}^{m_\alpha} g_i A_\alpha^{(n,0)}\right), \end{aligned}$$

since

$$A_\alpha = A_\alpha^{(1)} \cup A_\alpha^{(2)} \cup \cdots \cup A_\alpha^{(n-1)} \cup A_\alpha^{(n,0)} \cup A_\alpha^{(n,1)},$$
$$A_{\alpha_0}^{(n,0)} \cap A_\alpha^{(n,1)} = \phi \quad \text{and} \quad A_{\alpha_0}^{(n,0)} \cap A_\alpha^{(k)} = \phi$$

if $k < n$. Hence any (FI)-form set relative to $\{A_\beta^{(n,0)} : \beta \in \Lambda_0 \cup \{\alpha\}\}$ has positive measure. Therefore

$$\left\{A_\beta^{(n,0)} : \beta \in \Lambda_0 \cup \{\alpha\}\right\}$$

is an S.P. family. Similarly,

$$\left\{A_\beta^{(n,1)} : \beta \in \Lambda_0 \cup \{\alpha\}\right\}$$

is an S.P. family. This contradicts that $\mathscr{A}_{\lambda_0}$ is maximal. Therefore, for any $\alpha \in \Lambda$,

$$A_\alpha = A_\alpha^{(1)} \cup A_\alpha^{(2)} \cup \cdots \cup A_\alpha^{(n-1)} \cup A_\alpha^{(n,0)} \cup A_\alpha^{(n,1)}$$

satisfies the property ($**$). By induction, we finish the proof. □

LEMMA 2.4. *For any nondiscrete locally compact amenable group G, there are S.P. subsets E_n ($n = 1, 2, \ldots\ \ldots$) in G such that $E_n \cap E_m = \phi$ ($n \neq m$) and $\psi(\bigcup_{n=1}^{\infty} E_n) < 1$ for each $\psi \in TLIM$.*

Proof. If G is compact, there is an open dense subset V_0 in G with $\lambda(V_0) < 1$ by Proposition 2 of [7]. We can find disjoint S.P. subsets E_n of G such that $V_0 = \bigcup_{n=1}^{\infty} E_n$ by Proposition 3.4 of [15]. Since $TLIM = \{\lambda\}$, $\psi(\bigcup_{n=1}^{\infty} E_n) < 1$ for $\psi \in TLIM$. We use all the notation as in Lemma 2.3. Put $E_n = \bigcup_{\alpha \in \Lambda} x_\alpha A_\alpha^{(n)}$. Then by Lemma 2.2 and Lemma 2.3 E_n is an S.P. subset in G. Since

$$\bigcup_{n=1}^{\infty} E_n = \left(\bigcup_{\alpha \in \Lambda} x_\alpha V_0 \right)^{-1}$$

with $\lambda(V_0) < 1$ (see Lemma 2.3 for $A_\alpha^{(n)}$ and V_0),

$$\psi\left(\bigcup_{n=1}^{\infty} E_n \right) = 0 < 1$$

for all $\psi \in TLIM$ by the last proposition of [7]. □

As in Chou [3], let

$$\mathscr{F}_1 = \Big\{ \theta \in l^\infty(\mathbf{N})^* : \theta \geq 0,\ \|\theta\| = 1 \text{ and } \theta(f) = 0 \text{ if } f \in l^\infty(\mathbf{N}) \text{ with } \lim_n f(n) = 0 \Big\}$$

then $\beta\mathbf{N} \sim \mathbf{N} \subset \mathscr{F}_1$ and $|\mathscr{F}_1| = 2^c$. We are going to prove our first main result.

THEOREM 2.5. *Let G be a nondiscrete locally compact group which is amenable as a discrete group. Then there exists a positive mapping of $L^\infty(G)$ onto $l^\infty(\mathbf{N})$, say π, such that $\|\pi\| = 1$ and its conjugate π^* is a linear isometry of $l^\infty(\mathbf{N})^*$ into $L^\infty(G)^*$ with $\pi^* \mathscr{F}_1 \subset LIM \sim TLIM$. Moreover, elements of $\pi^*(\beta\mathbf{N} \sim \mathbf{N})$ are mutually singular and $\pi^*\theta$ is singular to every $\psi \in TLIM$ for any $\theta \in \mathscr{F}_1$.*

