ILLINOIS JOURNAL OF MATHEMATICS
Volume 36, Number 1, Spring 1992

A LOW INTENSITY MAXIMUM PRINCIPLE FOR BI-BROWNIAN MOTION

BY

THOMAS S. SALISBURY[1]

1. Introduction

Bi-Brownian motion is the process $Z(s,t) = (X(s), Y(t))$, where X and Y are independent d-dimensional Brownian motions started from $\mathbf{0}$. Here $d \geq 3$. Studying a pair of Brownian motions as a two-parameter process has proved useful both in probability (for example path intersections—see [DS] or [FS]) and analysis (see [GS] or [W1]).

Estimates of the hitting probabilities for Z were given in [FS]. They use capacity (defined analytically) for the kernel $u = v \otimes v$, where $v(x, y) = c(d)|x - y|^{2-d}$ is the Green function for Brownian motion. In other words, if $z = (x, y)$ and $z' = (x', y')$ then

$$u(z, z') = v(x, x')v(y, y').$$

While the capacity theory for u behaves well (see also [F] and [O]), the same cannot be said for other potential theoretic objects involving u. The principal cause is that the maximum principle fails badly (see §4). Since the maximum principle is closely tied to the strong Markov property this failure is not unexpected. It is however possible to retain it in a weakened form.

We will establish a "low-intensity" version of the bounded maximum principle for Z, to the effect that if $U\mu = \int u(\cdot, z)\mu(dz)$ is bounded on K and $\mathbf{0}$ is far from K, in the sense that the probability of hitting K starting from $\mathbf{0}$ is small, then $U\mu(\mathbf{0})$ can't be large. This would not be hard if "far" were interpreted using the Euclidean distance. In contrast, our condition can be thought of as allowing K to be thin at $\mathbf{0}$ (in a fairly stringent sense). This permits consideration of sets like thorns or fractal dusts, and forces us to

Received September 15, 1989.

1980 Mathematics Subject Classification (1985 Revision). Primary 60J65; Secondary 60G60, 31B30.

[1]Research supported in part by a grant from NSERC.

achieve a deeper understanding of the issues involved. Our formal statement is:

(1.1) THEOREM. *There exist* $c_1 > 0$ *and* $c_2 > 0$ *such that if* $\mu(K^c) = 0$, $U\mu \le 1$ *on* K, *and* $P(Z \text{ hits } K) \le c_1$ *then*

$$U\mu(\mathbf{0}) \le c_2 P(Z \textit{ hits } K).$$

We will not keep track of the values of the constants. This could in fact be done, but it is not expected that the values obtained would be of the correct order of magnitude. In §4 we will consider the following conjecture:

$$U\mu(\mathbf{0}) \le \log(1/P(Z \text{ misses } K)). \tag{1.2}$$

There we will show by example that if the conjecture is true then it is sharp, in which case the optimal c_2 would be $c_1^{-1}\log((1-c_1)^{-1})$.

The main argument is given in §2. It uses preliminary estimates, postponed to §3, that follow from results of [FS]. For completeness, the relevant arguments of the latter are given in §3, adapted to the present setting.

2. The maximum principle

We prove Theorem (1.1) using a good point/bad point argument. Loosely, a point z is good if, conditional on Z hitting z, it is unlikely that Z will hit K except near z. For example, let $d = 3$ and $e = (1, 1, 1)$. Then $z = (e, e)$ might be considered a good point of the set

$$D = \{(e + x, e - x);\ |x_k| \le 1\ \forall k\},$$

but a bad point of the set

$$D = \{(x, y);\ 0 \le x_k \le 2\ \forall k,\ |y - e| \le 10^{-10}\}.$$

Figure 1 gives such sets D schematically. Each class of points requires its own energy-related estimate, proved in §3.

We write $B(r)$ for the open ball in $\mathbf{R}^d$ with centre $\mathbf{0}$ and radius r. As in §1, P will be a probability under which X and Y are independent Brownian motions, both started at the origin $\mathbf{0}$, and $Z(s,t) = (X(s), Y(t))$. We may assume that Z is realised on an appropriate path space, so that the shift operators $\theta(\cdot)$ are at our disposal. Of course there are two shifts possible, one for each parameter, but it will be clear from the context which is meant.

