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BANACH LATTICES WITH PROPERTY (H) AND WEAK HILBERT SPACES

BY

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Introduction

The notions of weak type 2 and weak cotype 2 were introduced and studied by Milman and Pisier [10]. In [12] and [13] Pisier defined a weak Hilbert space to be a Banach space, which is both of weak type 2 and weak cotype 2 and developed an extensive theory of these spaces and weak properties in general. In [12] he defined the so-called property (H) for Banach spaces (which roughly says that for every normalized unconditional basic sequence (x_j) and for every integer $n, \|\sum_{j=1}^{n} x_j\|$ behaves like $\sqrt{n}$) and proved that weak Hilbert spaces have this property; it was left as an open problem whether property (H) is actually equivalent to the space in question being a weak Hilbert space.

One of the major problems of the theory is the scarcity of known examples; basically the only known weak Hilbert spaces are variations of the Tsirelson construction (see e.g., [2]) and this raises the question whether every weak Hilbert space has a basis.

In this paper we study the structure of unconditional sequences in Banach spaces with property (H) and we give strong estimates of the tail behaviour of such sequences. The estimates have the same order of magnitude as those obtained for the unit vector basis of the 2-convexified Tsirelson space and its dual. We then use these results to show that a Banach lattice has property (H) if and only if it is a weak Hilbert space, thus solving the above question of Pisier in the affirmative for Banach lattices. We also combine our estimates with the results of W.B. Johnson [4] to investigate the structure of subspaces of quotients of a Banach lattice with property (H). We show that every such space has a basis and give estimates for the uniformity function of the uniform approximation property. Again these estimates have the same order of magnitude as in the Tsirelson case.

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We now wish to discuss the arrangement and contents of this paper in greater detail.

In Section 1 we study property (H) for unconditional sums of Banach spaces. In particular we discuss a quantitative finite dimensional version of an interesting method, due to W.B. Johnson and presented in [12] and [13], which often can be used in the study of weak Hilbert spaces and other weak and asymptotic properties. Some related applications can be found in [5]. The main result of the section states that if we have a "long" unconditional direct sum of subspaces of equal finite dimension in a Banach space with property (H), then at least one of the subspaces is close to a Hilbert space. This result will be used heavily to obtain our main results.

Section 2 is devoted to the construction of subsymmetric and unconditional direct sums from a direct sum of copies of a given Banach space. Constructions of this kind are typically done using Ramsey's theorem, following the Brunel-Sucheston approach. However, while standard arguments are mostly concerned with infinite sequences of vectors, we require here quantitative results on finite direct sums of copies of a fixed finite dimensional space. Our proofs are closely modelled on those of [11], Section 11. However, to keep track of all the integer functions involved, we present short arguments which refer directly to Ramsey's theorem.

Section 3 contains the main results of the paper, namely the investigation of the structure of unconditional basis sequences in Banach spaces with property (H), as described above. In section 4 we prove the characterization of Banach lattices with property (H). We also combine the results of Section 3 with [4] to obtain the results on bases and the uniform approximation property of subspaces of quotients of Banach lattices with property (H).

Section 5 contains some additional properties of spaces with property (H) and some open problems.

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0. Notation and terminology

In this paper we shall use the notation and terminology commonly used in Banach space theory as it appears in [8], [9], [11], [13] and [15].

Let $G = \{-1, 1\}^{\mathbf{N}}$, let $\mathbf{m}$ denote the normalized Haar measure on G and define the sequence (r_n) of Rademacher functions on G by $r_n(\varepsilon) = \varepsilon(n)$ for all $\varepsilon \in G$ and all $n \in \mathbf{N}$.

Let X be a Banach space. By $\mathrm{Rad}(X)$ we denote the closed linear span of $\{r_i \otimes x_i | i \in \mathbf{N},\ x_i \in X\}$ in $L_2(\mathbf{m}, X)$, while if $n \in \mathbf{N}$, then $\mathrm{Rad}_n(X)$ denotes the closed linear span of $\{r_i \otimes x_i | 1 \leq i \leq n,\ x_i \in X\}$ in $L_2(\mathbf{m}, X)$.

If E and F are isomorphic Banach spaces, we let $d(E, F)$ denote the Banach–Mazur distance between E and F and if F is a Hilbert space, we put $d(E) = d(E, F)$.

One of the fundamental notions of this paper is that of a weak Hilbert space. For our purpose the following definition will be convenient.

0.1 DEFINITION. *A Banach space X is said to be a weak Hilbert space if there exist $\delta > 0$ and $C \geq 1$ such that for every finite-dimensional subspace E of X there exist a subspace $F \subset E$ and a projection P: $X \to F$ such that* $\dim F \geq \delta \dim E$, $d(F) \leq C$ *and* $\|P\| \leq C$.

The definition is not the original one but is chosen out of many equivalent characterizations proved by Pisier (cf. [12]).

Another notion which is basic for our investigations is the property (H), also introduced in [12].

0.2 DEFINITION. *A Banach space X is said to have property (H) if for every $\lambda \geq 1$ there is $c(\lambda)$ so that for every $n \in \mathbf{N}$, whenever $\{u_1, u_2, \ldots, u_n\} \subseteq X$ is a λ-unconditional normalized basic sequence, then*

$$c(\lambda)^{-1}\sqrt{n} \leq \left\| \sum_{j=1}^{n} u_j \right\| \leq c(\lambda)\sqrt{n}. \tag{1}$$

The smallest $c(\lambda)$ which satisfies (1) is denoted by $\kappa_X(\lambda)$, or by $\kappa(\lambda)$ if no ambiguity can occur. The constant $\kappa_X(1)$ is called the property (H) constant of X and it is also denoted by $\kappa(X)$.

It was proved in [12] that weak Hilbert spaces have property (H). In general it is not known whether these two notions are equivalent, but as mentioned in the introduction, one of the main results of this paper is that they are indeed equivalent for Banach lattices.

A basis (x_j) in an n-dimensional Banach space X is said to be C-equivalent to the unit vector basis in l_2^n if for any sequence $(t_j)_{j=1}^n$ of scalars one has

$$C^{-1}\left(\sum_{j=1}^{n} |t_j|^2\right)^{1/2} \leq \left\| \sum_{j=1}^{n} t_j x_j \right\| \leq C\left(\sum_{j=1}^{n} |t_j|^2\right)^{1/2}.$$

If A is a subset of a Banach space X, then $[A]$ will denote the closed linear span of A.