Proof. Let $\{E_n : n = 1, 2, \ldots\}$ be the subsets of Lemma 2.4. Since G is amenable as a discrete group and E_n is an S.P. subset of G there is an $m_n \in LIM$ such that $m_n(1_{E_n}) = 1$ for each n (see [2] p. 48, the proof of (3) $\Rightarrow$ (4)). Define $\pi : L^\infty(G) \to l^\infty(\mathbf{N})$ by $\pi(f)(n) = m_n(f)$ for $f \in L^\infty(G)$ and $n \in \mathbf{N}$. Then π is linear and nonnegative. Since $\pi(1) = 1$, and for each $f \in L^\infty(G)$

$$\|\pi(f)\| = \sup_n |m_n(f)| \leq \|f\|_\infty$$

$\|\pi\| = 1$. For each $F \in l^\infty(\mathbf{N})$, define $f(x) = F(n)$ if $x \in E_n$ and $f(x) = 0$ if $x \notin \bigcup_{n=1}^\infty E_n$. Then $f \in L^\infty(G)$ and $\pi(f)(n) = m_n(f) = m_n(f \cdot 1_{E_n}) = F(n)$ ($n \in \mathbf{N}$), i.e., $\pi(f) = F$ and $\|f\|_\infty = \|F\|_\infty$. Hence π is onto and π^* is a linear isometry.

For each $\theta \in \mathscr{F}_1$, $\pi^*\theta \in LIM$. Indeed, given $f \in L^\infty(G)$ and $x \in G$, since for each $n \in \mathbf{N}$,

$$\pi({}_xf)(n) = m_n({}_xf) = m_n(f) = \pi(f)(n)$$

i.e., $\pi({}_xf) = \pi(f)$, we have

$$\pi^*\theta({}_xf) = \theta(\pi_x f) = \theta(\pi f) = \pi^*\theta(f).$$

Hence $\pi^*\theta$ is left invariant. Since both π and θ are nonnegative, $\pi^*\theta$ is nonnegative. Also, $\pi^*\theta(1) = \theta(\pi(1)) = \theta(1) = 1$, hence $\pi^*\theta \in LIM$. Let $E = \bigcup_{n=1}^\infty E_n$, then

$$\pi(1_E)(n) = m_n(1_E) = 1 \quad (n \in \mathbf{N}),$$

i.e., $\pi(1_E) = 1$. Hence $\pi^*\theta(1_E) = 1$. By Lemma 2.4, $\pi^*\theta \notin TLIM$. If G is not compact, then $\pi^*\theta$ is singular to any $\psi \in TLIM$ since $\sup \widehat{\pi^*\theta} \subset \hat{E}$ and $\sup \hat{\psi} \subset \widehat{G \sim E}$ (see [15], p. 35). If G is compact, since $\pi^*\theta(1_{G \sim E}) = 0$ and $\lambda(G \sim E) > 0$, by Proposition 2.4 and Lemma 2.6 of [15], $\pi^*\theta$ is singular to λ. Let $\theta_1, \theta_2 \in \beta\mathbf{N} \sim \mathbf{N}$ and $\theta_1 \neq \theta_2$, then $\|\theta_1 - \theta_2\| = 2$ (see [3], page 208). Hence

$$\left\|\widehat{\pi^*\theta_1} - \widehat{\pi^*\theta_2}\right\| = \|\pi^*\theta_1 - \pi^*\theta_2\| = \|\theta_1 - \theta_2\| = 2.$$

By the Hahn decomposition theorem, for the signed measure $\pi = \widehat{\pi^*\theta_1} - \widehat{\pi^*\theta_2}$, there are subsets D^+ and D^- of $\mathscr{D}$ such that $\mu \geq 0$ on D^+ and $\mu \leq 0$ on D^-. Also $\mathscr{D} = D^+ \cup D^-$, $D^+ \cap D^- = \phi$. Since $\|\mu\| = 2$, $\|\widehat{\pi^*\theta_1}\| = \|\widehat{\pi^*\theta_2}\| = 1$, and $\|\mu\| = \mu(\mathscr{D}^+) - \mu(D^-)$, $\widehat{\pi^*\theta_1}(D^-) = 0$, $\widehat{\pi^*\theta_2}(D^+) = 0$. Hence $\pi^*\theta_1$ and $\pi^*\theta_2$ are mutually singular (see [19], p. 134). □

COROLLARY 2.6. *Let G be a nondiscrete locally compact group. If G is amenable as a discrete group, then there is a subset $E \subset LIM \sim TLIM$ with $|E| \geq 2^c$ and $\|m_1 - m_2\| = 2$ for any $m_1, m_2 \in E$. In particular,*

$$|LIM \sim TLIM| \geq 2^c.$$

Proof. As in the proof of Theorem 2.5, let $E = \pi^*(\beta\mathbf{N} \sim \mathbf{N})$. Then for any $\theta_1, \theta_2 \in \beta\mathbf{N} \sim \mathbf{N}$,

$$\|\pi^*\theta_1 - \pi^*\theta_2\| = \|\theta_1 - \theta_2\| = 2.$$

Since $|\beta\mathbf{N} \sim \mathbf{N}| = 2^c$, $|E| \geq 2^c$. □

Remark 1. Corollary 2.6 removes the condition of σ-compact for Corollary (7.20) of [13].