We will also use probabilities $P_J^z = P_J^{x,y}$ for $z = (x, y)$ and $J \subset \{0, 1\}$. Under $P_J^{x,y}$, X and Y are independent, and start at x and y respectively. If $0 \notin J$ then X is a Brownian motion, but if $0 \in J$ then X is an h-transform of

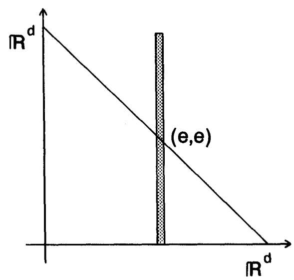

FIG. 1.

Brownian motion by $h = v(\cdot, \mathbf{0})$. The time reversal

$$\hat{X}(s) = X(L - s), \qquad 0 \le s < L \tag{2.1}$$

(where L is the last exit time of X from $B(r)$) will be such an h-transform under P, started uniformly on $\partial B(r)$—see [Do]. Likewise, Y is an h-transform or a Brownian motion under P_J^z, depending on whether 1 is or is not in J. We use a similar notation P_J^ν if the initial law (that of $(X(0), Y(0))$) is ν. In either case, we drop the subscript if $J = \emptyset$.

Throughout this section, we assume that $\mu(K^c) = 0$ and $U\mu \le 1$ on K. Since Z has continuous paths and $U\mu$ is lower semi-continuous (it is a potential for $2d$-dimensional Brownian motion), we will assume without loss of generality that K is closed. Let $R(i)$ be the closure of $[B(2^{2i+1}) \setminus B(2^{2i})]$, $R(i, j) = R(i) \times R(j)$, and $R = \bigcup\{R(i, j);\ i, j \in \mathbf{Z}\}$ (see Figure 2). Until the very end of this section, we'll restrict attention to K satisfying $K \subset R$.

(2.2) LEMMA.
(a) *If $K \subset R(i, j)$ then $P(Z$ hits $K) \ge c\int u(\mathbf{0}, z)\mu(dz)$.*
(b) *If $K \subset A \times \mathbf{R}^d$ and $A \subset R(i)$ then $P(X$ hits $A) \ge c\int u(\mathbf{0}, z)\mu(dz)$.*

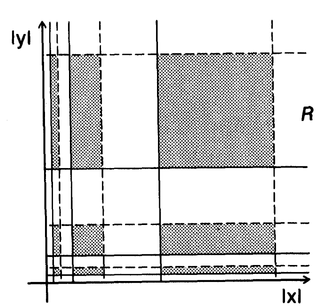

FIG. 2.

Note. Here and in the future, c denotes a generic constant, possibly depending on d but not on K, i, j or other parameters. Its value may change from line to line.

Proof. (a) Take ν to be the product of uniform probabilities on $\partial B(2^{2i+1})$ and $\partial B(2^{2j+1})$. Set

$$e(\mu) = \iint u(z, z')\mu(dz)\mu(dz').$$

Then $e(\mu) \leq \mu(K)$ since $U\mu \leq 1$ on K. Renormalize μ to give a probability measure $\mu' = \mu(K)^{-1}\mu$. By (3.1) and the strong Markov property of X and Y,

$$\begin{aligned} P(Z \text{ hits } K) &\geq P^{\nu}(Z \text{ hits } K) \\ &\geq c(2^{2i+1}2^{2j+1})^{2-d}/e(\mu') \\ &= c2^{(2-d)[(2i+1)+(2j+1)]}\mu(K)^2/e(\mu) \\ &\geq c2^{(2-d)[(2i+1)+(2j+1)]}\mu(K) \\ &\geq c\int u(\mathbf{0}, z)\mu(dz), \end{aligned}$$

showing (a).

(b) Let $\nu = \eta \times \delta_{\mathbf{0}}$, where η is the uniform probability on $\partial B(2^{2i+1})$. Then by (3.4),

$$\begin{aligned} P(X \text{ hits } A) &\geq P(Z \text{ hits } K) \\ &\geq c2^{(2-d)(2i+1)}\int v(\mathbf{0}, y)\mu(dx, dy) \\ &\geq c\int u(\mathbf{0}, z)\mu(dz), \end{aligned}$$

showing (b). □

Write $K(i, j)$ for $K \cap R(i, j)$ and $K(\widehat{i, j})$ for $\bigcup\{K(i', j'); i', j' \in \mathbf{Z}, (i', j') \neq (i, j)\}$. The following would be easy if there were any good strong Markov property for Z.