Throughout the paper, the function log denotes the logarithm with base 2, and $\exp_a x = a^x$, for $a > 1$ (in most cases, $a = 2$). If f: $\mathbf{R} \to \mathbf{R}$ and $m \in \mathbf{N}$ then $f^{(m)}$ denotes the m-th iteration of the function f. Finally, if A is a set, then $|A|$ stands for the cardinality of A, and if $t \in \mathbf{R}$, $[t]$ denotes the integer part of t.

1. Property (H) for unconditional sums of Banach spaces

As mentioned in the introduction, in this and the next section we shall discuss a quantitative finite dimensional version of a method due to W.B. Johnson, but recorded (and, according to Johnson, amplified) by G. Pisier in [12] and [13]. This approach has many applications to problems concerning property (H) and weak Hilbert spaces. It is based on an investigation of Schauder decompositions and unconditional direct sums of Banach spaces.

In this section we consider a Banach space which is a λ-unconditional direct sum of given subspaces E_i. This means that every vector x is of the form $x = \sum_i x_i$ with $x_i \in E_i$ $(i = 1, 2, \dots)$ and that $\|\sum_i \pm x_i\| \leq \lambda \|x\|$. The main result which is a finite-dimensional version of [12] Theorem 4.4 and [13] Theorem 14.3 states.

1.1 THEOREM. *Let X be a Banach space with property (H), which is a λ-unconditional direct sum of subspaces E_i, $1 \leq i \leq k$ such that* $\dim(E_i) =$ *$m < \infty$ for all $1 \leq i \leq k$. If $m^{m+(1+16m)^m} \leq k$, then there is a j_0, $1 \leq j_0 \leq k$, and a K so that $d(E_{i_0}) \leq K$. Moreover, one can take*

$$K = C\lambda^8\kappa(\lambda)^8\kappa\left(C\lambda^2\kappa(\lambda)^2\right)^4,$$

where $C \geq 1$ is a universal constant.

The proof follows the line of the original argument although it requires several modifications which we indicate below.

The next proposition is a finite dimensional version of [12] Proposition 4.3.

1.2 PROPOSITION. *Let X be an n-dimensional Banach space. There is a universal constant $C \geq 1$ such that $d(X) \leq C\kappa(\mathrm{Rad}_n(X))^4$.*

Proof. To simplify the notation set $\kappa = \kappa(\mathrm{Rad}_n(X))$. From the definition of property (H) we have

$$(1) \qquad (\kappa)^{-1}\sqrt{m} \leq \left(\int_G \left\| \sum_{j=1}^m r_j(t)x_j \right\|^2 d\mathbf{m}(t)\right)^{1/2} \leq \kappa\sqrt{m}$$

for all $\{x_1, x_2, \dots, x_m\} \subseteq X$, with $\|x_j\| = 1$ for $1 \leq j \leq m$, and for all $1 \leq m \leq n$.

The right hand side inequality, combined with the results of Tomczak [14] and Bourgain, Kalton and Tzafriri [1] Theorem 3.1, shows that there is a universal constant c_1 so that $T_2(X) \le c_1\kappa$, where $T_2(X)$ is the type 2 constant of X.

From König and Tzafriri [7], if $q = 2 + 256(c_1\kappa)^4$ then $C_q(X) \le 2$ where $C_q(X)$ is the cotype q constant of X. Applying [14] and [1] again and the left hand side of (1), together with known quantitative inequalities between Gaussian and Rademacher averages (for example, see [11] Appendix 2 or [15] Theorem 25.1), we have the existence of universal constants c_2 and c_2' so that $C_2(X) \le c_2\sqrt{q}\,\kappa \le c_2'\kappa^3$. Using a classical result of Kwapien (for example, see [15] Theorem 13.15) we get the required estimate

$$d(X) \le T_2(X)C_2(X) \le C\kappa^4 = \tilde{K}. \qquad \blacksquare$$

The following lemma is familiar to specialists in the field. Its proof is a straightforward modification of the known argument for compactness of the Minkowski compactum of all n-dimensional Banach spaces (for example, see [15] p. 278) and will be omitted.

1.3 LEMMA. *Let $m \in \mathbf{N}$ and $N(m) = m^{(1+16m)^m}$. There exists a family $\mathcal{F} = \{E_j \mid 1 \le j \le N(m)\}$ of m-dimensional Banach spaces so that if E is an arbitrary m-dimensional Banach space, then there is a j_0, $1 \le j_0 \le N(m)$ with $d(E, E_{j_0}) \le 4$.*

Proof of Theorem 1.1. Let $N = N(m)$ be defined as in Lemma 1.3. This lemma then implies that there is a subset $I \subseteq \{1, 2, \dots, k\}$ of cardinality at least kN^{-1} so that $d(E_i, E_j) \le 16$ for all $i, j \in I$.

Consider the subspace $Z = \sum_{i\in I} \oplus E_i \subseteq X$, which clearly forms a λ-unconditional direct sum; moreover, $|I| \ge 2^m$. We will show that if $i_0 \in I$ then $\mathrm{Rad}_m(E_{i_0})$ is $C\lambda^2\kappa(\lambda)^2$-isomorphic to a subspace of Z. So it has property (H) with the constant $\kappa(\mathrm{Rad}_m(E_{i_0})) \le C\lambda^2\kappa(\lambda)^2\kappa_Z(C\lambda^2\kappa(\lambda)^2)$. Since $\kappa(Z) \le \kappa(X)$, an application of Proposition 1.2 will conclude the proof of the theorem with $K = (C\lambda^2\kappa(\lambda)^2\kappa(C\lambda^2\kappa(\lambda)^2))^4$.

Assume for simplicity that $I = \{1, \dots, |I|\}$ and let us describe some details of the construction of an isomorphism T of $\mathrm{Rad}_n(E_1)$ into Z with $\|T\|\,\|T^{-1}\| \le C\lambda^2\kappa(\lambda)^2$, where C is a universal constant.