2. Let V_0 be an open dense subset of G_0 with $\lambda(V_0) < 1$. Then $V = \bigcup_{\alpha \in \Lambda} x_\alpha V_0$ is an open dense subset of G. Since V_0 can be divided into disjoint S.P. subsets $V_0^{(0)}$ and $V_0^{(1)}$, V can be divided into disjoint S.P. subsets $V^{(0)} = \bigcup_{\alpha \in \Lambda} x_\alpha V_0^{(0)}$ and $V^{(1)} = \bigcup_{\alpha \in \Lambda} x_\alpha V_0^{(1)}$, and so on (see Lemma 2.2). Therefore we can remove the condition of σ-compact for Rosenblatt's theorem of Proposition 3.5 of [15].

3. The size of *LIM* for a noncompact metrizable locally compact group

By comparing $|LIM|$ with $|TLIM|$ for a metrizable noncompact locally compact group, we obtain the cardinality of LIM as the following (see [13], Chapter 7).

THEOREM 3.1. *If G is a metrizable noncompact locally compact amenable group, then*

$$|LIM| = |TLIM| = 2^{2^{d(G)}},$$

where $d(G)$ is the smallest possible cardinality for a covering of G by compact subsets.

Proof. Let G_0 be an open and closed σ-compact subgroup of G (see [14], Proposition 22.24) and let $\{x_\alpha G_0 : \alpha \in \Lambda\}$ be all the left cosets of G_0 in G. Since G_0 is σ-compact, we can find compact subsets K_n of G_0 such that $K_n \subset K_{n+1}$, $K_n \neq K_{n+1}$ $(n = 1, 2, \ldots)$ and $G_0 = \bigcup_{n=1}^{\infty} K_n$. Let $E_n = K_n \sim K_{n-1}$ $(n = 1, 2, \ldots)$, where we assume that $K_0 = \phi$. Then $E_n \cap E_m = \phi$ if

$n \neq m$, E_n is λ-measurable and $\overline{E}_n$ is compact $(n = 1, 2, \ldots)$. Since $G_0 = \bigcup_{n=1}^{\infty} E_n$,

$$G = \bigcup_{\alpha \in \Lambda} x_\alpha G_0 = \bigcup_{\alpha \in \Lambda} \bigcup_{n=1}^{\infty} x_\alpha E_n = \bigcup_{(n,\alpha) \in \mathbf{N} \times \Lambda} x_\alpha E_n$$

and

$$x_\alpha E_n \cap x_{\alpha'} E_{n'} = \phi \quad \text{if } (n, \alpha) \neq (n', \alpha').$$

We first show that $d(G) = |\mathbf{N} \times \Lambda|$. Since $\{x_\alpha \overline{E}_n : (n, \alpha) \in \mathbf{N} \times \Lambda\}$ is a compact cover of G,

$$d(G) \leq |\mathbf{N} \times \Lambda|.$$

To prove that $d(G) \geq |\mathbf{N} \times \Lambda|$, let $\mathscr{C}$ be a compact cover of G with $|\mathscr{C}| = d(G)$ and let

$$\mathscr{C}_n = \left\{ C \cap (x_{\alpha_i} G_0) : C \in \mathscr{C}, \alpha_1, \alpha_2, \ldots, \alpha_n \in \Lambda \right.$$
$$\left. \text{with } C \subset \bigcup_{i=1}^{n} x_{\alpha_i} G_0 \text{ and } i = 1, 2, \ldots, n \right\}.$$

Note that the mapping $C \cap (x_{\alpha_i} G_0) \to (C, x_1, x_2, \ldots, x_n)$ from $\mathscr{C}_n$ to a subset of $\mathscr{C} \times \{0, 1\}^n$ is 1-1, where $x_i = 1$, $x_j = 0$ $(j \neq i)$. Hence $|\mathscr{C}_n| \leq |\mathscr{C}| = d(G)$. Since for each $C \in \mathscr{C}$, there are $\alpha_1, \alpha_2, \ldots, \alpha_n \in \Lambda$ such that $C \subseteq \bigcup_{i=1}^{n} x_{\alpha_i} G_0$, $\bigcup_{n=1}^{\infty} \mathscr{C}_n$ is a compact cover of G and

$$\left| \bigcup_{n=1}^{\infty} \mathscr{C}_n \right| \leq |\mathscr{C}| = d(G).$$

Therefore we can assume that for each $C \in \mathscr{C}$, there is an $\alpha \in \Lambda$ such that $C \subset x_\alpha G_0$. For each $\alpha \in \Lambda$, there is $C_\alpha \in \mathscr{C}$ with $C_\alpha \subseteq x_\alpha G_0$. So the mapping $\alpha \to C_\alpha$ is 1-1 from Λ to a subset of $\mathscr{C}$. Hence $|\mathbf{N} \times \Lambda| = |\Lambda| \leq |\mathscr{C}| = d(G)$.

