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UNIQUENESS IN ERGODIC DECOMPOSITION OF INVARIANT PROBABILITIES

BY

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Abstract

We show that for any set of transition probabilities on a common measurable space and any invariant probability, there is at most one representing measure on the set of extremal, invariant probabilities with the σ-algebra generated by the evaluations. The proof uses nonstandard analysis.

1. Results

There are some well known results in ergodic decomposition of invariant measures, e.g., J. Kerstan and A. Wakolbinger [9], G. Winkler [22], and R.H. Farrell [7]. But there are either tightness assumptions (Winkler), or there are assumptions on the cardinality or on the special structure of the set of transition probabilities (Kerstan and Wakolbinger or Farrell). Using nonstandard analysis, we do not need such assumptions to prove uniqueness of the representing measure.

A transition probability on a measurable space $(\Omega, \mathscr{F})$ is a function P: $\Omega \times \mathscr{F} \to [0,1]$ whose partial functions $P(\cdot, F)$, $F \in \mathscr{F}$, are $\mathscr{F}$-measurable, and whose partial functions $P(\omega, \cdot)$, $\omega \in \Omega$, are (perhaps finitely additive) probabilities. Usually, transition probabilities are assumed to be countably additive, but we never use this assumption. In contrast, we always assume a probability to be countably additive and announce if there is an exception.

DEFINITION 1. (a) Let P be a transition probability on a measurable space $(\Omega, \mathscr{F})$. A finitely additive probability p on this measurable space is called P-invariant if

$$p \cdot P := \int P(\omega, \cdot)\, dp(\omega) = p.$$

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(b) Given a set $\mathscr{P}$ of transition probabilities, a finitely additive probability measure p is called $\mathscr{P}$-invariant if it is P-invariant for all $P \in \mathscr{P}$.

Let $\mathscr{E}(\mathscr{P})$ be the set of all $\mathscr{P}$-invariant probabilities and let $\tilde{\mathscr{E}}(\mathscr{P})$ be the set of all finitely additive, $\mathscr{P}$-invariant probabilities.

We say that a measure μ on a set of finitely additive probabilities endowed with the σ-algebra generated by the evaluations *represents* p if $p(F) = \int q(F)\, d\mu(q)$ for any measurable set F.

THEOREM 2. *Let $\mathscr{P}$ be a set of transition probabilities on a common measurable space with the σ-algebra generated by the evaluations and let p be a $\mathscr{P}$-invariant (countably additive) probability measure. If p has a representing measure μ on the set of extremal, $\mathscr{P}$-invariant, countably additive probabilities, this representing measure is unique.*

The proof of this theorem, like the proofs of the following corollary and the following proposition, are postponed to Sections 3 and 4.

This theorem does not state that there exists a representing measure. Indeed, there is no additional structure on Ω assumed; thus one cannot apply any theorem of Choquet type for the existence of a representing measure, and, in [5], L.E. Dubins and D.A. Freedman even gave an example of an invariant measure which has no representing measure. But if one gives up countable additivity, the set of invariant probabilities is compact in the topology generated by the evaluations, and one can apply Bishop-deLeeuw's theorem.

COROLLARY 3. *If p is a finitely additive, $\mathscr{P}$-invariant probability measure, then there exists an unique representing measure for p on the set of extremal, $\mathscr{P}$-invariant, finitely additive probabilities.*

The main tool in proving the theorem is the following proposition, which in the case of inverse limits of measure spaces gives us a representation of the representing measure as, in some sense, a weak limit of an independently constructed net of measures. For this proposition we need a dual definition of invariance.

DEFINITION 4. For any transition probability P and any P-invariant probability p on $(\Omega, \mathscr{F})$, the set

$$\mathscr{I}_{P,p} := \{F \in \mathscr{F} | \mathbf{1}_F = P(\cdot, F)\, p\text{-a.s.}\}$$

is called the invariance-σ-algebra for P and p. The elements of $\mathscr{I}_{P,p}$ are called P-invariant. We further define

$$\mathscr{I}_P := \bigcap_{p:\, p \text{ is } P\text{-invariant}} \mathscr{I}_{P,p}.$$

Remark 5. It is well known and easy to see that the invariance-σ-algebra is a σ-algebra, even if the transition probability is only finitely additive. Nevertheless we cannot drop the countable additivity of the invariant probability.

Let $\mathscr{M}_1(\Omega, \mathscr{F})$ be the set of all finitely additive probabilities endowed with the topology generated by the evaluations. It is well known that $\mathscr{M}_1(\Omega, \mathscr{F})$ is a compact space. Now we are able to formulate the aforementioned proposition.

PROPOSITION 6. *Let $(I, \leq)$ be a directed index set, $(P_i)_{i \in I}$ a family of transition probabilities on a common measurable space $(\Omega, \mathscr{F})$, $(\mathscr{F}_i)_{i \in I}$ a decreasing family of sub-σ-algebras of $\mathscr{F}$, and $\mathscr{E}$ a set of extremal, $\mathscr{P}$-invariant probability measures such that for any $q \in \mathscr{E}$, we have*

$$\bigcap_{i \in I} \mathscr{F}_i \subseteq \bigcap_{i \in I} \mathscr{I}_{P_i, q}.$$

Further, let $\gamma_i\colon \Omega \to \mathscr{M}_1(\Omega, \mathscr{F})$, $i \in I$, be defined by

$$\gamma_i(\omega) := P_i(\omega, \cdot)$$

and let ι be the embedding of $\mathscr{E}$ into the set $\mathscr{M}_1(\Omega, \mathscr{F})$. Now if μ is a probability measure on $\mathscr{E}$ and if the probability $p := \int_{\mathscr{E}} q \, d\mu(q)$ satisfies

$$P_i(\cdot, F) = E_p(\mathbf{1}_F | \mathscr{F}_i) \quad p\text{-a.s.}$$

for any $F \in \mathscr{F}$ and any $i \in I$, then the net $(p \circ \gamma_i^{-1})_{i \in I}$ converges to $\mu \circ \iota^{-1}$ in the weak topology.

2. Tools from nonstandard analysis

To prove the results we will use some tools from nonstandard analysis. There are several well known introductions to nonstandard analysis, e.g., Stroyan and Luxembourg [21] and Lindstrøm [10].

In this article we only use a polysaturated enlargement of our standard universe. The symbol * denotes, as usual, the embedding of the standard world into our nonstandard universe.

2.1. Some topology.

The symbol st, perhaps with some index denoting a topological space, denotes the standard part mapping on the set of those points in a Hausdorff space where a standard part exists.

An element a in the enlargement of an ordered set A is called *infinitely large* if for any $b \in A$, we have $^*b \leq a$.