Since Z has property (H) with the constant $\kappa(\lambda)$, it follows from [12] formula (4.3) that

$$(1) \qquad \lambda^{-1}\kappa(\lambda)^{-1}\left\|(t_i)_{i=1}^m\right\|_{2,\infty} \le \left\|\sum_{i=1}^m t_i x_i\right\| \le \lambda\kappa(\lambda)\left\|(t_i)_{i=2}^m\right\|_{2,1}$$

for all $m \in \mathbf{N}$, all λ-unconditional normalized sequences $\{x_1, x_2, \dots, x_m\} \subseteq X$

and all $(t_i)_{i=1}^m \in \mathbf{R}^m$. (Here $\|\cdot\|_{2,\infty}$ and $\|\cdot\|_{2,1}$ denote the norms in the Lorentz sequence spaces $l_{2,\infty}$ and $l_{2,1}$ respectively.)

Let now $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{2^m}$ be an enumeration of the elements of $\{-1,1\}^m$ and choose isomorphisms T_i of E_1 onto E_i with $\|T_i\| \le 16$, $\|T_i^{-1}\| = 1$ for all $1 \le i \le |I|$. Define T: $\mathrm{Rad}_m(E_1) \to Z$ by

$$T\left(\sum_{j=1}^m r_j \otimes x_j\right) = 2^{-m/2}\sum_{i=1}^{2^m}\left(\sum_{j=1}^m r_j(\varepsilon_i)T_i x_j\right) \tag{2}$$

for all $\sum_{j=1}^m r_j \otimes x_j \in \mathrm{Rad}_m(E_1)$.

Formulas (1) and (2) give

$$\begin{aligned}(3)\qquad (\lambda\kappa(\lambda))^{-1}&\left\|\sum_{j=1}^m r_j \otimes x_j\right\|_{L_{2,\infty}(m,E_1)}\\ &= (\lambda\kappa(\lambda))^{-1}2^{-m/2}\left\|\left(\left\|\sum_{j=1}^m r_j(\varepsilon_i)x_j\right\|\right)_{i=1}^{2^m}\right\|_{l_{2,\infty}}\\ &\le \left\|T\left(\sum_{j=1}^m r_j \otimes x_j\right)\right\|\\ &\le \lambda\kappa(\lambda)2^{-m/2}\left\|\left(\left\|\sum_{j=1}^m r_j(\varepsilon_i)T_i x_j\right\|\right)_{i=1}^{2^m}\right\|_{l_{2,1}}\\ &\le 16\lambda\kappa(\lambda)\left\|\sum_{j=1}^m r_j \otimes x_j\right\|_{L_{2,1}(\mathrm{m},E_1)}.\end{aligned}$$

By the classical result of Kahane (for example, see [11] Appendix 3, [15] (4.7)), there is a universal constant C' so that the $L_{2,1}$- and $L_{2,\infty}$-norms are C'-equivalent on $\mathrm{Rad}(E_1)$. Together with (3) this gives that

$$\|T\|\,\|T^{-1}\| \le C\lambda^2\kappa(\lambda)^2, \tag{4}$$

with $C = 16C'$. This completes the proof of the theorem. ∎

2. Subsymmetric and unconditional direct sums of Banach spaces

In this section we shall discuss how to construct subsymmetric and unconditional direct sums from a given "long" direct sum of copies of a given Banach space E. We shall only consider finite direct sums; the case of infinite sums was treated in Pisier [12], Section 3.

Let $(E, \|\cdot\|_E)$ be a Banch space and let $N \in \mathbf{N}$. Let E^N denote the product on N copies of E. Let $\|\cdot\|$ be a norm on E^N for which there are numbers $a > 0$ and $b > 0$, so that for all $x = (x(1), x(2), \ldots, x(N)) \in E^N$ we have

$$\text{(a)} \qquad a \sup_{n \le N} \|x(n)\|_E \le \|x\| \le b \sum_{n=1}^{N} \|x(n)\|_E.$$

Then $(E^N, \|\cdot\|)$ is called a *direct sum of N copies of E.*

If $n \le N$ we shall identify E^n with the subspace of E^N consisting of those $x \in E^N$ for which $x(j) = 0$ for $n < j \le N$.

If $n \le N$ and $x \in E^n$ and $(i_j)_{j=1}^n \subseteq \{1, 2, \ldots, N\}$ with $i_1 < i_2 < \cdots < i_n$, then the element $x(i_1, i_2, \ldots, i_n) \in E^N$ is defined by

$$x(i_1, i_2, \ldots, i_n)(k) = \begin{cases} x(j) & \text{for } k = i_j,\ 1 \le j \le n, \\ 0 & \text{otherwise.} \end{cases}$$

If $\lambda \ge 1$ then $(E^N, \|\cdot\|)$ is called a *λ-subsymmetric direct sum*, if for all $n \le N$, all $x \in E^n$ and all $(i_j)_{j=1}^n \subseteq \{1, 2, \ldots, N\}$, $i_1 < i_2 < \cdots < i_n$, we have

$$\text{(b)} \qquad \|x(i_1, i_2, \ldots, i_n)\| \le \lambda \|x\|.$$

$(E^N, \|\cdot\|)$ is called *λ-unconditional* provided that for every $\varepsilon \in \{-1, 1\}^N$ and every $x \in E^N$,

$$\text{(c)} \qquad \|(\varepsilon(j) x(j))\| \le \lambda \|x\|.$$

An infinite direct sum of copies of E is defined in a similar manner; indeed, consider a norm on the space $E^{(\mathbf{N})}$ of all finitely supported sequences satisfying (a) and define the infinite direct sum to be the completion of $E^{(\mathbf{N})}$ in that norm. Subsymmetricity and unconditionality are defined by (b), respectively (c).

If A is a set and $k \in \mathbf{N}$ we shall let $A^{[k]}$ denote the set of all subsets of A with k elements.

We shall use the finite version of Ramsey's theorem as follows (for example, see [3] Section 11.2, [11] Theorem 11.2).

2.1 THEOREM. *For all $k, n, m \in \mathbf{N}$ there is an $N \in \mathbf{N}$, $N = R(k, n, m)$ so that if*

$$f: \{1, 2, \ldots, N\}^{[k]} \to \{1, 2, \ldots, m\}$$

then there is an $M \subseteq \{1, 2, \ldots, N\}$ with $|M| = n$ such that $f(M^{[k]})$ is a singleton.