Since $|LIM| \geq 2^{2^{d(G)}}$ (see [13], p. 274) and $LIM \subseteq L^\infty(G)^*$, to show that $|LIM| = 2^{2^{d(G)}}$, it suffices to show that $|L^\infty(G)^*| \leq 2^{2^{d(G)}}$.

For any subset E of G, let $CB(E)$ be the set of all continuous functions on E. For each $(n, \alpha) \in \mathbf{N} \times \Lambda$, since $x_\alpha \overline{E}_n$ is compact and metrizable, $x_\alpha \overline{E}_n$ is separable and

$$\left| CB(x_\alpha \overline{E}_n) \right| \leq c.$$

Hence

$$|\mathscr{F}| \leq c|\mathbf{N} \times \Lambda| = c \cdot d(G) \quad \text{where } \mathscr{F} = \bigcup_{(n,\alpha)\in \mathbf{N}\times\Lambda} CB(x_\alpha E_n).$$

Let $f \in L^\infty(G)$ and $(n, \alpha) \in \mathbf{N} \times \Lambda$. By Lusin's theorem (see [19], p. 55), for each $k \in \mathbf{N}$, there is $f_{(k,n,\alpha)} \in \mathscr{F}$ such that

$$\lambda\{x \in x_\alpha E_n : f_{(k,n,\alpha)}(x) \neq f(x)\} < \frac{1}{k}.$$

If λ-measurable function g on G such that $F = \{x \in G, f(x) \neq g(x)\}$ is locally null, then

$$\lambda(F \cap x_\alpha E_n) = 0$$

for any $(n, \alpha) \in \mathbf{N} \times \Lambda$, i.e.,

$$\lambda\{x \in x_\alpha E_n : f_{(k,n,\alpha)}(x) \neq g(x)\} < \frac{1}{k}$$

for $(k, n, \alpha) \in \mathbf{N} \times \mathbf{N} \times \Lambda$. So $f_{(k,n,\alpha)}$ is well-defined. Let mapping ϕ: $L^\infty(G) \to \mathscr{F}^{\mathbf{N}\times\mathbf{N}\times\Lambda}$ be defined by

$$\phi(f) = (f_{(k,n,\alpha)})_{(k,n,\alpha)\in\mathbf{N}\times\mathbf{N}\times\Lambda}.$$

Then ϕ is a 1-1 mapping from $L^\infty(G)$ to a subset of $\mathscr{F}^{\mathbf{N}\times\mathbf{N}\times\Lambda}$. Indeed, let $g \in L^\infty(G)$ with $f \neq g$. Then there is $\alpha \in \Lambda$ such that $f \neq g$ on $x_\alpha G_0$. Since $x_\alpha G_0 = \bigcup_{n=1}^\infty x_\alpha E_n$, there is $n \in \mathbf{N}$ such that $f \neq g$ on $x_\alpha E_n$. Hence there is $k \in \mathbf{N}$ such that

$$\lambda\{x \in x_\alpha E_n : f(x) \neq g(x)\} > \frac{2}{k}$$

so $f_{(k,n,\alpha)} \neq g_{(k,n,\alpha)}$ by the definition of $f_{(k,n,\alpha)}$ and $g_{(k,n,\alpha)}$, i.e. $\phi(f) \neq \phi(g)$. Hence

$$|L^\infty(G)| \leq |\mathscr{F}^{\mathbf{N}\times\mathbf{N}\times\Lambda}| \leq (cd(G))^{d(G)}$$

since $2 \leq cd(G) \leq 2^{d(G)}$, $(cd(G))^{d(G)} = 2^{d(G)}$. Therefore

$$|L^\infty(G)| \leq 2^{d(G)} \quad \text{and} \quad |L^\infty(G)^*| \leq 2^{2^{d(G)}}. \qquad \square$$

COROLLARY 3.2. *Let G be a σ-compact metrizable locally compact group. If G is amenable as a discrete group, then*

$$|LIM \sim TLIM| = 2^c.$$

Proof. By Corollary 2.6, $|LIM \sim TLIM| \geq 2^c$. By Theorem 3.1, $|LIM \sim TLIM| \leq 2^c$. Hence $|LIM \sim TLIM| = 2^c$. □

As in the proof of Theorem 3.1, we have the following.

THEOREM 3.3. *Let G be a locally compact amenable group. If G is metrizable, then*

$$|LIM(CB(G))| = 1$$

when G is compact and

$$|LIM(CB(G))| = 2^{2^{d(G)}}$$

when G is not compact, where d(G) is the smallest possible cardinality for a covering of G by compact sets.

Unfortunately, Theorem 3.1 does not hold without the metrizability.