Remark 7. Notice that a net $(x_i)_{i \in I}$ in a (standard) Hausdorff space converges to an element x if and only if there exists an infinitely large index i_0 such that for all $i \geq i_0$, we have $st(^*x_i) = x$ (cf. Robinson [20], Theorem 4.2.4).

2.2. Loeb measures.

If $\mathscr{F}$ is an internal algebra of subsets of the enlargement $^*\Omega$ of Ω, then there always exists a σ-algebra generated by the (standard) algebra $\mathscr{F}$, which we denote by $L(\mathscr{F})$ and call the *Loeb algebra* of $\mathscr{F}$. The difference from the usual definition, where $L(\mathscr{F})$ is further completed by a Loeb measure, is not deep, but we need this little difference because, in general, we handle more than one measure on the same measurable space.

If $\mathscr{F}$ is a σ-algebra in the standard universe, we denote by $\tilde{\mathscr{F}}$ the sub-σ-algebra of $L(^*\mathscr{F})$ generated by the standard sets. (Notice that the standard sets in $^*\mathscr{F}$, in general, form an external algebra but not a σ-algebra.)

By the saturation principle every internal algebra $\mathscr{F}$ is compact in the measure theoretic sense. So, if p is an internal, finitely additive measure on $\mathscr{F}$, the mapping $F \mapsto st(p(F))$ is automatically countably additive, and we are able to extend it to $L(\mathscr{F})$. This extension is denoted by $L(p)$ and called *Loeb measure* of p.

It is well known from Loeb's work (see also Cutland [4], Theorem 3.1) that for any Loeb measurable set, there exists an internal set which differs from the first only by a nullset. In proving our theorem, especially in the proof of Lemma 15, we need a strengthened version of this fact.

LEMMA 8. *If M is a set of probability measures on a common measurable space $(\Omega, \mathscr{F})$, then for any set in $\tilde{\mathscr{F}}$, there exists a standard set which differs from the first only by a set which is an $L(^*p)$-nullset for all $p \in M$.*

Proof. Let

$$\mathscr{D} := \{F \in L(^*\mathscr{F}) | \text{there exists a } G \in \mathscr{F} \text{ with } L(*p)(F \Delta {}^*G) = 0 \text{ for any } p \in M\}.$$

Then it is easy to see that $\mathscr{D}$ is an algebra which contains the standard $^*\mathscr{F}$-measurable sets. We only have to show that $\mathscr{D}$ is closed under countable unions. Therefore, let $(F_i)_{i\in\mathbf{N}}$ be any sequence in $\mathscr{D}$ and let $F := \bigcup_{i\in\mathbf{N}} F_i$. We know that there are standard sets G_i with $L(^*p)(F_I \Delta ^*G_i) = 0$. We only have to show that $G := \bigcup_{i\in\mathbf{N}} G_i$ is a good choice to approximate F. For any $p \in M$, we have

$$\begin{aligned} L(*p)(F\Delta ^*G) &\leq L(*p)\Big(\bigcup_{i\in\mathbf{N}} F_i \Delta \bigcup_{i\in\mathbf{N}} {}^*G_i\Big) + L(*p)\Big(\bigcup_{i\in\mathbf{N}} {}^*G_i \Delta ^*G\Big) \\ &\leq L(*p)\Big(\bigcup_{i\in\mathbf{N}} (F_i \Delta ^*G_i)\Big) + L(*p)(^*G) \\ &\quad - \lim_{k\to\infty} L(*p)\Big(\bigcup_{i=1}^{k} {}^*G_i\Big) = 0. \end{aligned}$$ □

We will use many other properties of Loeb measures but we cannot treat them all here. For the details, Cutland [13] is a good reference for nonstandard measure theory.

2.3. Loeb kernels and their relation to invariant sets.

DEFINITION 9. For any internal transition probability P on the extension $(^*\Omega, ^*\mathscr{F})$, we define the Loeb kernel $L(P)$ by

$$L(P)(\omega, \cdot) := L(P(\omega, \cdot)).$$

We have to prove that by our definition Loeb kernels really are transition kernels.

LEMMA 10. (a) *Let P be an internal transition probability on $(^*\Omega, ^*\mathscr{F})$. Then the Loeb kernel $L(P)$ is a transition probability on the measurable space $(^*\Omega, L(^*\mathscr{F}))$.*

(b) *If p is a probability on $(^*\Omega, \mathscr{F})$, then $L(p \cdot P) = L(p) \cdot L(P)$.*

(c) *If the probability p is P-invariant, then $L(p)$ is $L(P)$-invariant.*

(d) *If P is a standard transition probability on the measurable space $(\Omega, \mathscr{F})$, the restriction $L(^*P)_{|^*\Omega\times\tilde{\mathscr{F}}}$ is a transition probability on $(^*\Omega, \tilde{\mathscr{F}})$.*

(e) *If p is a standard, finitely additive probability, it is P-invariant if and only if $L(^*p)|_{\tilde{\mathscr{F}}}$ is $L(^*P)_{|^*\Omega\times\tilde{\mathscr{F}}}$-invariant.*

Proof. (a) and (d) In part (a) the only thing to prove is that for any $F \in L(^*\mathscr{F})$, the function $L(^*P)(\cdot, F)$ is $L(^*\mathscr{F})$-measurable. For part (d) one has to show that for any $F \in \tilde{\mathscr{F}}$, the function $L(^*P)(\cdot, F)$ is $\tilde{\mathscr{F}}$-measurable.

Because of the countable additivity of $L({}^*P)$ it is sufficient to prove part (a) only for internal sets and part (d) only for standard sets. Since $L({}^*P)(\cdot, F) = st({}^*P(\cdot, F))$ both are easily seen to be true (cf. Loeb [11], Theorem 2).

(b) Using Loeb [11], Theorem 3, we know for any set $F \in {}^*\mathscr{F}$,

$$L(p \cdot P)(F) = st\left(\int P(\cdot, F)\, dp\right) = \int L(P)(\cdot, F)\, dL(p).$$

Continuity of both sides of this equation yields part (b).

(c) Follows from (b).

(e) The 'if' part follows easily from (b). The other direction follows from (c) and (d). □

Loeb kernels yield a characterization for the invariance of sets.

LEMMA 11. (a) *An $\mathscr{F}$-measurable set F is P-invariant with respect to a P-invariant probability p if and only if ${}^*F$ is $L({}^*P)$-invariant for $L({}^*p)$.*

(b) *Any set $F \in \tilde{\mathscr{I}}_{P,p}$ is $L({}^*P)$-invariant with respect to $L({}^*p)$.*

Indeed, instead of (b) we are able to prove that any $F \in L({}^*\mathscr{I}_{P,p})$ is $L({}^*P)$-invariant, but we do not need this fact in the sequel.