Estimates of the function R can be found in [3]. For convenience of the notation let $\exp_2(x) = 2^x$. One has [3, pp. 90–91]

$$R(k,l,2) \le \exp_2^{(k)}((k-1)!l).$$

Moreover, a simple combinatorial argument yields

$$R(k,l,m) \le R^{(m)}(k,l,2).$$

Thus,

$$R(k,l,m) \le \exp_2^{(km)}((k-1)!l). \tag{d}$$

2.2 THEOREM. *For all d and $n \in \mathbf{N}$ there is $N = N(d,n) \in \mathbf{N}$ with the following property. Whenever E is a d-dimensional Banach space and $(E^N, \|\cdot\|)$ is a direct sum, then there is a subset $A \subseteq \{1,2,\dots,N\}$ with $|A| = n$, so that $(E^A, \|\cdot\|)$ is a 3-subsymmetric direct sum. The function N can be taken as $N = \exp_2^{(s)}(n)$, where $s = (3n)^{dn}$.*

Proof. For simplicity we shall assume that $a = b = 1$ in (a). By induction we shall construct subsets $\{1,2,\dots,N\} = A_1 \supseteq A_2, \dots, A_m \supseteq \cdots$ so that for every $x \in E^{A_m}$ and $1 \le r \le m$, and for any $(i_k)_{k=1}^r \subseteq A_m$, $(j_k)_{k=1}^r \subseteq A_m$ with $i_1 < i_2 < \cdots < i_r$, $j_1 < j_2 < \cdots < j_r$ we have

$$\|x(i_1,i_2,\dots,i_r)\| \le 3\|x(j_1,j_2,\dots,j_r)\|. \tag{1}$$

We shall continue the process as long as

$$|A_m| \ge m. \tag{2}$$

If we can ensure, by a suitable choice of N, that the integer n satisfies (2), then any subset A of A_n with $|A| = n$ will satisfy the requirements of the theorem.

Assume that $A_1, A_2, \dots, A_{m-1}$ have been constructed so that (1) and (2) hold. To construct A_m we proceed as follows.

Let $x \in E^{A_{m-1}}$, $x \neq 0$ be a fixed element supported on m coordinates of $E^{A_{m-1}}$. Divide the interval $[\|x\|_\infty, \|x\|_1]$ into at most m disjoint intervals of length at most $\|x\|_\infty$. (Here $\|x\|_\infty = \max_{j \in A_{m-1}} \|x(j)\|$, $\|x\|_1 = \sum_{j \in A_{m-1}} \|x(j)\|$.) Define $f\colon A_{m-1}^{[m]} \to \{1,2,\dots,m\}$ to be the function which to each (i_k), $i_1 < i_2 < \cdots < i_m$ assigns the number of the interval to which $\|x(i_1,i_2,\dots,i_m)\|$ belongs. By Ramsey's theorem there is a subset $M_1 \subseteq A_{m-1}$ such that all $\|x(i_1,i_2,\dots,i_m)\|$ belong to the same interval for $(i_k) \subseteq M_1$.

Hence if $(i_k) \subseteq M_1, (j_k) \subseteq M_1$, $i_1 < \cdots < i_m$, $j_1 < \cdots < j_m$, we obtain

$$(3) \qquad \begin{aligned} \|x(i_1, i_2, \dots, i_m)\| &\le \|x\|_\infty + \|x(j_1, j_2, \dots, j_m)\| \\ &\le 2\|x(j_1, j_2, \dots, j_m)\|. \end{aligned}$$

For this to be possible we need, by Ramsey, that

$$(4) \qquad |A_{m-1}| \ge R(m, |M_1|, m).$$

Let $\mathcal{N}$ be an $(1/m)$-net of the unit sphere of E of cardinality at most $r_m = (1 + 2m)^d$. (The existence of such a net is well known, e.g., see [11], Lemma 2.6.) We now repeat the above construction for every x of the form considered, the coordinates of which are in $\mathcal{N}$. That is, we have to repeat it $\tilde{r} = r_m^m$ times to get subset $A_{m-1} \supseteq M_1 \supseteq \cdots \supseteq M_{\tilde{r}} = A_m$. An easy approximation argument shows that A_m has the property

$$(5) \qquad \begin{aligned} &\forall (i_k), (j_k) \subseteq A_m, i_1 < \cdots < i_m, j_1 < \cdots < j_m \\ &\forall x \in E^{A_{m-1}}: \|x(i_1, i_2, \dots, i_m\| \le 3\|x(j_1, j_2, \dots, j_m\|. \end{aligned}$$

Moreover, by (4),

$$(6) \qquad |A_{m-1}| \ge R^{(r_m^m)}(m, |A_m|, m).$$

To complete the proof we have to estimate the number N. By (6) and (d) it follows that the required condition $|A_n| \ge n$ is ensured by

$$|A_{n-1}| = R^{(r_n^n)}(n, n, n) \le \exp_2^{(r_n^n n^2)}(n!) \le \exp_2^{(r_n^n n^2 + 2)}(n).$$

By iteration we get

$$N \le \exp_2^{(s)}(n)$$

where $s = \sum_{k=2}^{n-1}((1 + 2k)^{dk}k^2 + 2) \le (3n)^{dn}$. ■

Theorem 2.2 can be used to construct unconditional direct sums. First we need some more notation, stemming from [12].

Let E be a Banach space, $n \in \mathbf{N}$ and $(E^N, \|\cdot\|)$ a λ-subsymmetric direct sum, where N is even. If $x \in E$ let $\tilde{x} = (-x, x) \in E^2$, and for $1 \le n \le N/2$ define $S_n(x) \in E^N$ by

$$S_n(x) = \tilde{x}(2n - 1, 2n) = (0, 0, \dots, -x, x, 0, 0, \dots),$$

where the non-zero coordinates are on the $(2n-1)$-th and $2n$-th place.

