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THE SYMMETRIC GENUS OF THE HIGMAN-SIMS GROUP *HS* AND BOUNDS FOR CONWAY'S GROUPS Co_1, Co_2

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Introduction

By a *surface* we shall always mean a closed connected compact orientable 2 manifold. For G a finite group, the *symmetric genus* $\sigma(G)$ *of* G is, by definition, the least integer g such that there exists a surface of genus g on which G acts in a conformal manner. It is well known that any such action of G on a surface S must be accompanied by an orientation-preserving action of G^0 on S, where G^0 is a subgroup of index at most 2 in G. In particular, if G is simple, its conformal action on S must be orientation-preserving. In this case we have $\sigma(G) = \sigma^0(G)$, where $\sigma^0(G)$ denotes the *strong symmetric genus of* G, defined to be the least integer g such that there is a surface of genus g on which G acts in an orientation-preserving manner.

In this paper we determine the symmetric genus of the Higman-Sims sporadic group *HS* and substantially improve existing bounds for the sporadic groups Co_1 and Co_2 of Conway. To do this we rely on the theory of triangular tesselations of the hyperbolic plane (e.g. see [2], [3], [4]), as well as a theorem of Tucker on partial presentations of groups which admit cellularly embedded Cayley graphs in surfaces of prescribed genus (see [7]). This reduces the problem to one of group generation, which can be handled in principal by computing relevant structure constants for the group, as well as for a variety of its subgroups, by means of character tables. (See [9] for additional details on all of the above remarks.) Throughout, we adopt the notation used in [1] and [8]. In particular, $\Delta_G(K_1, K_2, K_3)$ denotes the structure constant whose value is the cardinality of the set

$$\{(a, b): a \in K_1, b \in K_2, ab = c\},$$

where c is a fixed element of the conjugate class K_3 of G. Also all conjugate classes are understood to be G-classes unless otherwise inferred.

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1. The Higman-Sims group

In [8] it was shown that $G = HS$ could be generated by two elements, of respective orders 2 and 3, whose product was of order 11, i.e. that G is $(2,3,11)$-generated. This sufficed to prove that

$$\sigma(G) \leq 1 + \tfrac{1}{2}|G|\left(1 - \tfrac{1}{2} - \tfrac{1}{3} - \tfrac{1}{11}\right) = 1680001.$$

It was also proved there that G could not be $(2,3,7)$-generated, giving the lower bound

$$\sigma(G) \geq 1 + \tfrac{1}{2}|G|\left(1 - \tfrac{1}{2} - \tfrac{1}{3} - \tfrac{1}{8}\right) = 924001.$$

In fact the only possible values for $\sigma(G)$ are

$$1 + \tfrac{1}{2}|G|\left(1 - \frac{1}{r} - \frac{1}{s} - \frac{1}{t}\right)$$

where

$$(r,s,t) \in \{(2,3,8),(2,4,5),(2,3,10),(2,3,11)\}.$$

In this section we eliminate the first three possibilities, proving that $\sigma(G) = 1680001$.

By a theorem of Ree on permutations [5] applied to the rank 3 action of G on the 22-regular graph on 100 vertices, we see that G cannot be $(2,3,8)$-generated, and that $(2,4,5)$- and $(2,3,10)$-generation can arise only from the following class structures:

$$(2B,4A,5A),(2B,4A,5B),(2B,4C,5A),(2B,4C,5B),$$
$$(2B,3A,10A),(2B,3A,10B).$$

Computing the structure constants $\Delta_G(K_1, K_2, K_3)$ for the relevant classes K_1, K_2 and K_3, we see that $\Delta_G(K_1, K_2, K_3)$ exceeds the order of the centralizer $C(z)$, $z \in K_3$, only for the constant

$$\Delta_G(2B,3A,10B) = 70.$$

Thus G can only be $(2B,3A,10B)$-generated (see [8]). But a maximal $U_3(5):2$ contributes a value of 25 to this constant. There are two classes of

$U_3(5):2$ in G; choose representatives U and W with $t \in U \cap W \cap 10B$. Then it is easy to show that

$$U \cap W \cong 5_+^{1+2}:8:2,$$

whence $U \cup W$ contributes a total value of 50. But the centralizer in $\mathrm{Aut}(G) \cong HS:2$ of a $10B$ element is of order 40. This means that any $(2B, 3A, 10B)$-subgroup of G, not contained in a $U_3(5):2$, must have nontrivial centralizer in $\mathrm{Aut}(G)$. We conclude that G cannot be $(2B, 3A, 10B)$-generated.