Proof. (a) If $\mathbf{1}_F = P(\cdot, F)$ p-a.s., we know $\mathbf{1}_{{}^*F} = {}^*P(\cdot, {}^*F)$ ${}^*p$-a.s., and then

$$\mathbf{1}_{{}^*F} = L({}^*P)(\cdot, {}^*F) \quad L({}^*p)\text{-a.s.}$$

Conversely, if F is not P-invariant, then there exists an $\varepsilon > 0$ such that

$$\begin{aligned} L({}^*p)\big(\{|\mathbf{1}_{{}^*F} - L({}^*P)(\cdot, {}^*F)| \geq \varepsilon\}\big) &\geq L({}^*p)\big(\{|\mathbf{1}_{{}^*F} - {}^*P(\cdot, {}^*F)| \geq {}^*\varepsilon\}\big) \\ &= st\big({}^*p(\{|\mathbf{1}_{{}^*F} - {}^*P(\cdot, {}^*F)| \geq {}^*\varepsilon\})\big) \\ &= p\big(\{|\mathbf{1}_F - P(\cdot, F)| \geq \varepsilon\}\big) \geq \varepsilon. \end{aligned}$$

Hence $\mathbf{1}_{{}^*F}$ is not $L({}^*P)$-invariant too.

(b) Follows from (a) by the remark that the invariant sets form a σ-algebra. □

3. Proof of Proposition 6

First, we need a lemma.

LEMMA 12. *Let $\mathscr{P}$ be a set of transition probabilities on a common measurable space and let p be a $\mathscr{P}$-invariant probability. Then any extremal, $\mathscr{P}$-invariant probability p is atomic on the σ-algebra $\bigcap_{P \in \mathscr{P}} \mathscr{I}_{P,p}$.*

Proof. It is well known and easy to see that for any $F \in \mathscr{I}_{P,p}$ with $0 < p(F) < 1$, the probabilities $p(\cdot|F)$ and $p(\cdot|F^c)$ are two different, $\mathscr{P}$-invariant probabilities with a weighted mean p. $\square$

Proof of the proposition. There is a theorem in Burkholder [1], with a similar proof, which tells us that the intersection of a decreasing sequence of sufficient σ-algebras is sufficient. Of course, Burkholder's statement is not true for any index set because he uses pointwise martingale convergence. But we can use his ideas. Since our index set is not necessarily countable, we have to use L^2-martingale convergence, instead of pointwise martingale convergence.

Let ε be any infinitesimally small number. The L^2-martingale convergence theorem tells us that there is a standard family $(f_F)_{F\in\mathscr{F}}$ of $\bigcap_{i\in I}\mathscr{I}_{P_i}$-measurable functions such that the net of functions $(P_i(\cdot, F))_{i\in I}$ converges in quadratic mean, and in probability too, to f_F for any $F \in \mathscr{F}$. For luck of nonstandard analysis there is an internal *finite algebra $\mathscr{F}_0 \subset {}^*\mathscr{F}$ containing all standard sets of ${}^*\mathscr{F}$. Hence we can find an index $i_0 \in {}^*I$ such that for any $i \geq i_0$,

$$^*p\big(\{|^*P_i(\cdot, F) - {}^*f_F| > \varepsilon \text{ for some } F \in \mathscr{F}_0\}\big) < \varepsilon.$$

Now take any of these indices $i \geq i_0$. The functions f_F were chosen to be $\bigcap_{i\in I}\mathscr{I}_{P_i}$-measurable. So, by Lemma 12, for any $q \in {}^*\mathscr{E} \subset {}^*ex\mathscr{C}((P_i)_{i\in I})$, the functions ${}^*f_F$, $F \in \mathscr{F}$, are all q-a.s. constant; i.e.

$$^*f_F = \int {}^*f_F \, dq \quad q\text{-a.s.}$$

Thus

$$q\left(\left\{\left|^*P_i(\cdot, F) - \int {}^*f_F\, dq\right| > \varepsilon \text{ for some } F \in \mathscr{F}_0\right\}\right)$$
$$= q\big(\{|^*P_i(\cdot, F) - {}^*f_F| > \varepsilon \text{ for some } F \in \mathscr{F}_0\}\big).$$

Now we have

$$\int_{^*\mathscr{E}} q\left(\left\{\left|^*P_i(\cdot, F) - \int {}^*f_F\, dq\right| > \varepsilon \text{ for some } F \in \mathscr{F}_0\right\}\right) d\mu(q)$$
$$= {}^*p\big(\{|^*P_i(\cdot, F) - {}^*f_F| > \varepsilon \text{ for some } F \in \mathscr{F}_0\}\big) < \varepsilon.$$

For any $q \in {}^*\mathscr{E}$, define the set G_q by

$$G_q := \left\{\left|^*P_i(\cdot, F) - \int {}^*f_F\, dq\right| \leq \varepsilon \text{ for all } F \in \mathscr{F}_0\right\}.$$

Then

$$\begin{aligned}
&\int_{^*\mathscr{E}} L(q)(G_q)\, dL(^*\mu)(q)\\
&\quad = \int_{^*\mathscr{E}} st\big(q(G_q)\big)\, dL(^*\mu)(q) = st\left(\int_{^*\mathscr{E}} q(G_q)\, d^*\mu(q)\right)\\
&\quad = st\Bigg(\int_{^*\mathscr{E}}\Bigg(1 - q\Bigg(\Bigg\{\left|{}^*P_i(\cdot, F) - \int {}^*f_F\, dq\right|\\
&\qquad\qquad > \varepsilon \text{ for some } F \in \mathscr{F}_0\Bigg\}\Bigg)\Bigg)\, d^*\mu(q)\Bigg)\\
&\quad \geq st(1 - \varepsilon) = 1.
\end{aligned}$$

This yields $L(q)(G_q) = 1$ for $L(^*\mu)$-a.a. $q \in {}^*\mathscr{E}$. For these $q \in {}^*\mathscr{E}$, all $\omega \in G_q$, and all $F \in \mathscr{F}_0$, we know

$$st\big({}^*P_i(\omega, F)\big) = st\left(\int {}^*f_F\, dq\right) = st\big({}^*f_F\big) = st\big({}^*P_i(\cdot, F)\big) \quad L(q)\text{-a.s.},$$

and, since $q \in {}^*\mathscr{E} \subset {}^*\mathscr{C}(\mathscr{P})$, we have

$$st\big(({}^*P_i(\omega, F)\big) = \int st\big({}^*P_i(\cdot, F)\big)\, dL(q) = st\left(\int {}^*P_i(\cdot, F)\, dq\right) = q(F).$$