(2.3) LEMMA. *Suppose that $K \subset R$, and that*

$$P_J^z\left(Z \text{ hits } K(\widehat{i, j})\right) \leq \theta \qquad \forall z \in K(i, j), \quad \forall J \subset \{0, 1\}. \tag{2.4}$$

Then

$$P\left(Z \text{ hits } K(i,j) \text{ and } K\left(\widetilde{i,j}\right)\right) \leq c\theta P(Z \text{ hits } K(i,j)).$$

Proof. Consider first

$$(2.5)\quad K' = \bigcup_{i',j' \in \mathbf{Z},\, j' > j} K(i',j')$$

$$\rho = P(\exists s \leq s',\, t \leq t' \text{ s.t. } Z(s,t) \in K(i,j),\, Z(s',t') \in K').$$

Let

$$T = \inf\{t > 0;\, |Y(t)| = 3 \cdot 2^{2i}\} \quad \text{and} \quad T' = \inf\{t > 0;\, |Y(t)| = 2^{2i+1}\}.$$

Let $T'(0) = 0$,

$$T(n) = T'(n) + T \circ \theta(T'(n)), \qquad T'(n+1) = T(n) + T' \circ \theta(T(n)).$$

Then $Z(s,t)$ can be in $K(i,j)$ only if t is in some interval $[T'(n), T(n)]$. We know that $P(T(n) < \infty) = (3/2)^{n(2-d)}$ and that $Y(T'(n))$ is uniform when $T(n)$ is conditioned to be finite. Set

$$\sigma(n) = \inf\{s > 0;\, \exists t \in [T'(n), T(n)] \text{ s.t. } Z(s,t) \in K(i,j)\},$$
$$\tau(n) = \inf\{t > T'(n);\, Z(\sigma(n),t) \in K(i,j)\}.$$

Suppose that s, t, s', t' are as in (2.5). Let $t \in [T'(n), T(n)]$. Then $\sigma(n) \leq s \leq s'$ and $\tau(n) \leq T(n) \leq t'$. As a consequence,

$$\rho \leq \sum_{n \geq 0} P(\sigma(n) < \infty,\, T(n) < \infty,\, Z \text{ hits } K' \text{ during } [\sigma(n), \infty[\times [T(n), \infty[).$$

But $(\sigma(n), T(n))$ is a stopping point. That is,

$$\{\sigma(n) \leq s,\, T(n) \leq t\} \in \sigma\{Z(s',t');\, s' \leq s,\, t' \leq t\}.$$

At such two-parameter times, Z has the strong Markov property. To see this, just approximate from the upper right by discrete stopping points, as in the one-parameter case. See [M] or [W2] for a discussion of the general two-parameter theory. Now

$$P(\sigma(n) < \infty,\, T(n) < \infty,\, Z \text{ hits } K' \text{ during } [\sigma(n), \infty[\times [T(n), \infty[)$$
$$= P\left(\sigma(n) < \infty,\, T(n) < \infty,\, P^{X(\sigma(n)),\, Y(T(n))}(Z \text{ hits } K')\right).$$

But harmonic measure for $B(2^{2i+2})$ starting from $Y(T(n))$ is bounded by a constant multiple of that starting from $Y(\tau(n))$. Therefore

$$(2.6)\quad P^{X(\sigma(n)),Y(T(n))}(Z \text{ hits } K') \leq cP^{X(\sigma(n)),Y(\tau(n))}(Z \text{ hits } K') \leq c\theta,$$

and hence

$$P(\sigma(n) < \infty, T(n) < \infty, Z \text{ hits } K' \text{ during } [\sigma(n),\infty[\, \times [T(n),\infty[) \\ \leq c\theta P(\sigma(n) < \infty, T(n) < \infty).$$

By the strong Markov property of Y at $T'(n)$ the latter equals

$$c\theta(3/2)^{n(2-d)}P(Z \text{ hits } K(i,j) \text{ during } [0,\infty[\, \times [T',T]) \\ \leq c\theta(3/2)^{n(2-d)}P(Z \text{ hits } K(i,j)),$$

if $n \geq 1$, with a similar inequality for $n = 0$. Summing, we get

$$\rho \leq c\theta P(Z \text{ hits } K(i,j)).$$

Similar arguments give that each of

$$P(\exists s \geq s', t \leq t' \text{ s.t. } Z(s,t) \in K(i,j), Z(s',t') \in K')$$
$$P(\exists s \leq s', t \geq t' \text{ s.t. } Z(s,t) \in K(i,j), Z(s',t') \in K')$$
$$P(\exists s \geq s', t \geq t' \text{ s.t. } Z(s,t) \in K(i,j), Z(s',t') \in K')$$