THEOREM 3.4. *For any cardinal numbers η_1 and η_2, if η_2 is infinite, then there is a locally compact group G such that $|LIM| \geq \eta_1$ and $|TLIM| = 2^{2^{\eta_2}}$. Moreover, there is a compact group G with $|LIM| \geq \eta_1$.*

Proof. Let S be a compact nondiscrete abelian group and let A and B be S.P. subsets in S such that $\lambda(A) < 1$, $\lambda(B) < 1$ and $A \cap B = \varnothing$ (see [7], Proposition 2 and [15], Proposition 3.4). Let $G_0 = \prod_{\gamma \in \eta_1} S_r$ where $S_r = S$ for any $\gamma \in \eta_1$. Take a discrete abelian group U with $|U| = \eta_2$. Let $G = U \times G_0$. Then G is a nondiscrete abelian group. Note that G_0 is an open and closed subgroup of G and $\{uG_0 : u \in U\}$ is the set of all cosets of G_0 in G. For each finite subset Δ of η_1 and $\beta \in \eta_1 \sim \Delta$, let

$$E_{(\Delta,\beta)} = \bigcup_{u \in U} u\left(\prod_{\gamma \in \eta_1} E_r\right)$$

where $E_\gamma = B$ if $\gamma = \beta$, $E_\gamma = A$ if $\gamma \in \Delta$ and $E_\gamma = S$ if $\gamma \notin \Delta \cup \{\beta\}$. Since $\prod_{\gamma \in \eta_1} E_\gamma$ is an S.P. subset in G_0 (see [10], 13.22) $E_{(\Delta,\beta)}$ is an S.P. set in G by

Lemma 2.2. Note that for any $\beta \in \eta_1$, finite subsets Δ_i of $\eta_1 \sim \{\beta\}$ and $x_i \in G$ $(i = 1, 2, \ldots, n)$,

$$\bigcap_{i=1}^{n} x_i E_{(\Delta_i, \beta)} \supseteq \bigcap_{i=1}^{n} x_i E_{(\Delta, \beta)}$$

is not locally null since $E(\Delta, \beta)$ is an S.P. subset, where $\Delta = \bigcup_{i=1}^{n} \Delta_i$. Also, the maximal ideal space $\mathscr{D}$ of G is compact (see the beginning of section 2 and [15], p. 35). Hence the set

$$D_\beta = \bigcap \left\{ \widehat{xE}_{(\Delta, \beta)} : \Delta \text{ is a finite subset of } \eta_1 \sim \{\beta\},\ x \in G \right\}$$

is a nonempty left invariant and closed subset of $\mathscr{D}$. By Proposition 3.4 of [16], there is a left invariant probability measure μ_β on D_β. Hence there is $m_\beta \in LIM$ such that $\hat{m}_\beta = \mu_\beta$ (see the beginning of Section 2). If $\beta, \beta' \in \eta_1$ with $\beta \neq \beta'$, let $\Delta = \{\beta'\}$, $\Delta' = \{\beta\}$. Then

$$E_{(\Delta, \beta)} \cap E_{(\Delta', \beta')} = \varnothing$$

by the definition of $E_{(\Delta, \beta)}$. Hence $D_\beta \cap D_{\beta'} = \varnothing$ and $m_\beta \neq m_{\beta'}$. Therefore $|LIM| \geq \eta_1$. Also, as in the proof of Theorem 3.1, $|U| = d(G)$. Hence

$$|TLIM| = 2^{2^{d(G)}} \quad \text{where } d(G) = \inf\{|\mathscr{C}| : \mathscr{C} \text{ is a compact cover of } G\}.$$

If we take U such that $|U| = 1$ or $G = G_0$, then G is compact and $|LIM| \geq \eta_1$. □

Let $f \in L^\infty(G)$ and $I(f)$ denote the smallest closed left invariant ideal containing f. In [15] Rosenblatt showed that if a subset E of G satisfies $\lambda(E^{-1}) < 1$, then any $m \in LIM$ with $\ker m \supseteq I(1_{G \sim E})$ is singular to every $\psi \in TLIM$. He asked if the converse is true and he proved that it is for a compact group. Our Theorem 3.6 shows that for a class of groups it is not the case. We need a lemma first.

LEMMA 3.5. *Let G be a locally compact noncompact group and let G_0 be an open and closed compact subgroup of G. If $\{x_\alpha G_0 : \alpha \in \Lambda\}$ is the set of all left cosets of G_0 in G and V_0 is an open dense subset of G_0, then $\bigcap_{i=1}^{n} x_i V$ is not locally null and*

$$\lambda\left(x_\alpha G_0 \cap \bigcap_{i=1}^{n} x_i V \right) \geq \varepsilon_n$$

for any $x_1, x_2 \dots x_n \in G$ *and* $\alpha \in \Lambda$, *where* $V = \bigcup_{\alpha \in \Lambda} x_\alpha V_0$ *and* $\varepsilon_n > 0$ *depends on n only.*