If we now take only standard sets for F, we get

$$st_{\mathscr{M}_1(\Omega, \mathscr{F})}\big({}^*\gamma_i(\omega)\big) = st_{\mathscr{M}_1(\Omega, \mathscr{F})}\big({}^*P_i(\omega, \cdot)\big) = st_{\mathscr{M}_1(\Omega, \mathscr{F})}(q)$$

for all $\omega \in G_q$. Because $L(q)(G_q) = 1$ we can conclude that

$$st_{\mathscr{M}_1(\Omega, \mathscr{F})} \circ {}^*\gamma_i = st_{\mathscr{M}_1(\Omega, \mathscr{F})}(q) \quad L(q)\text{-a.s.}$$

for $L(^*\mu)$-a.a. $q \in {}^*\mathscr{E}$, and

$$L(q) \circ {}^*\gamma_i^{-1} \circ st^{-1}_{\mathscr{M}_1(\Omega, \mathscr{F})} = \varepsilon_{st_{\mathscr{M}_1(\Omega, \mathscr{F})}(q)}$$

for $L(^*(\mu \circ \iota^{-1}))$-a.a. $q \in \mathscr{M}_1(\Omega, \mathscr{F})$.

By Lemma 10, part (b), it follows that

$$\begin{aligned} L(*p) \circ {}^*\gamma_i^{-1} \circ st^{-1}_{\mathscr{M}_1(\Omega, \mathscr{F})} &= \int L(q) \circ {}^*\gamma_i^{-1} \circ st^{-1}_{\mathscr{M}_1(\Omega, \mathscr{F})} dL\left({}^*\left(\mu \circ \iota^{-1}\right)\right) \\ &= \int \varepsilon_{st_{\mathscr{M}_1(\Omega, \mathscr{F})}(q)} dL\left({}^*\left(\mu \circ \iota^{-1}\right)\right) \\ &= L\left({}^*\left(\mu \circ \iota^{-1}\right)\right) \circ st^{-1}_{\mathscr{M}_1(\Omega, \mathscr{F})}. \end{aligned}$$

Using [14, Corollary 1], we see that $L(*p) \circ {}^*\gamma_i^{-1} \circ st^{-1}_{\mathscr{M}_1(\Omega, \mathscr{F})}$ is the standard part of ${}^*p \circ {}^*\gamma_i$ in the weak topology, and, further,

$$L\left({}^*\left(\mu \circ \iota^{-1}\right)\right) \circ st^{-1}_{\mathscr{M}_1(\Omega, \mathscr{F})} = \mu \circ \iota^{-1}.$$

Since $i \geq i_0$ is arbitrary, we can use Remark 5 to get the weak convergence of the net $(p \circ \gamma_i^{-1})_{i \in I}$ to $\mu \circ \iota^{-1}$. □

4. Proof of the theorem

The main problem in proving the theorem now is how to satisfy the assumptions of Proposition 6. One easily sees that for the index set I the set of non-empty, finite subsets of $\mathscr{P}$ would be a pleasant choice. For the σ-algebras $\mathscr{F}_i$ we would like to take the intersections of the invariance σ-algebras for the $P \in i$. But if we look at Burkholder's paper [1] (by Dynkin's results [6] there are some parallels of our problem to the existence of H-sufficient statistics), we notice that there is still some work to be done.

4.1. How to use nonstandard analysis on our problem.

LEMMA 13. *Let $\mathscr{C}_0$ be any set of $\mathscr{P}$-invariant probabilities and let the ideal $\mathscr{N}$ be defined by*

$$\mathscr{N} := \left\{N \in L({}^*F) | L(*q)(N) = 0 \textit{ for any } q \in \mathscr{C}_0\right\}.$$

Further let $\overline{\mathscr{F}} := \tilde{\mathscr{F}} \vee \mathscr{N}$ and

$$\overline{\mathscr{P}} := \left\{L({}^*P)_{|{}^*\Omega \times \overline{\mathscr{F}}} | P \in \mathscr{P}\right\}.$$

Then:

(a) *$\overline{\mathscr{P}}$ is a set of transition probabilities on $({}^*\Omega, \overline{\mathscr{F}})$ and any probability p is $\mathscr{P}$-invariant if and only if $L({}^*p)_{|\overline{\mathscr{F}}}$ is $\overline{\mathscr{P}}$-invariant.*

(b) *For any $p \in \mathscr{C}_0$, the probability p is extremal in the set of $\mathscr{P}$-invariant probabilities if and only if $L({}^*p)_{|\overline{\mathscr{F}}}$ is extremal in the set of $\overline{\mathscr{P}}$-invariant probabilities.*

(c) *The mapping* $\vartheta\colon ex\mathscr{C}(\mathscr{P}) \cap \mathscr{C}_0 \to ex\mathscr{C}(\overline{\mathscr{P}})$ *defined by*

$$p \mapsto L(*p)_{|\overline{\mathscr{F}}}$$

is injective and bimeasurable.

Proof. First, we have to prove that for any $P \in \mathscr{P}$, the mapping $\overline{P} := L(^*P)_{|^*\Omega \times \overline{\mathscr{F}}}$ is a transition probability on the measurable space $(^*\Omega, \overline{\mathscr{F}})$, i.e., for all sets $G \in \overline{\mathscr{F}}$, the functions $L(^*P)(\cdot, G)$ are already $\overline{\mathscr{F}}$-measurable. For $G \in \mathscr{N}$ this is trivial, since by invariance it is easy to see that for any $p \in \mathscr{C}_0$, the function $\overline{P}(\cdot, G)$ is an $L(^*p)$-nullfunction. For standard G (a standard measurable set in the nonstandard universe) we can use Lemma 10. Well known measure theoretic arguments are just remaining.

Now we come to the main problems of this lemma.

(a) If p is a $\mathscr{P}$-invariant probability, using Lemma 10, it is easy to see that $L(^*p)$ is an $\{L(^*P) | P \in \mathscr{P}\}$-invariant probability too. Now one direction is easy to see.

If, conversely, $L(^*p)_{|\overline{\mathscr{F}}}$ is $\overline{\mathscr{P}}$-invariant, we conclude for any $P \in \mathscr{P}$ and any $F \in \mathscr{F}$,

$$\int P(\cdot, F)\, dp = st\left(\int {}^*P(\cdot, {}^*F)\, d^*p\right) = \int st({}^*P(\cdot, {}^*F))\, dL(*p)$$

$$= \int L(^*P)(\cdot, {}^*F)\, dL(*p) = L(*p)(^*F) = p(F).$$

Hence p is $\mathscr{P}$-invariant.