Denote $S_1(E)$ by $\tilde{E}$. We can now prove:

2.3 PROPOSITION. *Let $n \in \mathbf{N}$, let $N = n(n+3)$ and let $(E^N, \|\cdot\|)$ be a λ-subsymmetric sum. Then $\{\sum_{k=1}^n S_k(x_k) | 1 \le k \le n,\ x_k \in E\}$ is a $(\lambda^2 + 2\lambda a^{-1}b)$-unconditional sum of n copies of $\tilde{E}$. Moreover, $d(E, \tilde{E}) \le 2a^{-1}b$.*

Proof. We will show that if $A \subseteq B \subseteq \{1, 2, \ldots, n\}$ and $x_1, x_2, \ldots, x_n \in E$ then

$$\left\| \sum_{i\in A} S_i(x_i) \right\| \le (\lambda^2 + 2\lambda a^{-1}b) \left\| \sum_{i\in B} S_i(x_i) \right\|. \tag{1}$$

To prove (1) define a sequence $\{\sigma_i\}_{i\in B}$ of subsets of $\{1, 2, \ldots, N\}$ such that:

$$i < i' \Rightarrow \max \sigma_i < \min \sigma_{i'}. \tag{2}$$

$$\text{If } i \in A \text{ then } |\sigma_i| = 2, \text{ say } \sigma_i = \{j(i,1), j(i,2)\}. \tag{3}$$

$$\text{If } i \in B \setminus A \text{ then } |\sigma_i| = n+1, \tag{4}$$

$$\text{say } \sigma_i = \{j(i,1), j(i,2), \ldots, j(i,n+1)\}.$$

Note that to find subsets with these properties we need at least $N = n(n+3)$ terms in the direct sum.

By the λ-subsymmetricity of $(E^N, \|\cdot\|)$ we get for every $1 \le k \le n$

$$\left\| \sum_{i\in A} \tilde{x}_i(j(i,1), j(i,2)) + \sum_{i\in B\setminus A} \tilde{x}_i(j(i,k), j(i,k+1)) \right\| \le \lambda \left\| \sum_{i\in B} S_i(x_i) \right\| \tag{5}$$

Averaging (5) over all $1 \le k \le n$ we obtain

$$\left\| \sum_{i\in A} \tilde{x}_i(j(i,1), j(i,2)) + \frac{1}{n} \sum_{i\in B\setminus A} S_1(x_i)(j(i,n), j(i,n+1)) \right\| \le \lambda \left\| \sum_{i\in B} S_i(x_i) \right\|. \tag{6}$$

Therefore

$$\begin{aligned}
(7)\quad \Big\| \sum_{i\in A} S_i x_i \Big\| &\le \lambda \Big\| \sum_{i\in A} \tilde{x}_i(j(i,1), j(i,2)) \Big\| \\
&\le \lambda^2 \Big\| \sum_{i\in B} S_i x_i \Big\| + \frac{\lambda}{n} \Big\| \sum_{i\in B\setminus A} S_1(x_i)(j(i,n), j(i,n+1)) \Big\| \\
&\le \lambda^2 \Big\| \sum_{i\in B} S_i x_i \Big\| + \frac{2\lambda b}{n} \sum_{i\in B\setminus A} \|x_i\| \\
&\le \lambda^2 \Big\| \sum_{i\in B} S_i x_i \Big\| + 2\lambda b a^{-1} \Big\| \sum_{i\in B} S_i x_i \Big\| \\
&= (\lambda^2 + 2\lambda b a^{-1}) \Big\| \sum_{i\in B} S_i x_i \Big\|.
\end{aligned}$$

It is clear that $d(E, \tilde{E}) \le 2a^{-1}b$ and that the considered subspace is a direct sum of n copies of $\tilde{E}$. ■

Combining Theorem 2.2 with Proposition 2.3 we obtain the following corollary which is one of the basic tools used in this paper.

2.4 COROLLARY. *For all d and $n \in \mathbf{N}$ there is $N = N(d, n) \in \mathbf{N}$ with the following property. Whenever E is a d-dimensional Banach space and $(E^N, \|\cdot\|)$ is a direct sum, then there is a subspace of $(E^N, \|\cdot\|)$ which is a $(9 + 3a^{-1}b)$-unconditional direct sum of n copies of a space $\tilde{E}$ such that $d(\tilde{E}, E) \le 2a^{-1}b$. The function N can be chosen to satisfy $N \le \exp_2^{(s)}(n(n+3))$, where $s = (3n)^{dn}$.*

3. Unconditional bases and property (H); main estimates

We are ready now to pass to the main results of this paper concerning quantitative behaviour of unconditional bases and blocks of these in Banach spaces with property (H). It turns out that in this general case the obtained estimates coincide with the known theorems on the unit vector basis in the 2-convexified Tsirelson space [2].

We begin with some observations which are the starting points for all further constructions. The first lemma is well known.

3.1 LEMMA. *Let E and F be m-dimensional Banach spaces with normalized 1-unconditional bases $(x_j)_{j=1}^m$, respectively $(y_j)_{j=1}^m$. If $T\colon E \to F$ is the linear map defined by $Tx_j = y_j$ for all $1 \le j \le m$, then there exists an $x \in E$ such that*

(i) $1 \le \|x\| \le 3$, $\|T\| \le \|Tx\| \le 3\|T\|$;

(ii) *x belongs to the span of at most $[\log m] + 2$ mutually disjoint blocks of the x_j's with constant coefficients.*

Proof. Pick $y \in E$ with $\|y\| = 1$ such that $\|Ty\| = \|T\|$ and write $y = \sum_{j=1}^{m} t_i x_i$. By unconditionality we may assume that $t_i \geq 0$ for all $1 \leq i \leq m$. For $1 \leq j \leq k = [\log m] + 1$ define the level sets

$$E_j = \{i \in \mathbf{N} | 2^{-j} < t_i \leq 2^{-(j-1)}\} \tag{1}$$

$$E_{k+1} = \{1, 2, \ldots, m\} \Big\backslash \bigcup_{j=1}^{k} E_j. \tag{2}$$

It is easy to see that

$$x = \sum_{j=1}^{k} 2^{-(j-1)} \sum_{i \in E_j} x_i + \frac{1}{m} \sum_{i \in E_{k+1}} x_i \tag{3}$$

satisfies the requirements of the lemma. ∎

Repeating the lemma several times we get the following corollary. In its statement we denote by $\psi \colon \mathbf{N} \to \mathbf{N}$ the function defined by $\psi(n) = [\log n] + 2$ for all $n \in \mathbf{N}$.