is bounded by $c\theta P(Z \text{ hits } K(i,j))$. In each case we first reverse time for one or both of X, Y and in (2.6) use our hypothesis on P_J^z for some $J \subset \{0,1\}$ other than $J = \emptyset$. For example, to handle the first of these, we reverse X from its last exit from $B(2^{2i+1})$ and apply the preceding argument to $\hat{X}$, which is now a $v(\cdot,\mathbf{0})$-transform started uniformly on $\partial B(2^{2i+1})$—see (2.1). Combining these four bounds, we get such a bound on

$$P(Z \text{ hits } K(i,j) \text{ and } K').$$

Similar arguments apply if K' is replaced by $\bigcup\{K(i',j');\ j' < j\}$, or $\bigcup\{K(i',j');\ i' > i\}$, or $\bigcup\{K(i',j');\ i' < i\}$. Combining the bounds obtained yields the lemma. □

Proof of (1.1). By Lemma (2.3), we may choose θ so small that if (2.4) holds then

$$P\Big(Z \text{ hits } K(i,j) \text{ but misses } K\big(\widehat{i,j}\big)\Big) \\ \geq P(Z \text{ hits } K(i,j)) - P\Big(Z \text{ hits } K(i,j) \text{ and } K\big(\widehat{i,j}\big)\Big) \\ \geq cP(Z \text{ hits } K(i,j)).$$

Fix such a θ and let

$$\alpha(J;i) = \bigcup_{j\in\mathbf{Z}} \left\{ z \in K(i,j);\ P_J^z\Big(Z \text{ hits } \bigcup_{j'\neq j} K(i',j') \Big) > \theta/8 \right\},$$

$$\beta(J;j) = \bigcup_{i\in\mathbf{Z}} \left\{ z \in K(i,j);\ P_J^z\Big(Z \text{ hits } \bigcup_{i'\neq i} K(i',j') \Big) > \theta/8 \right\},$$

$$\alpha(J) = \bigcup_{i\in\mathbf{Z}} \alpha(J;i), \quad \beta(J) = \bigcup_{j\in\mathbf{Z}} \beta(J;j),$$

$$G = K \setminus \bigcup_{J\subset\{0,1\}} [\alpha(J)\cup\beta(J)].$$

Then

$$U\mu(\mathbf{0}) \le \int_G u(\mathbf{0},z)\mu(dz) + \sum_{J\subset\{0,1\}} \int_{\alpha(J)} u(\mathbf{0},z)\mu(dz) + \sum_{J\subset\{0,1\}} \int_{\beta(J)} u(\mathbf{0},z)\mu(dz). \tag{2.7}$$

But (2.4) holds for G, so by Lemma (2.3) and (a) of Lemma (2.2),

$$\begin{aligned} \int_G u(\mathbf{0},z)\mu(dz) &\le c \sum_{i,j\in\mathbf{Z}} P(Z \text{ hits } G(i,j)) \\ &\le c \sum_{i,j\in\mathbf{Z}} P\big(Z \text{ hits } G(i,j) \text{ but misses } G(\widehat{i,j})\big) \\ &\le cP(Z \text{ hits } K). \end{aligned} \tag{2.8}$$

For the time being, fix $J = \emptyset$ and let

$$A(i) = \{x;\ \exists y \text{ s.t. } (x,y) \in \alpha(\emptyset;i)\},$$

$$A = \bigcup_{i\in\mathbf{Z}} A(i).$$

Harmonic measure for $\{y;\ 2^{2j-1} < |y| < 2^{2j+2}\}$, starting from any $y \in R(j)$, is bounded by a constant times that starting from any other $y' \in R(j)$. Let T be the first time Y hits $R(j)$. If $(x,y) \in \alpha(\emptyset;i)$ and $y \in R(j)$ then by the strong Markov property of Y at T,

$$\begin{aligned} \theta/8 &\le P^{x,y}\big(Z \text{ hits } K(\widehat{i,j})\big) \\ &\le cP^{x,\mathbf{0}}\Big(P^{x,Y(T)}\big(Z \text{ hits } K(\widehat{i,j})\big)\Big) \\ &\le cP^{x,\mathbf{0}}(Z \text{ hits } K). \end{aligned}$$