(b) Now let $p \in \mathscr{C}_0$. If p is extremal in the set of $\mathscr{P}$-invariant probabilities and q_1 and q_2 are two $\overline{\mathscr{P}}$-invariant probabilities with $L(^*p)_{|\overline{\mathscr{F}}} = \frac{1}{2} \cdot (q_1 + q_2)$, there is no problem in seeing that

$$q_{1|\mathscr{N}} \equiv q_{2|\mathscr{N}} \equiv 0.$$

Let us further define p_1 and p_2 by

$$p_1\colon F \mapsto q_1(^*F) \quad \text{and} \quad p_2\colon F \mapsto q_2(^*F)$$

It is easy to see that p_1 and p_2 are additive (indeed, countably additive but we do not need this) $\mathscr{P}$-invariant probabilities which satisfy $p = \frac{1}{2} \cdot (p_1 + p_2)$. So, by the extremality of p we see that $p_1 = p_2$, which shows that q_1 and q_2 are equal on all standard sets. By countable additivity of q_1 and q_2 it follows that

$$q_{1|\tilde{\mathscr{F}}} = q_{2|\hat{\mathscr{F}}}.$$

Both results together show that q_1 and q_2 are equal. Thus $L(^*p)_{|\overline{\mathscr{F}}}$ is extremal.

Conversely, if $L(^*p)_{|\overline{\mathscr{F}}}$ is assumed to be extremal in the set of $\overline{\mathscr{P}}$-invariant probabilities and p_1 and p_2 are two P-invariant probabilities with $p = \frac{1}{2} \cdot (p_1 + p_2)$, we know

$$L(^*p) = \tfrac{1}{2} \cdot \left(L(^*p_1) + L(^*p_2)\right).$$

By part (a), $L(^*p_1)_{|\overline{\mathscr{F}}}$ and $L(^*p_2)_{|\overline{\mathscr{F}}}$ are $\overline{\mathscr{P}}$-invariant. Extremality of $L(^*p)_{|\overline{\mathscr{F}}}$ yields the equality of $L(^*p_1)_{|\overline{\mathscr{F}}}$ and $L(^*p_2)_{|\overline{\mathscr{F}}}$, and we easily see that $p_1 = p_2$.

(c) Part (b) tells us that ϑ is well defined. Because $\overline{\mathscr{F}}$ contains all standard sets, it is easy to see that ϑ is injective.

Furthermore, for any standard set $F \in \mathscr{F}$, it is easy to see that $p \mapsto L(^*p)_{|\overline{\mathscr{F}}}(^*F) = p(F)$ is measurable, and for any $N \in \mathscr{N}$, it is trivial that $p \mapsto L(^*p)_{|\overline{\mathscr{F}}}(N) = 0$ is measurable. This yields the measurability of ϑ.

Conversely, ϑ^{-1} is measurable because for any $F \in \mathscr{F}$, the mapping

$$L(^*p)_{|\overline{\mathscr{F}}} \mapsto p(F) = L(^*p)_{|\overline{\mathscr{F}}}(^*F)$$

is measurable. □

4.2. How the ergodic theorem is used.

The ergodic theorem and martingale convergence are the main tools in proving our theorem. The assumptions of Proposition 6 deal with some system of transition kernels representing conditional expectations, while our theorem does not make such special assumptions. It is a little tricky but ergodic theorem helps in this situation.

LEMMA 14. *Let P be a transition probability on $(\Omega, \mathscr{F})$, $\mathscr{C}_0$ be any set of P-invariant probabilities and let $\mathscr{N}$ and $\overline{\mathscr{F}}$ be defined as in the last lemma. Further, suppose $n \in {}^*\mathbf{N} \setminus \mathbf{N}$ is an infinitely large number, and let Q be defined by*

$$Q := L\left(\frac{1}{n} \cdot \sum_{i=0}^{n-1} {}^*P^i\right)_{|^*\Omega \times \overline{\mathscr{F}}}.$$

Then Q is a transition probability on $(^*\Omega, \overline{\mathscr{F}})$ which satisfies

$$E_{L(^*p)}\left(\mathbf{1}_F | \tilde{\mathscr{I}}_{P,p}\right) = Q(\cdot, F) \quad L(^*p)\text{-a.s.}$$

for any $p \in \mathscr{C}_0$ and any set $F \in \overline{\mathscr{F}}$. Furthermore, for any probability p, the probability $L(^*p)_{|\overline{\mathscr{F}}}$ is Q-invariant, and $Q \cdot L(^*P)_{|\overline{\mathscr{F}}} = Q$ is valid.

Proof. By Remark 7, the ergodic theorem (cf. Neveu [16], Proposition V.6.4) shows that for any $p \in \mathscr{C}_0$ and any $F \in \mathscr{F}$,

$$st\left(\left\|\frac{1}{n}\cdot\sum_{i=0}^{n-1}{}^*P^i(\cdot,{}^*F) - {}^*E_p\left(\mathbf{1}_F|\mathscr{I}_{P,p}\right)\right\|_{\mathbf{L}^1(^*p)}\right) = 0.$$

Thus it is easy to see that

$$Q(\cdot,{}^*F) = L\left(\frac{1}{n}\cdot\sum_{i=0}^{n-1}{}^*P^i\right)(\cdot,{}^*F) = st\left({}^*E_p\left(\mathbf{1}_F|\mathscr{I}_{P,p}\right)\right) \quad L(^*p)\text{-a.s.}$$

is an $\tilde{\mathscr{F}} \vee \mathscr{N}$-measurable function. It is further easy to see that this statement is true if ${}^*F$ is replaced by any set $F \in \mathscr{N}$. Monotone class arguments and continuity of Q show that this statement is also true for any set $F \in \overline{\overline{\mathscr{F}}}$. We now conclude that Q is a transition probability on $({}^*\Omega, \overline{\overline{\mathscr{F}}})$.

For standard $F \in \mathscr{F}$, the function $E_p(\mathbf{1}_F|\mathscr{I}_{P,p})$ is finite. Hence it is easy to see (cf. Cutland [4], Theorem 3.5) that

$$\begin{aligned} Q(\cdot,{}^*F) &= st\left({}^*E_p\left(\mathbf{1}_F|\mathscr{I}_{P,p}\right)\right) = st\left(E^*_p\left(\mathbf{1}_{^*F}|{}^*\mathscr{I}_{P,p}\right)\right) \\ &= E_{L(^*p)}\left(\mathbf{1}_{^*F}|\tilde{\mathscr{I}}_{P,p}\right) \quad L(^*p)\text{-a.s.} \end{aligned}$$

Both sides are continuous so for any $F \in \tilde{\mathscr{F}}$,

$$E_{L(^*p)}\left(\mathbf{1}_F|\tilde{\mathscr{I}}_{P,p}\right) = Q(\cdot, F) \quad L(^*p)\text{-a.s.}$$

If p is a P-invariant probability, we know that for any set $F \in {}^*\mathscr{F}$,

$$st\left(\int\frac{1}{n}\cdot\sum_{i=0}^{n-1}{}^*P^i(\cdot,F)\,d^*p\right) = st(^*p(F)) = L(^*p)(F),$$

which yields the Q-invariance of $L(^*p)_{|\overline{\overline{\mathscr{F}}}}$.