3.2 Lemma. *Let E and F be n-dimensional Banach spaces with 1-unconditional normalized bases $(x_j)_{j=1}^{n}$, respectively $(j_j)_{j=1}^{n}$ and let $T \colon E \to F$ be the linear map defined by $Tx_j = y_j$ for all $1 \leq j \leq n$. For every $m \in \mathbf{N}$ there exist normalized block basic sequences $\{f_j | 1 \leq j \leq \psi^{(m)}(n)\}$ of (x_j) and $\{z_j | 1 \leq j \leq \psi^{(m)}(n)\}$ of (y_j) so that if $T_m \colon [f_j] \to [z_j]$ is the linear map defined by $T_m f_j = z_j$ for all $1 \leq j \leq \psi^{(m)}(n)$ then*

$$(3\kappa(E)\kappa(F))^{-m} \|T\| \leq \|T_m\| \leq (\kappa(E)\kappa(F))^{m} \|T\| \tag{i}$$

where $\kappa(E)$ and $\kappa(F)$ are the property (H) constants of E and F respectively.

Proof. Let $(E_j)_{j=1}^{\psi(n)}$ be as in the proof of Lemma 3.1. Put $u_j = \sum_{i \in E_j} x_i$, $v_j = \sum_{i \in E_j} y_i$ for all $1 \leq j \leq \psi(n)$. Observe that

$$(\kappa(E)\kappa(F))^{-1} \leq \|u_j\| \|v_j\|^{-1} \leq \kappa(E)\kappa(F) \quad \text{for all } 1 \leq j \leq \psi(n). \tag{1}$$

Let $f_j = u_j / \|u_j\|$, $z_j = v_j / \|v_j\|$ for all $1 \leq j \leq \psi(n)$ (if $E_j = \emptyset$ set $f_j = z_j = 0$). Define the linear map $T_1 \colon [f_j] \to [z_j]$ by $T_1(f_j) = z_j$ for all $1 \leq j \leq \psi(n)$. By (1) we get

$$(\kappa(E)\kappa(F))^{-1} \|Tu\| \leq \|T_1 u\| \leq \kappa(E)\kappa(F) \|Tu\| \quad \text{for all } u \in [f_j]. \tag{2}$$

By Lemma 3.1 this implies

$$(3) \qquad (3\kappa(E)\kappa(F))^{-1}\|T\| \le \|T_1\| \le \kappa(E)\kappa(F)\|T\|.$$

Applying Lemma 3.1 once more we find a subspace $Y_2 \subseteq E$ spanned by $\psi^{(2)}(n)$ blocks of (f_j) with constant coefficients, so that T_1 attains its norm up to the constant 3 on Y_2 and we proceed with an obvious induction. ■

The following technical definitions are fundamental in this section.

3.3 DEFINITION. *Let X be a Banach space with an unconditional basis $(x_j)_{j\in J}$ and let $K \ge 1$, $m \in \mathbf{N}$.*

(i) *(x_j) is called (m, K)-Euclidean, if for every subset $A \subseteq J$ with $|A| \le m$, $(x_j)_{j\in A}$ is K-equivalent to the unit vector basis of $l_2^{|A|}$.*

(ii) *Let $n \in \mathbf{N}$; (x_j) is said to have property $E(n, m, K)$, if there is a set $I \subseteq J$, $|I| = n$, so that $\{x_j | j \in J \setminus I\}$ is (m, K)-Euclidean.*

3.4 DEFINITION. *Let (x_j) be a 1-unconditional basis for a Banach space X. We shall say that X has property (H) for blocks of the basis, if the conditions of Definition 0.4 are satisfied for $\lambda = 1$ and all block basic sequences (u_i) of (x_j).*

If (x_i) is a normalized 1-unconditional basis for a Banach space X and $x \in X$ has the form $x = \sum_i t_i x_i$ then we shall write $\|x\|_2 = (\sum_i |t_i|^2)^{1/2}$.

Our main aim in this section is to prove, in Theorem 3.11 below, that if a Banach space X has property (H) and (x_j) is a 1-unconditional basis in X, then for a suitable K and every $n \in \mathbf{N}$, (x_j) has $E(n, m(n), K)$, where $m(\cdot)$ is a fast growing function of n and K depending only on the property (H) constant. The proof of this fact is done in several steps which consist of improving estimates for $m(\cdot)$ using blocking procedures.

Let us describe the first one. It requires the following lemma.

3.5 LEMMA. *Let X be a Banach space with a 1-unconditional normalized basis (x_j) satisfying property (H) for blocks of the basis with constant κ.*

Let $(u_i)_{i=1}^m$ be disjointly supported finite blocks of (x_j), say $u_i = \sum_{j\in\sigma_i} t_j x_j$ for $1 \le i \le m$. Assume that there is a $K \ge 1$ so that $\{x_j | j \in \sigma_i\}$ is K-equivalent to the unit vector basis of $l_2^{|\sigma_i|}$ for all $1 \le i \le m$. Let $\alpha = \max_i \max_{j\in\sigma_i} |t_j|$ and $\beta = \min_i \|u_i\|_2$. There exists a universal constant $C \ge 1$ such that

$$(CK\kappa(\alpha+\beta)\beta^{-1})^{-1}\left\|\sum_{i=1}^m u_i\right\|_2 \le \left\|\sum_{i=1}^m u_i\right\| \le CK\kappa(\alpha+\beta)\beta^{-1}\left\|\sum_{i=1}^m u_i\right\|_2.$$

Proof. Our assumptions imply

$$(1) \qquad K^{-1}\|u_i\|_2 \le \|u_i\| \le K\|u_i\|_2 \quad \text{for all } 1 \le i \le m.$$

We shall define a blocking of $\Sigma_{i=1}^m u_i$ as follows: Let $\delta_1 \subseteq \sigma_1$ be a set of smallest cardinality such that

$$\beta \leq \left(\sum_{j \in \delta_1} t_j^2 \right)^{1/2}. \tag{2}$$

If we let $z_1 = \Sigma_{j \in \delta_1} t_j x_j$, then from the minimality of δ_1 we get

$$\beta \leq \|z_1\|_2 \leq (\alpha + \beta). \tag{3}$$

Continuing in this manner we construct mutually disjoint subsets $\delta_1, \delta_2, \ldots, \delta_{n_1}$ of σ_1 so that if we let $\delta_1' = \sigma_1 \setminus \bigcup_{r=1}^{n_1} \delta_r$, then

$$\beta \leq \left(\sum_{j \in \delta_r} |t_j|^2 \right)^{1/2} \leq (\alpha + \beta) \qquad 1 \leq r \leq n_1 \tag{4}$$

and

$$\left(\sum_{j \in \delta_1'} |t_j|^2 \right)^{1/2} < \beta. \tag{5}$$

Now let δ be a subset of σ_2 of smallest cardinality so that

$$\left(\sum_{j \in \delta} |t_j|^2 + \sum_{j \in \delta_1'} |t_j|^2 \right)^{1/2} \geq \beta.$$