Let $S = \inf\{s > 0;\ X(s) \in A\}$. Then by the strong Markov property, now of X,

$$\begin{aligned} P(Z \text{ hits } K) &\geq P(X \text{ hits } A,\ P^{X(S),\mathbf{0}}(Z \text{ hits } K)) \\ &\geq c_{00} P(X \text{ hits } A) \end{aligned}$$

for some constant $c_{00} > 0$. A simple argument using the strong Markov property of X shows that there is a $c_0 > 0$ such that

$$P(X \text{ hits } A(i) | X \text{ misses } A(j)\ \forall j < i) \geq c_0 P(X \text{ hits } A(i)).$$

Thus

$$P(X \text{ misses } A) \leq \prod_{i \in \mathbf{Z}} (1 - c_0 P(X \text{ hits } A(i)).$$

Now let c_1' be a constant less than c_{00}. Use (b) of Lemma (2.2) and the inequality

$$r \leq \log(1/[1-r])$$

to get a constant c_2' such that

$$\begin{aligned} \int_{\alpha(\emptyset)} u(\mathbf{0}, z)\mu(dz) &\leq c \sum_{i \in \mathbf{Z}} P(X \text{ hits } A(i)) \\ &\leq c \sum_{i \in \mathbf{Z}} \log(1/[1 - c_0 P(X \text{ hits } A(i))]) \\ &\leq c \log(1/P(X \text{ misses } A)) \\ &\leq c \log\big(1/\big[1 - c_{00}^{-1} P(Z \text{ hits } K)\big]\big) \\ &\leq c_2' P(Z \text{ hits } K) \quad \text{if } P(Z \text{ hits } K) < c_1'. \end{aligned}$$

The same inequality holds for $\beta(\emptyset)$. By reversing time for one or both of X, Y, similar inequalities can be proven for arbitrary $\alpha(J)$, $\beta(J)$. Together with (2.7) and (2.8), these give constants c_1 and c_2 for which the theorem holds, at least for $K \subset R$. By scaling, we have it as well for

$$\begin{aligned} &K \subset \{(x, y); (2x, y) \in R\} \text{ or} \\ &\qquad \{(x, y); (x, 2y) \in R\}, \text{ or } \{(x, y); (2x, 2y) \in R\}. \end{aligned}$$

Dividing (resp. multiplying) these c_1 (resp. c_2) by four then gives the general result. □

3. Energy

The following result is an immediate consequence of Theorem 4.16 of [FS]. For completeness, we give a direct proof. Recall that

$$e(\mu) = \iint u(z, z')\mu(dz)\mu(dz') \text{ and } v(x, x') = c(d)|x - x'|^{2-d}.$$

(3.1) PROPOSITION. *Let $K \subset B(q) \times B(r)$, and let ν be the uniform probability on $\partial B(q) \times \partial B(r)$. Then*

$$P^{\nu}(Z \text{ hits } K) \geq 4^{-1}c(d)^2(qr)^{2-d}/I(K),$$

where

$$I(K) = \inf\{e(\mu); \mu(K^c) = 0, \mu(K) = 1\}.$$

Note. The principal result of [FS] was a general reverse inequality, but this is not needed for our purposes. The argument uses ideas of Murali Rao [Ra] and E.B. Dynkin [Dy]. See also [E] and [Ro].

Proof. Without loss of generality, K is compact. Let $K(\varepsilon)$ be the set of points within distance ε of K, and choose ε so small that $K(\varepsilon) \subset B(q) \times B(r)$. Fix a measure μ satisfying $\mu(K^c) = 0$ and $\mu(K) = 1$. Without loss of generality $e(\mu) < \infty$. Let

$$g_\varepsilon(x, y) = c1_{[0,\varepsilon]}(|x|)1_{[0,\varepsilon]}(|y|),$$

where c is chosen to make g_ε integrate to 1, and then let

$$\begin{aligned}
f_\varepsilon &= \int g_\varepsilon(\cdot - z)\mu(dz) \\
\mu_\varepsilon(dz) &= f_\varepsilon(z)\, dz \\
v_\varepsilon(x, x') &= c\int_{B(\varepsilon)}\int_{B(\varepsilon)} v(x + a, x' + a')\, da\, da' \\
u_\varepsilon &= v_\varepsilon \otimes v_\varepsilon .
\end{aligned}$$

Then it is easily checked that

$$e(\mu_\varepsilon) = \iint u_\varepsilon(z, z')\mu(dz)\mu(dz').$$

Since v is superharmonic in both components, $v_\varepsilon \uparrow v$ as $\varepsilon \downarrow 0$. Thus $e(\mu_\varepsilon) \to e(\mu)$ by monotone convergence. Simple computations also show that