Proof. For each n, the function $\lambda(\bigcap_{i=1}^n x_i V_0)$ of $(x_1, x_2, \dots, x_n)$ on the compact space G_0^n is continuous. Since $\lambda(\bigcap_{i=1}^n x_i V_0) > 0$ for any $(x_1, x_2, \dots, x_n) \in G_0^n$ and G_0^n is compact, there is $\varepsilon_n > 0$ such that $\lambda(\bigcap_{i=1}^n x_i V_0) \geq \varepsilon_n$ for any $(x_1, x_2, \dots, x_n) \in G_0^n$. If $\alpha \in \Lambda$, for each $1 \leq i \leq n$, there is $\alpha_i \in \Lambda$ such that $y_i = x_\alpha^{-1} x_i x_{\alpha_i} \in G_0$. Hence

$$x_\alpha G_0 \cap \bigcap_{i=1}^n x_i V \supseteq x_\alpha G_0 \cap \bigcap_{i=1}^n (x_i x_{\alpha_i} V_0) = x_{\alpha_0}\left(\bigcap_{i=1}^n y_i V_0 \right).$$

This implies that

$$\lambda\left(x_\alpha G_0 \cap \bigcap_{i=1}^n x_i V \right) \geq \lambda\left(\bigcap_{i=1}^n y_i V_0 \right) \geq \varepsilon_n.$$

Therefore $\bigcap_{i=1}^n x_i V$ is not locally null. □

THEOREM 3.6. *If G is an abelian locally compact noncompact group which contains an open and closed compact subgroup* G_0, *then there is an* $m \in LIM$ *such that m is singular to every* $\psi \in TLIM$ *and* $m(1_E) = 0$ *for any subset E of G with* $\lambda(E^{-1}) < \infty$. *In particular*, ker m *does not contain* $I(1_{G \sim E})$ *for any subset E with* $\lambda(E^{-1}) < 1$.

Proof. For each n, we can find an open dense subset V_{n_0} in G_0 such that $\lambda(V_{n_0}) < 1/n$ (see [7], Proposition 2). Let $V_n = \bigcup_{\alpha \in \Lambda} x_\alpha V_{n_0}$; then V_n is an open dense subset in G (see Lemma 3.5 for α, Λ and x_α).

For each $x \in G$, let $x = x_{\alpha_0} g_0$ for some $\alpha_0 \in \Lambda$ and $g_0 \in G_0$, then

$$1_{G_0} * 1_{V_n}(x) = \int_G 1_{G_0}(t) 1_{V_n}(t^{-1}x)\, dt = \lambda(G_0 \cap x V_n^{-1}) = \lambda\left(g_0 V_{n_0}^{-1}\right) = \lambda\left(V_{n_0}\right) < \frac{1}{n}$$

since G is abelian. Hence for any $\psi \in TLIM$,

$$\psi(1_{V_n}) = \psi(1_{G_0} * 1_{V_n}) \leq \frac{1}{n}.$$

Let I_n be the smallest left invariant ideal of $L^\infty(G)$ containing $1_{G \sim V_n}$ and all 1_A for the subset A of G with $\lambda(A) < \infty$. Then it is clear that

$$I_n = \operatorname{span}\{ f \cdot {}_x 1_{G \sim V_n} + g \cdot 1_A : f, g \in L^\infty(G),\ x \in G,\ A \subseteq G \text{ with } \lambda(A) < \infty \}.$$

Then $\bar{I}_n \neq L^\infty(G)$. Indeed, for any $f \in I_n$, there are $g_1, g_2, \ldots, g_m \in G$ and subset A in G such that $\lambda(A) < \infty$ and

$$|f| \leq \|f\|_\infty \left(\sum_{i=1}^{m} 1_{G \sim g_i V_n} + 1_A \right).$$

Let

$$E = \left(G \sim \left(\bigcup_{i=1}^{n} G \sim g_i V_n \right) \right) \sim A = \bigcap_{i=1}^{m} g_i V_n \sim A.$$

Since there is $\varepsilon_m > 0$ such that $\lambda(x_\alpha G_0 \cap \bigcap_{i=1}^{m} g_i V_n) \geq \varepsilon_m$ for any $\alpha \in \Lambda$ by Lemma 3.5, also G is not compact, E is not locally null. Since $f = 0$ on E, $\|f - 1\|_\infty \geq 1$, i.e., $1 \notin \bar{I}_n$. By Proposition 2.5 of [15], there is an $m_n \in LIM$ with $\ker m_n \supseteq \bar{I}_n$. We can assume that $V_1 \supseteq V_2 \supseteq V_3 \supseteq \cdots$. Let m be a w^*-cluster point of net $\{m_n\}$. Then there is a subnet $\{m_{n_\beta}\}$ of $\{m_n\}$ such that

$$m = \lim_\beta m_{n_\beta} \quad \text{in } w^*\text{-topology.}$$

For each n, there is β_n such that $n_\beta \geq n$ for all $\beta \geq \beta_n$. Hence