2. Conway's group Co_1

The best previous known bounds for the symmetric genus of $G = Co_1$ are

$$1 + \tfrac{1}{2}|G|(1 - \tfrac{1}{2} - \tfrac{1}{3} - \tfrac{1}{8}) \leq \sigma(G) \leq 1 + \tfrac{1}{2}|G|(1 - \tfrac{1}{2} - \tfrac{1}{3} - \tfrac{1}{23}).$$

(See [8], where $(2,3,23)$-generation and $(2,3,7)$-non-generation are established.) Presently, we prove G is $(2,3,11)$-generated, which lowers the upper bound to $1 + \frac{1}{2}|G|(1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{11})$.

Let $\lambda = \Delta_G(2C, 3D, 11A)$. We compute $\lambda = 18546$ and observe that the only maximal subgroups of G which meet each of the classes $2C$, $3D$ and $11A$ are $2^{11}:M_{24}$, Co_1 and $3^6:2M_{12}$. The contribution of each of these classes of groups to the full structure constant λ is handled in a separate lemma.

LEMMA A. *Let $z \in 11A$. Then z is contained in precisely six distinct conjugates of $V:K \cong 2^{11}:M_{24}$ in G, each of which contributes at most* 1122 *to λ. Thus the total contribution from the class $\{2^{11}:M_{24}\}$ is at most* 6732.

Proof. We apply the method of *little groups* (see [6]) to obtain vital information on the characters of $V:K$. First observe that the action of K on the irreducible characters $\mathrm{Irr}(V)$ of V is contragredient to that of K on V. Thus $\mathrm{Irr}(V):K$ is the splitting extension of 2^{11} by M_{24} which occurs in Janko's sporadic group J_4. This means that K has three orbits on $\mathrm{Irr}(V)$ of respective sizes 1, 276 and 1771. As 11 divides 1771, which in turn divides the degree of all irreducible characters of VK which induce from $2^{11}:2^6:3^{\cdot}S_6$, such characters vanish at z so may be ignored in the structure constant computation. Consider next characters which induce to VK irreducibly from $2^{11}:M_{22}:2$. As $2^{11}:M_{22}:2$ fails to meet $3D$, all such characters vanish on this conjugate class, so too may be ignored. This leaves only the faithless characters of VK, i.e., those irreducible characters with V in their kernel. Let

b and c represent the two K-classes of elements of order 2, with b Sylow-central. It is immediate from the permutation character corresponding to the action of K on V that $|C_V(b)| = 2^7$ and $|C_V(c)| = 2^6$. (Note that the two inequivalent irreducible actions of M_{24} on 2^{11} admit the same permutation character.) Thus the coset Vb contains precisely 2^7 involutions of which 2^4 are conjugate to b, while Vc contains precisely 2^6 involutions of which 2^5 are conjugate to c. Moreover, the remaining involutions in Vb are fused under VP where P is a Sylow 7-subgroup of $C_K(b)$, while the remaining involutions in Vc are fused under V. By character restriction we see that $b \in 2A$. We assume the worst case, i.e., that the three remaining classes of involutions in $VK \setminus V$ all fuse to $2C$ in G. Letting $[g]$ denote the VK-class which contains g, we now compute

$$\Delta_{VK}([c],[t],[z]) = 484$$
$$\Delta_{VK}([eb],[t],[z]) = 154$$
$$\Delta_{VK}([e_1c],[t],[z]) = 484$$

where t represents the unique VK-class which meets $3D$, and $[eb]$ and $[e_1c]$ are the aforementioned VK-classes which differ from $[b]$ and $[c]$. This gives the value of 1122 as the maximal contribution of VK to λ. As the distinct VK-classes $[z]$ and $[z^{-1}]$ fuse in G, and as $|C_{VK}(z)| = 22$ and $|C_G(z)| = 66$, z is in precisely six distinct conjugates of VK. The result follows.