(3.2)

$$E^\nu\left[\left(\int_0^\infty\int_0^\infty f_\varepsilon(Z(s,t))\,ds\,dt\right)^2\right]$$
$$= \int\cdots\int f_\varepsilon(x,y)f_\varepsilon(x',y')[v(a,x)v(b,y)v(x,x')v(y,y')$$
$$+v(a,x)v(b,y')v(x,x')v(y',y) + v(a,x')v(b,y)v(x',x)v(y,y')$$
$$+v(a,x')v(b,y')v(x',x)v(y',y)]\nu(da)\nu(db)\,dx\,dy\,dx'\,dy'$$
$$= 4c(d)^2(qr)^{2-d}e(\mu_\varepsilon)$$

and that

(3.3)
$$E^\nu\left[\int_0^\infty\int_0^\infty f_\varepsilon(Z(s,t))\,ds\,dt\right]$$
$$= \iiiint f_\varepsilon(x,y)v(a,x)v(b,y)\nu(da)\nu(db)\,dx\,dy$$
$$= c(d)^2(qr)^{2-d}\int \mu_\varepsilon(dz)$$
$$= c(d)^2(qr)^{2-d}.$$

If the integrand on the left hand side of (3.3) is non-zero then Z must have hit $K(\varepsilon)$, so by the Cauchy-Schwarz inequality

$$P^\nu(Z \text{ hits } K(\varepsilon)) \geq \frac{E[\int_0^\infty\int_0^\infty f_\varepsilon(Z(s,t))\,ds\,dt,\ Z \text{ hits } K(\varepsilon)]^2}{E\left[\left(\int_0^\infty\int_0^\infty f_\varepsilon(Z(s,t))\,ds\,dt\right)^2\right]}$$
$$= 4^{-1}c(d)^2(qr)^{2-d}/e(\mu_\varepsilon).$$

Letting ε tend to zero then completes the proof. □

(3.4) Proposition. *Suppose $K \subset B(r) \times \mathbf{R}^d$, $\mu(K^c) = 0$, and $U\mu \leq 1$ on K. Let $\nu = \eta \times \delta_{\mathbf{0}}$, where η is the uniform probability on $\partial B(r)$. Then*

$$P^\nu(Z \text{ hits } K) \geq 4^{-1}c(d)r^{2-d}\int v(\mathbf{0},y)\mu(dx,dy)$$

Proof. Now let $\nu = \eta \times \delta_{\mathbf{0}}$, where η is the uniform probability on $\partial B(r)$. Let $\mu(K^c) = 0$ and $U\mu \le 1$ on K. Then μ cannot charge $\mathbf{R}^d \times \{\mathbf{0}\}$ as if it did, $U\mu$ would be infinite on this set, hence infinite somewhere in K. Thus without loss of generality μ does not charge $\mathbf{R}^d \times B(\delta)$ for some $\delta > 0$, so that $\int v(\mathbf{0}, y)\mu(dx, dy) < \infty$.

Define f_ε, μ_ε, and v_ε as in Prop. (3.1). Let

$$\tilde{v}_\varepsilon(y, y') = c\int_{B(\varepsilon)}\int_{B(\varepsilon)} v(y + b, y' + b')v(\mathbf{0}, y + b)\, db\, db'.$$

Computing as before now gives that

$$\begin{aligned} E^\nu&\left[\left(\int_0^\infty\int_0^\infty f_\varepsilon(Z(s,t))\, ds\, dt\right)^2\right] \\ &= 2c(d)r^{2-d}\int\cdots\int f_\varepsilon(x, y)f_\varepsilon(x', y')v(x, x')v(y, y') \\ &\qquad \times\left[v(\mathbf{0}, y) + v(\mathbf{0}, y')\right] dx\, dy\, dx'\, dy' \\ &= 4c(d)r^{2-d}\iint v_\varepsilon(x, x')\tilde{v}_\varepsilon(y, y')\mu(dz)\mu(dz'). \end{aligned}$$