$$m(G \sim V_n) = \lim_\beta m_{n_\beta}(G \sim V_n) \leq \lim_{\beta \geq \beta_n} m_{n_\beta}(G \sim V_n) \leq \lim_{\beta \geq \beta_n} m_{n_\beta}(G \sim V_{n_\beta})$$
$$= 0$$

i.e., $m(V_n) = 1$ and $\sup \hat{m} \subseteq \hat{V}_n$. If $\psi \in TLIM$, then

$$\hat{\psi}(\sup \hat{m}) \leq \hat{\psi}(\hat{V}_n) = \psi(V_n) \leq \frac{1}{n}$$

for any n. So $\hat{\psi}(\sup \hat{m}) = 0$ and m is singular to ψ. If A is a subset of G with $\lambda(A^{-1}) < \infty$, then $\lambda(A) < \infty$. So

$$m(A) = \lim_\beta m_{n_\beta}(A) = 0.$$

Therefore $\ker m$ can not contain $I(1_{G \sim A})$. □

Liu and Rooji in [12] showed that if G is noncompact, nondiscrete and amenable as a discrete group, then $LIM(CB(G)) \neq TLIM(CB(G))$. Rosenblatt in [17] showed the following.

THEOREM (Rosenblatt). *Assume G is a noncompact σ-compact nondiscrete locally compact group which is amenable as a discrete group. Then there exist $f \in CB(G)$ with $0 \le f \le 1$ and $\theta \in LIM(CB(G))$ such that $\theta(f) = 1$ and $\psi(f) = 0$ for any $\psi \in TLIM(CB(G))$.*

And, he asked if the discrete amenability assumption is necessary. Chou in [4] speculates that if G is noncompact, nondiscrete and amenable, then

$$LIM(CB(G)) \neq TLIM(CB(G))$$

and he showed in [2] that there exist compact groups which are not amenable as discrete groups such that $LIM \neq TLIM$. Since there are amenable groups which are not amenable as discrete groups, our theorem 3.8 answers the problem of Rosenblatt negatively and confirms Chou's conjecture for some locally compact groups.

DEFINITION 3.7. For $f \in CB(G)$ with $0 \le f \le 1$, f is called permanently near one if for any $\varepsilon > 0$ and $x_i \in G$ $(i = 1, 2, \dots, n)$, there is $x_0 \in G$ such that

$$\left|1 - {}_{x_i}f(x_0)\right| < \varepsilon \quad (i = 1, 2, \dots, n).$$

The function $f \in CB(G)$ in the theorem of Rosenblatt above can be taken as a permanently near one function with the property that for any $\varepsilon > 0$ there is a $\varphi \in P(G)$ and $f_M \in CB(G)$ such that $\|\varphi * f_M\|_\infty < \varepsilon$ and the support of $f - f_M$ is compact.

THEOREM 3.8. *Let G_1 be a noncompact σ-compact nondiscrete group which is amenable as a discrete group. If G_2 is any amenable locally compact group and $G = G_1 \times G_2$, then there exist $F \in CB(G)$ with $0 \le F \le 1$ and $\theta \in LIM(CB(G))$ such that $\theta(F) = 1$ and $\psi(F) = 0$ for any $\psi \in TLIM(CB(G))$.*

Proof. Suppose $f_1 \in CB(G_1)$ is a permanently near one function as in Rosenblatt's theorem above. Let $F_{f_1} \in CB(G)$ be defined by $F_{f_1}(x, y) = f_1(x)$ for any $(x, y) \in G$ and let

$$H = \operatorname{span}\left\{{}_{(x,y)}F - F : (x, y) \in G, f \in CB(G)\right\}.$$

Note that for any $(x, y) \in G$ and $F \in CB(G)$,

$$\begin{aligned} {}_{(x,y)}F - F &= \left({}_{(x,y)}F - {}_{(e,y)}F\right) + \left({}_{(e,y)}F - F\right) \\ &= \left[{}_{(x,e)}\left({}_{(e,y)}F\right) - \left({}_{(e,y)}F\right)\right] + \left[{}_{(e,y)}F - F\right], \end{aligned}$$

where e is the group unit of G_1 or G_2. Hence for any $h \in H$, there are $x_i \in G_1$, $y_i \in G_2$, $F_i \in CB(G)$, $\overline{F}_i \in CB(G)$ and constants a_i $(i = 1, 2, \ldots, n)$ such that $h = h_1 + h_2$ where

$$h_1 = \sum_{i=1}^{n} a_i\left({}_{(x_i,e)}F_i - F_i\right), \quad h_2 = \sum_{i=1}^{n} a_i\left({}_{(e,y_i)}\overline{F}_i - \overline{F}_i\right).$$