Lemma B. *$z \in 11A$ is in precisely three distinct conjugates of $C \cong Co_3$ in G, each of which contributes 671 to λ. Thus the total contribution from the class $\{Co_3\}$ is at most 2013.*

Proof. That z is in three distinct conjugates of C is immediate as $|C_C(z)| = 22$. By character restriction each of $2C$ and $3D$ are seen to meet C in a single class (these C-classes are denoted $2B$ and $3C$ in [1]), and we let x and y be respective representatives. Then

$$\Delta_C([x],[y],[z]) = 671$$

and the result follows.

Lemma C. *$z \in 11A$ is in a unique conjugate of $E : M \cong 3^6 : 2M_{12}$, which contributes at most 891 to λ.*

Proof. We assume $z \in M$ and that $\langle t \rangle = Z(M)$. For $x, y \in M$ with $x \in 2C$ and y of order 3, we have $\Delta_M([x],[y],[z]) = 0$ if y is Sylow-central in M (in which case $y \in 3B$) and $\Delta_M([x],[y],[z]) = 11$ otherwise (in which case $y \in 3D$). Now let $a, b \in EM$ with $a \in 2C$, $b \in 3D$, $ab = z$. Then it is easy to show that $a = eg$, $b = eh$ ($e \in E$, $g, h \in M$) and that g and h have respective orders 2 and 3. (Note that M has no element of order 9.) One also sees that g inverts e and so $gh = z$. As $e = ag$, g is conjugate to a in $\langle a, g\rangle \cong S_3$, whence $g \in 2C$. Since $gh = z$ we now conclude from our opening remarks that $h \in 3D$. This establishes that the number of pairs (a, b) with $a, b \in EM$, $a \in 2C$, $b \in 3D$ and $ab = z$ is bounded above by $11k$ where

$$k = |\{e \in E\colon g \text{ inverts } e\}|.$$

(Note that k does not depend on g as M has a unique class of involutions, distinct from $[t]$, which fuses to $2C$ in G.) But, as M is perfect, each of its elements acts with determinant 1 on E, hence inverts an even dimensional subspace of E. As g does not act as $-I$ on E, $k \leq 81$. The result now follows.

By Lemmas A, B and C, we see that the total contribution of the classes $\{2^{11}: M_{24}\}$, $\{Co_3\}$ and $\{3^6: 2M_{12}\}$ to the full structure constant $\lambda = 18546$ is at most 7392. This proves that Co_1 is (2, 3, 11)-generated.

3. Conway's group Co_2

As in the case of Co_1, the best previous known bounds for $\sigma(G)$, $G = Co_2$, arise from (2, 3, 23)-generation and (2, 3, 7)-non-generation of G, established in [8]. So again it is the case that

$$1 + \tfrac{1}{2}|G|(1 - \tfrac{1}{2} - \tfrac{1}{3} - \tfrac{1}{8}) \leq \sigma G \leq 1 + \tfrac{1}{2}|G|(1 - \tfrac{1}{2} - \tfrac{1}{3} - \tfrac{1}{23}),$$

and again we lower the upper bound to $1 + \tfrac{1}{2}|G|(1 - \tfrac{1}{2} - \tfrac{1}{3} - \tfrac{1}{11})$ by establishing (2, 3, 11)-generation. Only here the task is much simpler.

We compute $\Delta_G(2C, 3A, 11A) = 55$ and observe that the only maximal subgroups of G which have order divisible by 11 are $U_6(2)\colon 2$, $2^{10}: M_{22}: 2$, McL, $HS: 2$ and M_{23}. Clearly then, any proper (2, 3, 11)-subgroup of G must lie in one of $U_6(2)$, $2^{10}: M_{22}$, McL, HS or M_{23}. But $2^{10}: M_{22}$, HS and M_{23} each fails to meet $3A$, while McL fails to meet $2C$. One easily checks that $U_6(2)$ meets each of $2C$ and $3A$ in a single class (these classes are denoted by $2C$ and $3B$ in [1], respectively). An easy computation reveals that $\Delta_U(2C, 3B, 11A) = 0$. Thus G has no proper $(2C, 3B, 11A)$-subgroup, so is itself (2, 3, 11)-generated.

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