For fixed y and y', we have $\tilde{v}_\varepsilon(y, y') \to v(y, y')v(\mathbf{0}, y)$ as $\varepsilon \downarrow 0$. Moreover, if $\varepsilon < \delta/2$ then $v(\mathbf{0}, y + b)$ is bounded for $y \notin B(\delta)$ and $b \in B(\varepsilon)$. Using superharmonicity of v as before, we see that $\tilde{v}_\varepsilon$ is dominated by a multiple of v on $\mathbf{R}^d \times (\mathbf{R}^d \setminus B(\delta))$. As before we have $v_\varepsilon \uparrow v$, so by dominated convergence

$$\begin{aligned} E^\nu&\left[\left(\int_0^\infty\int_0^\infty f_\varepsilon(Z(s,t))\, ds\, dt\right)^2\right] \\ &\to 4c(d)r^{2-d}\iint u(z, z')v(\mathbf{0}, y)\mu(dz)\mu(dz') \\ &\le 4c(d)r^{2-d}\int v(\mathbf{0}, y)\mu(dz), \end{aligned}$$

the latter since $U\mu \le 1$ on K. Similarly

$$\begin{aligned} E^\nu\left[\int_0^\infty\int_0^\infty f_\varepsilon(Z(s,t))\, ds\, dt\right] &= c(d)r^{2-d}\iint f_\varepsilon(x, y)v(\mathbf{0}, y)\, dx\, dy \\ &\to c(d)r^{2-d}\int v(\mathbf{0}, y)\mu(dz). \end{aligned}$$

Now use the Cauchy-Schwarz inequality as before, to give the result. □

4. Examples and problems

(4.1) *Example.* The maximum principle fails.

Let $S(r) = \partial B(r)$. For fixed n and $\varepsilon > 0$ let

$$\Sigma(k) = S(\varepsilon^k) \times S(\varepsilon^{n-k})$$

$$K = \bigcup_{k=0}^{n} \Sigma(k)$$

(see Figure 3). Let μ_k be the product of the equilibrium measures on $S(\varepsilon^k)$ and $S(\varepsilon^{n-k})$, so that $U\mu_k(z) = P^z(Z \text{ hits } \Sigma(k))$. Thus $U\mu_k$ can be made arbitrarily small on $\Sigma(k')$ for $k' \neq k$, by making ε sufficiently small. Let

$$\mu = \sum_{k=0}^{n} \mu_k.$$

We have that $U\mu(\mathbf{0}) = n$, but $U\mu$ can be made arbitrarily close to 1 on K. □

Recall the inequality (1.2):

Conjecture. $U\mu(\mathbf{0}) \leq \log(1/P(Z \textit{ misses } K))$.

The right hand side arises as follows. Suppose that K can be partitioned into pieces $K(k)$, and that if N of them are hit then $U\mu(z) = E^z[N]$. If N has a Poisson distribution then (1.2) is just the statement that $P(N = 0) = \exp(-E[N])$.

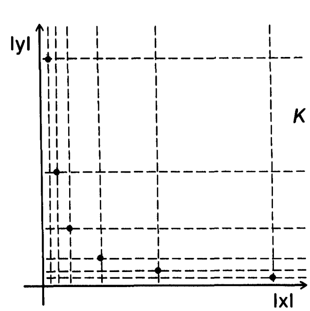

FIG. 3.

To support the conjecture, let $\lambda\colon \mathbf{R} \to \mathbf{R}$ and consider bounds of the form

$$U\mu(\mathbf{0}) \le \lambda(P(Z \text{ misses } K)). \tag{4.2}$$

(4.3) *Example.* Suppose (4.2) is valid for every compact set K and every finite measure μ on K satisfying $U\mu \le 1$ on K. Then

$$\lambda(p) \ge \log(1/p).$$

Let $p = e^{-\gamma}$. Replace $S(r)$ from the last example, by $r\Gamma$, where $\Gamma \subset S(1)$ and $P(X \text{ hits } \Gamma) = \sqrt{\gamma/n}$. Again let $U\mu_k(z) = P^z(Z \text{ hits } K(k))$, where $K(k) = [\varepsilon^k\Gamma] \times [\varepsilon^{n-k}\Gamma]$. Then $U\mu(\mathbf{0}) = \gamma$. By choosing ε sufficiently small, $U\mu$ can be made arbitrarily close to 1 on K, and $P(Z \text{ misses } K)$ can be made arbitrarily close to $(1 - \gamma n^{-1})^n$. Letting $n \to \infty$ establishes the result. □

Finally, let us return to the context of (3.1), so that attention is restricted to sets $K \subset B(q) \times B(r)$. Call $\Gamma(K) = c(d)^{-2}(qr)^{d-2}P^\nu(Z \text{ hits } K)$ the capacity of K, so that K has capacity zero if and only if it is almost surely missed by Z.