Set $\delta_{n_1+1} = \delta \cup \delta_1'$ and continue as before. In this way we construct mutually disjoint subsets δ_r, $1 \leq r \leq n_m$, such that if $z_r = \Sigma_{j \in \delta_r} t_j x_j$, then

$$\beta \leq \|z_r\|_2 \leq (\alpha + \beta) \tag{6}$$

and if $\delta_m' = \sigma_m \setminus \bigcup_{r=m_{m-1}+1}^{n_m} \delta_r$ and $z_0 = \Sigma_{j \in \delta_m'} t_j x_j$, then

$$\|z_0\|_2 < \beta. \tag{7}$$

This clearly implies that if $r_2 \neq r_1 \neq 0$ then

$$\| z_{r_2} \|_2 \leq (\alpha + \beta) \beta^{-1} \| z_{r_1} \|_2. \tag{8}$$

From the construction it follows that each δ_r intersects at most two σ_i's; therefore by (1),

$$(\sqrt{2} K)^{-1} \|z_r\|_2 \leq \|z_r\| \leq \sqrt{2} K \|z_k\|_2 \quad \text{for all } 0 \leq r \leq n_m. \tag{9}$$

If κ denotes the property (H) constant, then by (8) and (9)

$$(10)\quad \left(\sqrt{2}K\kappa(\alpha+\beta)\beta^{-1}\right)^{-1}\left\|\sum_{r>0} z_r\right\|_2 \le \left\|\sum_{r>0} z_r\right\| \le \sqrt{2}K\kappa(\alpha+\beta)\beta^{-1}\left\|\sum_{r>0} z_r\right\|_2.$$

Let $u=\sum_{i=1}^m u_i$. It follows from (1) and (10) that

$$(11)\quad \|u\| \le \|z_0\| + \left\|\sum_{r>0} z_r\right\| \le K\|z_0\|_2 + \sqrt{2}K\kappa(\alpha+\beta)\beta^{-1}\left\|\sum_{r>0} z_r\right\|_2 \le K\left(1+\sqrt{2}\kappa(\alpha+\beta)\beta^{-1}\right)\|u\|_2.$$

Similarly,

$$(12)\quad \|u\| \ge \left\|\sum_{r>0} z_r\right\| \ge \left(\sqrt{2}K\kappa(\alpha+\beta)\beta^{-1}\right)^{-1}\left\|\sum_{r>0} z_r\right\|_2 \ge \left(2\sqrt{2}K\kappa(\alpha+\beta)\beta^{-1}\right)^{-1}\|u\|_2,$$

completing the proof. ∎

3.6 PROPOSITION. *Let X be a Banach space with a 1-unconditional normalized basis (x_j) satisfying property (H) for blocks of the basis (with constant κ). Let $m\in\mathbf{N}$ and $K\ge 1$ and assume that (x_j) is (m,K)-Euclidean.*

Let $(u_i)_{i=1}^m$ be disjointly supported finite blocks of (x_j), say $u_i=\sum_{j\in\sigma_i} t_jx_j$ for $1\le i\le m$. Assume that $\{x_j \mid j\in\sigma_i\}$ is K-equivalent to the unit vector basis of $l_2^{|\sigma_i|}$ for all $1\le i\le m$. There exists a universal constant $C\ge 1$ such that

$$(CK\kappa)^{-1}\left\|\sum_{i=1}^m u_i\right\|_2 \le \left\|\sum_{i=1}^m u_i\right\| \le CK\kappa\left\|\sum_{i=1}^m u_i\right\|_2.$$

Proof. Put $u=\sum_{i=1}^m u_i$; without loss of generality we may assume that $\|u\|_2=1$. Let $J_1=\{j \mid |t_j|\ge m^{-1/2}\}$, $J_2=\bigcup_{i=1}^m \sigma_i\setminus J_1$ and define

$$(1)\quad z_1=\sum_{j\in J_1} t_jx_j,\quad v_i=\sum_{j\in\sigma_i\cap J_2} t_jx_j \quad \text{for all } 1\le i\le m,$$

$$(2)\quad I_1=\left\{i \mid \|v_i\|_2<\frac{1}{2m^{1/2}}\right\},\quad I_2=\left\{i \mid \|v_i\|_2\ge\frac{1}{2m^{1/2}}\right\}$$

$$(3)\quad z_2=\sum_{i\in I_2} v_i,\quad z_3=\sum_{i\in I_1} v_i.$$

Clearly $u = z_1 + z_2 + z_3$. Since (x_i) is (m, K)-Euclidean and $|J_1| \leq m\|z_1\|_2^2 \leq m$, we have

$$(4) \qquad K^{-1}\|z_1\|_2 \leq \|z_1\| \leq K\|z_1\|_2.$$

Further,

$$(5) \qquad K^{-1}\|v_i\|_2 \leq \|v_i\| \leq K\|v_i\|_2 \quad \text{for all } 1 \leq i \leq m.$$

The norm of z_2 is estimated by Lemma 3.5, with $\beta = 1/2m^{1/2}$ and $\alpha = 1/m^{1/2}$. It follows that there is a universal constant $C \geq 1$ so that

$$(6) \qquad (CK\kappa)^{-1}\|z_2\|_2 \leq \|z_2\| \leq CK\kappa\|z_2\|_2.$$

By the definition of z_3,

$$\|z_3\|_2 < m^{1/2}/2m^{1/2} \leq 1/2.$$

Moreover, by (5) and property (H) we get

$$(7) \qquad \|z_3\| \leq \max_i \|v_i\| \left\| \sum_{i \in I_1} v_i/\|v_i\| \right\| \leq (K/2m^{1/2})\kappa\sqrt{m} = K\kappa/2.$$

By (4), (6) and (7) we get

$$(8) \quad \|u\| \leq \|z_1\| + \|z_2\| + \|z_3\| \leq CK\kappa\big(\|z_1\|_2 + \|z_2\|_2 + 1\big) \leq 3CK\kappa$$

and

$$(9) \qquad \|u\| \geq \max\{\|z_1\|, \|z_2\|\} \geq \frac{1}{\sqrt{2}}(CK\kappa)^{-1}\big(\|z_1\|_2^2 + \|z_2\|_2^2\big)^{1/2}$$
$$\geq \frac{1}{\sqrt{2}}(CK\kappa)^{-1}\left(1 - \frac{1}{4}\right)^{1/2} \geq (2CK\kappa)^{-1}.$$

This proves the result. ■

As an immediate consequence we get the following blocking principle.