The last equality is valid because, using Lemma 10, for any set $F \in {}^*\mathscr{F}$ we have

$$\begin{aligned} &L\left(\frac{1}{n}\sum_{i=0}^{n-1}{}^*P^i\right)\cdot L(^*P)(\cdot,F) \\ &\quad= st\left(\frac{1}{n}\sum_{i=0}^{n-1}{}^*P^i\,.{}^*P(\cdot,F)\right) \\ &\quad= st\left(\frac{1}{n}\sum_{i=0}^{n-1}{}^*P^i(\cdot,F)\right) - st\left(\frac{1}{n}\cdot\mathbf{1}_F\right) + st\left(\frac{1}{n}\cdot{}^*P^n(\cdot,F)\right) \\ &\quad= st\left(\frac{1}{n}\sum_{i=0}^{n-1}{}^*P^i(\cdot,F)\right). \end{aligned}$$

Our proof of Lemma 14 is now finished. □

4.3. How to treat finite sets of transition probabilities.

The next step in proving our theorem is to construct transition probabilities which relate in some sense to given finite sets of transition probabilities. Following Burkholder's paper [1], we easily see that nullsets cause some trouble. The main idea now in the proof of the following lemma is well known in Hilbert space theory and already used in Burkholder's paper [1] and Dynkin's paper [6].

LEMMA 15. *Let $\mathscr{P} := \{P_1, \dots, P_k\}$ be a finite set of transition probabilities on a common measurable space $(\Omega, \mathscr{F})$. For any $\mathscr{P}$-invariant probability p, we denote the invariance-σ-algebra for P_i and p by $\mathscr{I}_{i,p}$. We further assume, as in the last lemma, that $\mathscr{C}_0$ is any set of $\mathscr{P}$-invariant probabilities but with the additional property*

$$E_p\big(\mathbf{1}_F | \mathscr{I}_{i,p}\big) = P_i(\cdot, F) \quad p\text{-a.s.}$$

for any $p \in \mathscr{C}_0$ and any $F \in \mathscr{F}$. The ideal $\mathscr{N}$ and the σ-algebra $\overline{\mathscr{F}}$ shall be defined as in the lemma above. Further, we denote by $\mathscr{J}$ the σ-algebra

$$\mathscr{J} := \bigcap_{i=1}^{k} \left(\bigcap_{p \in \mathscr{C}_0} \left(\tilde{\mathscr{I}}_{i,p} \vee \mathscr{N} \right) \right).$$

Then there is a transition probability Q on $({}^*\Omega, \overline{\mathscr{F}})$ with the property

$$E_{L(^*p)}(\mathbf{1}_F | \mathscr{J}) = Q(\cdot, F) \quad L(^*p)\text{-a.s.}$$

for any $p \in \mathscr{C}_0$ and any $F \in \overline{\mathscr{F}}$. Furthermore, for any $p \in \mathscr{C}_0$, the probability $L(^*p)_{|\overline{\mathscr{F}}}$ is Q-invariant.

Proof. Let P be the transition kernel defined by

$$P := P_k \cdots P_1.$$

Lemma 14 assures that there is a transition probability Q with the properties

$$L(^*p)_{|\overline{\mathscr{F}}} \cdot Q = L(^*p)_{|\overline{\mathscr{F}}}$$

and

$$E_{L(^*p)}\big(\mathbf{1}_F | \tilde{\mathscr{I}}_{P,p}\big) = E_{L(^*p)}\big(\mathbf{1}_F | \tilde{\mathscr{I}}_P\big) = Q(\cdot, F) \quad L(^*p)\text{-a.s.}$$

for any set $F \in \overline{\mathscr{F}}$ and any probability $p \in \mathscr{C}_0$.

The only thing left to show now is

$$\mathscr{J} = \tilde{\mathscr{I}}_{P,p} \bmod L(*p),$$

to get the last equation of the statement.

Part 1. '$\subseteq$'. For any $G \in \mathscr{J}$, we shall show

$$G \in \tilde{\mathscr{I}}_{P,p} \mid \mathscr{N}.$$

We know that $G \in \tilde{\mathscr{F}} \vee \mathscr{N}$. Hence there exists a set $G' \in \tilde{\mathscr{F}}$ satisfying $L({}^*q)(G \Delta G') = 0$ for any $q \in \mathscr{C}_0$. By Lemma 8 this yields the existence of a standard set $F \in \mathscr{F}$ with $L({}^*q)(G \Delta {}^*F) = 0$. For $i \leq k$, it is now easy to see that $G \in \tilde{\mathscr{I}}_{P,p} \vee \mathscr{N}$ and ${}^*F \in \tilde{\mathscr{I}}_{i,p} \vee \mathscr{N}$. Using part (b) of Lemma 11, we conclude that ${}^*F$ is $L({}^*P_i)$-invariant for $L(*p)$. Using part (a) of the same lemma, we see that F is P_i-invariant for p. Because $i \leq k$ is arbitrary, we have shown

$$G \in \tilde{\mathscr{I}}_{P,p} \vee \mathscr{N}.$$

Part 2. '$\supseteq$'. On the other hand, let $G \in \tilde{\mathscr{I}}_{P,p} \vee \mathscr{N}$. Then we know, again, that there is a set $G' \in \tilde{\mathscr{F}}$ with $G \Delta G' \in \mathscr{N}$, and by Lemma 8 we know, again, that there is a standard set F with $L(*p)({}^*G \Delta {}^*F) = 0$. Using Lemma 11, we see that the sets G and ${}^*F$ are $L({}^*P)$-invariant for $L(*p)$. Because of Lemma 11, again, we know that the set F is P-invariant for p.