(4.4) Proposition. *Let K be a compact subset of $B(q) \times B(r)$. Then $\Gamma(K) = 0$ if and only if there is a finite measure μ of finite energy such that $U\mu = \infty$ on K.*

Proof. Let $\Gamma(K) = 0$. By Proposition (3.1), $I(K) = \infty$. By Lemma 2.3.4 of [F] we can find open sets $K_n \downarrow K$ with $I(K_n) \uparrow \infty$. Moreover, by Theorem 2.4 of [F], the probability μ_n on K of minimal energy satisfies $U\mu_n \ge e(\mu_n)$ on K_n. By passing to a subsequence, we may assume that $\Sigma e(\mu_n)^{-1} < \infty$.
The desired measure μ is then just $\Sigma e(\mu_n)^{-1}\mu_n$. The converse holds by the energy principle (see (1) on p. 164 of [F]), as it is shown in [FS] that if $\Gamma(K) > 0$ then K supports a measure ν of finite energy, yet $\int U\mu\, d\nu$ would be infinite. □

This criterion would be easier to apply if the restriction to finite energy measures was removed. Call a set K polar if there is a finite measure μ with $U\mu = \infty$ on K. The above shows that capacity zero implies polar.

(4.5) *Problem.* If K is polar, does it have capacity zero?

Note that if $\Gamma(K) > 0$ then, by the results in [FS], K supports a measure ν of finite energy. By Theorem 2.4 of [F], we may assume that $U\nu$ is bounded on the support of ν. If the bounded maximum principle held then $U\nu$ would be bounded and (4.5) would be answered in the affirmative (by the law of reciprocity, $\int U\nu\, d\mu = \int U\mu\, d\nu$). It would be interesting to know it there is a valid version of the maximum principle that is strong enough to resolve the problem.

REFERENCES

[DS] B. DAVIS and T.S. SALISBURY, *Connecting Brownian paths*, Ann. Probab., vol. 16 (1988), pp. 1428–1457.

[Do] J.L. DOOB, *Classical Potential Theory and its Probabilistic Counterpart*, Grundlehren Math. Wiss. no. 262, Springer Verlag, New York, 1984.

[Dy] E.B. DYNKIN, *Additive functionals of several time-reversible Markov processes*, J. Funct. Anal., vol. 42 (1981), pp. 64–101.

[E] S.N. EVANS, *Potential theory for a family of several Markov processes*, Ann. Inst. Henri Poincaré, vol. 23 (1987), pp. 499–530.

[FS] P.J. FITZSIMMONS and T.S. SALISBURY, *Capacity and energy for multi-parameter Markov processes*, Ann. Inst. Henri Poincaré, vol. 25 (1989), pp. 325–350.

[F] B. FUGLEDE, *On the theory of potentials in locally compact spaces*, Acta Math., vol. 103 (1960), pp. 139–215.

[GS] R.F. GUNDY and E.M. STEIN, *H^p theory for the poly-disc*, Proc. Nat. Acad. Sci. U.S.A., vol. 76 (1979), pp. 1026–1029.

[M] P.A. MEYER, Théorie élémentaire des processus à deux indices in *Processus Aléatoires à Deux Indices*, Lecture Notes in Mathematics, no. 868, Springer-Verlag, New York, 1981, pp. 1–39.

[O] M. OHTSUKA, *On potentials in locally compact spaces*, J. Sci. Hiroshima Univ. Ser. A-I Math., vol. 25 (1961), pp. 135–352.

[Ra] K.M. RAO, *On polar sets for Lévy processes*, J. London Math. Soc., vol. 35 (1987), pp. 569–576.

[Ro] L.C.G. ROGERS, *Multiple points of Markov processes in a complete metric space*, preprint, 1988.

[W1] J.B. WALSH, "Optional increasing paths" in *Processus Aléatoires à Deux Indices*, Lecture Notes in Mathematics, no. 868, Springer-Verlag, New York, 1981, pp. 179–201.

[W2] J.B. WALSH, "Martingales with a multidimensional time parameter and stochastic integrals in the plane" in *Lectures in probability and statistics (Santiago de Chile*, 1986), Lecture Notes in Mathematics No. 1215, Springer-Verlag, New York, 1986, pp. 329–491.

YORK UNIVERSITY
NORTH YORK ONTARIO, CANADA