Then $\|F_{f_1} - h\|_\infty \geq 1$. Indeed, for any $\varepsilon > 0$, by the Følner condition argument, there are $x'_k \in G$, $\lambda_k > 0$ $(k = 1, 2, \ldots, N)$ with $\sum_{k=1}^{N}\lambda_k = 1$ and

$$\left\| \sum_{k=1}^{N} \lambda_{k(x'_k,e)}h_1 \right\|_\infty < \varepsilon.$$

Hence

$$\begin{aligned} \|F_{f_1} - h\|_\infty &\geq \left\| \sum_{k=1}^{N} \lambda_{k(x'_k,e)}\left(F_{f_1} - h\right) \right\|_\infty \\ &\geq \left\| \sum_{k=1}^{N} \lambda_{k(x'_k,e)}F_{f_1} - \sum_{k=1}^{N} \lambda_{k(x'_k,e)}h_2 \right\|_\infty - \varepsilon. \end{aligned}$$

Note that

$$\begin{aligned} \sum_{k=1}^{N} \lambda_{k(x'_k,e)}h_2 &= \sum_{i=1}^{n} a_i\left[{}_{(e,y_i)}\left(\sum_{k=1}^{N} \lambda_{k(x'_k,e)}\overline{F}_i\right) - \left(\sum_{k=1}^{N} \lambda_{k(x'_k,e)}\overline{F}_i\right)\right] \\ &= \sum_{i=1}^{n} a_i\left({}_{(e,y_i)}T_i - T_i\right) \end{aligned}$$

where $T_i = \sum_{k=1}^{N}\lambda_{k(x'_k,e)}\overline{F}_i$. Since f_1 is a permanently near one function and G_2 is amenable, there is a $x_0 \in G_1$ such that

$$\left|1 - {}_{x'_k}f_1(x_0)\right| < \varepsilon \quad (k = 1, 2, \ldots, N)$$

and $m_2 \in LIM(CB(G_2))$. For any $F \in CB(G)$, let $F^{(x_0)} \in CB(G_2)$ be defined by $F^{(x_0)}(y) = F(x_0, y)$ for any $y \in G_2$. Then

$$\begin{aligned}
&\|F_{f_1} - h\|_\infty \\
&\geq \left\| \left(\sum_{k=1}^{N} \lambda_{k(x_k', e)} F_{f_1} \right)^{(x_0)} - \left(\sum_{i=1}^{n} a_i \left({}_{(e, y_i)}T_i - T_i \right) \right)^{(x_0)} \right\|_\infty - \varepsilon \\
&\geq \left\| 1 - \sum_{i=1}^{n} a_i \left({}_{y_i}(T_i)^{(x_0)} - (T_i)^{(x_0)} \right) \right\|_\infty - 2\varepsilon \\
&\geq m_2 \left(1 - \sum_{i=1}^{n} a_i \left({}_{y_i}(T_i)^{(x_0)} - (T_i)^{(x_0)} \right) \right) - 2\varepsilon \\
&= 1 - 2\varepsilon.
\end{aligned}$$

Therefore $\|F_{f_1} - h\|_\infty \geq 1$ for any $h \in H$.

Let $\theta \in LIM(CB(G))$ such that $\theta(F_{f_1}) = 1$. For any $\psi \in TLIM(CB(G))$, $\psi(F_{f_1}) = 0$. Indeed, for any $\varepsilon > 0$, let $f_M \in CB(G_1)$ and $\varphi_1 \in P(G_1)$ such that the support of $f_M - f_1$ is compact and $\|\varphi_1 * f_M\|_\infty < \varepsilon$. Take a $\varphi_2 \in P(G_2)$. Then φ defined by

$$\varphi(x, y) = \varphi_1(x)\varphi_2(y) \quad \text{for } (x, y) \in G$$

is an element of $P(G)$. Also, for any $(x, y) \in G$,

$$|\varphi * F_{f_M}(x, y)| = \left| \int_G \varphi_1(t_1)\varphi_2(t_2) f_M(t_1^{-1}x) \, dt_1 \, dt_2 \right| = |\varphi_1 * f_M(x)| < \varepsilon,$$

where $F_{f_M} \in G$ is defined by $F_{f_M}(x, y) = f_M(x)$ for $(x, y) \in G$. So $\|\varphi * F_{f_M}(x)\|_\infty < \varepsilon$. Since the support of $f_M - f_1$ is compact on G_1, the support of $F_{f_M} - F_{f_1}$ is contained in $C \times G_2$ for some compact subset C of G_1. Also, G_1 is not compact. Hence $m(F_{f_M}) = m(F_{f_1})$ for any $m \in LIM(CB(G))$. Therefore

$$\psi(F_{f_1}) = \psi(F_{f_M}) = \psi(\varphi * F_{f_M}) < \varepsilon$$

for any $\psi \in TLIM(CB(G))$; i.e., $\psi(F_{f_1}) = 0$.

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