Now define

$$F' := \left\{ \lim_{k \to \infty} \sum_{i=0}^{k-1} P^i(\cdot, F) = 1 \right\}.$$

The ergodic theorem tells us that $p(F \Delta F') = 1$ and $F' \in \mathscr{I}_{P,p}$ for all $q \in \mathscr{C}_0$. If we now recursively define

$$f_0 := \mathbf{1}_{F'} \quad \text{and} \quad f_{i+1} := \int f_i(\omega) P_{i+1}(\cdot, d\omega) \text{ for } 0 \leq i < k,$$

we know by the definition of P that $f_k = P(\cdot, F') = \mathbf{1}_{F'}$ q-a.s. The assumptions on the kernels P_i yield

$$f_{i+1} = E_q\left(f_i | \mathscr{I}_{i+1,q}\right)$$

for any $i \leq k$. Because conditional expectations are orthogonal projections in $\mathbf{L}^2(q)$, we know

$$\|\mathbf{1}_F\|^2 = \|f_0\|^2 = \left\| f_k + \sum_{i=0}^{k-1} (f_i - f_{i+1}) \right\|^2 = \|f_k\|^2 + \sum_{i=0}^{k-1} \|f_i - f_{i+1}\|^2$$
$$= \|\mathbf{1}_F\|^2 + \sum_{i=0}^{k-1} \|f_i - f_{i+1}\|^2,$$

where $\| \cdot \|$ denotes the norm in $\mathbf{L}^2(q)$. Hence

$$f_i = f_0 = \mathbf{1}_F \quad q\text{-a.s.}$$

for any $i \leq k$. This shows that F is P_i-invariant with respect to q, or

$$^*F \in \bigcap_{i=1}^{k} \left(\bigcap_{q \in \mathscr{C}_0} \left(\tilde{\mathscr{I}}_{i,q} \vee \mathscr{N} \right) \right).$$

Since $L(^*p)(G \Delta ^*F) = 0$ we can conclude that

$$G \in \bigcap_{i=1}^{k} \left(\bigcap_{q \in \mathscr{C}_0} \left(\tilde{\mathscr{I}}_{i,q} \vee \mathscr{N} \right) \right).$$

Thus we have proved

$$\tilde{\mathscr{I}}_{P,p} \subset \mathscr{J} \mod L(^*p). \qquad \square$$

4.4. Another proposition.

For technical reasons we prove the statement of our theorem under some stronger assumptions.

PROPOSITION 16. *Let $\mathscr{P}$ be a set of transition probabilities and let $\mathscr{C}_1$ be any set of $\mathscr{P}$-invariant probabilities satisfying*

$$E_q\left(\mathbf{1}_F | \mathscr{I}_{P,q}\right) = P(\cdot, F) \quad q\text{-a.s.}$$

for any $F \in \mathscr{F}$, any $P \in \mathscr{P}$ and any $q \in \mathscr{C}_1$. Then for any $p \in \mathscr{C}_1$, there is at most one representing measure for p on the set $ex\mathscr{C}(\mathscr{P}) \cap \mathscr{C}_1$ of extremal, $\mathscr{P}$-invariant measures in $\mathscr{C}_1$ with the σ-algebra generated by the evaluations.

Proof. Suppose $\mathscr{P} \neq \varnothing$; otherwise the statement is trivial.

We now define the ideal $\mathscr{N}$ by

$$\mathscr{N} := \left\{ N \in L({}^*F) | L({}^*q)(N) = 0 \text{ for any } q \in \mathscr{C}_1 \right\}$$

and, as usual, $\overline{\mathscr{F}} := \tilde{\mathscr{F}} \vee \mathscr{N}$. Furthermore, we define, as already suggested, the directed set I by

$$I := \{ i \subseteq \mathscr{P} | i \text{ is finite}, i \neq \varnothing \}$$

and the decreasing system of sub-σ-algebras $(\mathscr{F}_i)_{i \in I}$ of $\overline{\mathscr{F}}$ by

$$\mathscr{F}_i := \bigcap_{P \in i,\, q \in \mathscr{C}_1} \left(\tilde{\mathscr{I}}_{P,q} \vee \mathscr{N} \right).$$

Let us now construct a family $\Pi = (\Pi_i)_{i \in I}$ of transition probabilities on $({}^*\Omega, \overline{\mathscr{F}})$ which represents the conditional expectations under $\mathscr{F}_i$. If the set i has only one element P, we use Lemma 14 to construct Π_i, and if i has more than one element, we use Lemma 15 for the construction. If q is an i-invariant probability (i is a set of transition probabilities), we know by the lemmas that $L({}^*q)_{|\overline{\mathscr{F}}}$ is Π_i-invariant and that Π_i satisfies

$$E_{L(*p)}(\mathbf{1}_F | \mathscr{F}_i) = E_{L(*p)}\left(\mathbf{1}_F \Big| \bigcap_{P \in i,\, q \in \mathscr{C}_1} \left(\tilde{\mathscr{I}}_{P,q} \vee \mathscr{N} \right) \right) = \Pi_i(\cdot, F) \quad L(*p)\text{-a.s.}$$

Let the set $\overline{\mathscr{P}}$ be defined as in Lemma 13. By the definition of $\Pi_{\{P\}}$, $P \in \mathscr{P}$, and by Lemma 14 we know

$$\Pi_{\{P\}} \cdot L({}^*P)_{|{}^*\Omega \times \overline{\mathscr{F}}} = \Pi_{\{P\}}.$$

So it is easy to see that any Π-invariant probability is $\overline{P}$-invariant, too. Now we consider the mapping

$$\vartheta : ex\mathscr{C}(\mathscr{P}) \cap \mathscr{C}_1 \to ex\mathscr{C}(\overline{\mathscr{P}})$$

defined in Lemma 13. We have seen above that its image is a subset of $\mathscr{C}(\Pi)$. Thus we will see that our problem can be transferred from the set of $\mathscr{P}$-invariant probabilities to the set of Π-invariant probabilities. Let $p \in \mathscr{C}_1$ be given and let μ_1 and μ_2 be two representing measures for p on $ex\mathscr{C}(\mathscr{P}) \cap \mathscr{C}_1$. Then $\mu_1 \circ \vartheta^{-1}$ and $\mu_2 \circ \vartheta^{-1}$ are two measures on

$$\mathscr{E} := \vartheta\left(ex\mathscr{C}(\mathscr{P}) \cap \mathscr{C}_1\right) \subseteq ex\mathscr{C}(\Pi).$$

(There had to be a reason for introducing the set $\mathscr{E}$ in Proposition 6.) Since

$$\int_{\mathscr{E}} \rho(^*F) d(\mu_j \circ \vartheta^{-1})(\rho) = \int L(^*q)(^*F) d\mu_j(q)$$
$$= \int q(F) d\mu_j(q) = p(F) = L(^*p)(^*F)$$

for any $F \in \mathscr{F}$, and

$$\int_{\mathscr{E}} \rho(N) d(\mu_i \circ \vartheta^{-1}) = \int L(^*q)(N) d\mu_j(q) = 0 = L(^*p)(N)$$

for any $N \in \mathscr{N}$, we know that $\mu_1 \circ \vartheta^{-1}$ and $\mu_2 \circ \vartheta^{-1}$ are two representing measures for $L(^*p)_{|\overline{\mathscr{F}}}$.

Now it is time to apply Proposition 6. As in this proposition, we define the mappings $\gamma_i\colon {}^*\Omega \to \mathscr{M}_1({}^*\Omega, \overline{\mathscr{F}})$, $i \in I$, by

$$\gamma_i(\omega) := \Pi_i(\omega, \cdot),$$

and let ι be the embedding of $\mathscr{E}$ into $\mathscr{M}_1({}^*\Omega, \overline{\mathscr{F}})$. We can now apply the result of this proposition to our invariant probability $L(^*p)_{|\overline{\mathscr{F}}}$, and conclude that the net

$$\left(L(^*p)_{|\overline{\mathscr{F}}} \circ \gamma_i^{-1}\right)_{i \in I}$$

converges in the weak topology to both $\mu_1 \circ \vartheta^{-1} \circ \iota^{-1}$ and $\mu_2 \circ \vartheta^{-1} \circ \iota^{-1}$. But ι is injective and bimeasurable. Hence

$$\mu_1 \circ \vartheta^{-1} = \mu_2 \circ \vartheta^{-1}.$$

The same argument used for the mapping ϑ yields

$$\mu_1 = \mu_2. \qquad \square$$

4.5. The proof of the theorem.

Now we are ready to prove our theorem. Let $\mathscr{C}_0$ be the whole set of $\mathscr{P}$-invariant probabilities. The ideal $\mathscr{N}$, the σ-algebra $\overline{\mathscr{F}}$, the set $\overline{\mathscr{P}}$ of transition probabilities on $({}^*\Omega, \overline{\mathscr{F}})$, and the mapping ϑ are defined as in Lemma 13. Further, let $\mathscr{C}_1$ be defined by

$$\mathscr{C}_1 := \{L(^*q)_{|\overline{\mathscr{F}}} | q \text{ is } \mathscr{P}\text{-invariant}\}.$$

We already know from Lemma 13 that $\mathscr{C}_1$ is a set of $\overline{\mathscr{P}}$-invariant probabilities.

In the following, we will show why we introduced the set $\mathscr{C}_1$ in Lemma 15. For any $P \in \mathscr{P}$, by Lemma 14 we can find a transition probability Q_P on $({}^*\Omega, \overline{\mathscr{F}})$ such that

$$Q_P \cdot L({}^*P)_{|{}^*\Omega \times \overline{\mathscr{F}}} = Q_P$$

and

$$E_{L({}^*q)}\left(\mathbf{1}_F | \mathscr{I}_{Q_P, L({}^*q)_{|\overline{\mathscr{F}}}}\right) = Q_P(\cdot, F) \quad L({}^*q)\text{-a.s.}$$

for any $F \in \overline{\mathscr{F}}$, and $L({}^*q)_{|\overline{\mathscr{F}}}$ is Q_P-invariant for any $\mathscr{P}$-invariant probability q.

Further, we know that for any extremal $\mathscr{P}$-invariant probability q, $L({}^*q)_{|\overline{\mathscr{F}}}$ is extremal in the set of $(Q_P)_{P \in \mathscr{P}}$-invariant probabilities. By Lemma 15 ($\mathscr{C}_0$ here is the set of all $\mathscr{P}$-invariant probabilities) $L({}^*q)_{|\overline{\mathscr{F}}}$ is extremal in the set of $\overline{\mathscr{P}}$-invariant probabilities. If ρ_1 and ρ_2 are any two $(Q_P)_{P \in \mathscr{P}}$-invariant probabilities satisfying $L({}^*q)_{|\overline{\mathscr{F}}} = \frac{1}{2} \cdot (\rho_1 + \rho_2)$, we know by Lemma 13 that both ρ_1 and ρ_2 are $\overline{\mathscr{P}}$-invariant; thus they are equal.

We have now proved that ϑ maps the extremal $\mathscr{P}$-invariant probabilities into $ex\mathscr{C}((Q_P)_{P \in \mathscr{P}}) \cap \mathscr{C}_1$. If μ_1 and μ_2 are two representing measures for p on the set of extremal $\mathscr{P}$-invariant probabilities, we easily see that $\mu_1 \circ \vartheta^{-1}$ and $\mu_2 \circ \vartheta^{-1}$ are two representing measures on $ex\mathscr{C}((Q_P)_{P \in \mathscr{P}}) \cap \mathscr{C}_1$. But Proposition 16 tells us that

$$\mu_1 \circ \vartheta^{-1} = \mu_2 \circ \vartheta^{-1}.$$

By Lemma 13 the mapping ϑ is injective and bimeasurable. Hence we conclude

$$\mu_1 = \mu_2. \qquad \square$$

4.6. Proof of the corollary.

The existence of a representing measure is an easy consequence of Bishop-deLeeuw's theorem because the set of finitely additive, $\mathscr{P}$-invariant probabilities with the topology generated by the evaluations is compact.

For the proof of the uniqueness we have to realize the assumptions of our theorem. Let $\tilde{\mathscr{C}}(\mathscr{P})$ denote the set of $\mathscr{P}$-invariant finitely additive probabilities and, again, let

$$\tilde{\mathscr{P}} := \{L({}^*P)_{|{}^*\Omega \times \tilde{\mathscr{F}}} \mid P \in \mathscr{P}\}.$$

Then we can define a mapping κ: $\tilde{\mathscr{C}}(\mathscr{P}) \to \mathscr{C}(\tilde{\mathscr{P}})$ by

$$q \mapsto L({}^*q)_{|\tilde{\mathscr{F}}}.$$

By Lemma 10, part (e), κ is well defined. It is easy to see that κ is affine, injective, and bimeasurable. Thus we know that for any extremal $q \in \tilde{\mathscr{C}}(\mathscr{P})$, the probability $\kappa(q)$ is extremal in $\mathscr{C}(\tilde{\mathscr{P}})$.

Suppose μ_1 and μ_2 are two representing measures for p on $ex\tilde{\mathscr{C}}(\mathscr{P})$. Since

$$\int q(^*F)d(\mu_i \circ \kappa^{-1})(q) = \int \kappa(\rho)(^*F)d\mu_i(\rho) = \int \rho(F)d\mu(\rho)$$
$$= p(F) = L(*p)(F) = \kappa(p)(F)$$

for any set $F \in \mathscr{F}$, the measures $\mu_1 \circ \kappa^{-1}_{|ex\tilde{\mathscr{C}}(\mathscr{P})}$ and $\mu_2 \circ \kappa^{-1}_{|ex\tilde{\mathscr{C}}(\mathscr{P})}$ are representing measures for $\kappa(p)$ on the set $ex\mathscr{C}(\tilde{\mathscr{P}})$. Our theorem tells us that

$$\mu_1 \circ \kappa^{-1}_{|ex\tilde{\mathscr{C}}(\mathscr{P})} = \mu_2 \circ \kappa^{-1}_{|ex\tilde{\mathscr{C}}(\mathscr{P})}.$$

But by the properties of κ this yields $\mu_1 = \mu_2$. □

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