3.7 COROLLARY. *Let X be a Banach space with a 1-unconditional normalized basis (x_j) satisfying property (H) for blocks of the basis (with constant κ). If there is an $n \in \mathbf{N}$ and $K \geq 1$ so that (x_j) is (n, K)-Euclidean, then (x_j) is $(n^2, \tilde{K})$-Euclidean as well, where $\tilde{K} = CK\kappa$ and $C \geq 1$ is a universal constant.*

Proof. Let $J \subseteq \mathbf{N}$, $|J| = n^2$, and fix $x = \sum_{j\in J} t_j x_j$ with $\sum_{j \in J} |t_j|^2 = 1$. Consider any partition of J into n mutually disjoint subsets σ_i, $i = 1, 2, \ldots, n$ with $|\sigma_i| = n$ and define

$$u_i = \sum_{j\in\sigma_i} t_j x_j. \tag{1}$$

It is readily verified that (u_i) satisfies the conditions of Proposition 3.6 and hence there is a universal constant $C \geq 1$ so that

$$(CK\kappa)^{-1} \leq \left\| \sum_{j\in J} t_j x_j \right\| \leq CK\kappa. \tag{2}$$ ■

If X has (H) and $\lambda \geq 1$, we put $\kappa^*(\lambda) = \kappa(C\lambda^2\kappa(X)^2)$ and $\kappa^*(X) = \kappa(C\kappa(X)^2)$, where C is the constant from Theorem 1.1.

The next proposition gives our first estimate of the function $m(\cdot)$ mentioned earlier in this section.

3.8 Proposition. *Let X be a Banach space with property (H). There is a constant $K \geq 1$ such that whenever $(x_j)_{j=1}^M$, $1 \leq M \leq \infty$, is a normalized unconditional basis for X, then (x_j) has $E(n,n,K)$ for all $1 \leq n \leq M$. Moreover, $K \leq C\kappa^*(X)^{17}$, where $C \geq 1$ is a universal constant.*

Proof. Let us first show that there is a constant K_1 such that property $E(n^2, n, K_1)$ holds for all $1 \leq n \leq M$. Fix K' to be defined later and assume that for some $n, E(n^2, n, K')$ does not hold. By induction we can then construct n disjointly supported subspaces E_i, each spanned by n vectors of the basis (x_j) not K'-equivalent to the unit vector basis of l_2^n.

Let $m \in \mathbf{N}$ be chosen independently of n, so that

$$\exp_2(\psi^{(m)}(n))N(\psi^{(m)}) \leq n \tag{1}$$

where $N(\cdot)$ is the function defined in Lemma 1.3 (note that $m = 4$ will do). By Lemma 3.2 we can for each $1 \leq i \leq n$ find a $\psi^{(m)}(n)$-dimensional subspace $F_i \subseteq E_i$, spanned by a normalized 1-unconditional basis (a block basis of (x_j)), which is not $K'(3\kappa)^{-m}$-equivalent to the unit vector basis of $l_2^{\psi^{(m)}(n)}$, where κ is the property (H) constant of X.

Since the F_i's form a 1-unconditional direct sum, we get from (1) and Theorem 1.1 that there exist $K_2 \leq C_1\kappa^*(X)^{12}, C_1$ a universal constant, and an i_0, $1 \leq i_0 \leq n$, so that $d(F_{i_0}) \leq K_2$. It is well known and easy to see that this implies that every normalized 1-unconditional basis of F_{i_0} is K_2-equivalent to the unit vector basis of $l_2^{\psi^{(m)}(n)}$. Hence, by the choice of the F_i's

$$K'(3\kappa)^{-m} < K_2.$$

We can therefore conclude that whenever $K_1 \geq 3^m C_1 \kappa^*(X)^{m+12} \geq 81 C_1 \kappa^*(X)^{16}$ then (x_j) has $E(n^2, n, K_1)$ for every n. Corollary 3.7 and an easy calculation now show that there is a universal constant $C \geq 1$ such that (x_j) has $E(n, n, K)$ for all n and all $K \geq C\kappa^*(X)^{17}$. ■

The second blocking procedure is performed for a specific ordering of an unconditional basis with the intention of improving the tail behaviour of the basis. To be more specific let us introduce the following definition which is a modification of Definition 3.3.

3.9 DEFINITION. *Let X be a Banach space with an unconditional basis (x_j). Let $K \geq 1$ and let $n, m \in \mathbf{N}$. We say that (x_j) has $E_t(n, m, K)$ provided $[x_j | j \geq n+1]$ is (m, K)-Euclidean.*

We also require a certain fast growing hierarchy of functions on $\mathbf{N} \cup \{0\}$ defined as follows. Given a non-decreasing function Φ_0 on $\mathbf{N}$ such that $\lim_{n \to \infty} \Phi_0(n)/n = \infty$, define, for $j \geq 1$,

$$(*) \qquad \Phi_j(0) = 1 \quad \text{and} \quad \Phi_j(n) = \Phi_{j-1}^{(\Phi_{j-1}(n))}(n) \quad \text{for } n \in \mathbf{N}.$$

That is, for $n \in \mathbf{N}$, $\Phi_j(n)$ is the $\Phi_{j-1}(n)$-th iteration of the Φ_{j-1}.

The next result provides a general inductive tail blocking procedure.

3.10 PROPOSITION. *Let X be a Banach space with a 1-unconditional normalized basis (x_j) satisfying property (H) for blocks of the basis (with the constant κ). Let Φ_0 be a non-decreasing function on $\mathbf{N}$ such that $\lim_{n \to \infty} \Phi(n)/n = \infty$ and let K be a constant. Assume that (x_j) has $E_t(n, \Phi_0(n), K)$ for all $n \in \mathbf{N}$. Then for every $\nu \in \mathbf{N}$, (x_j) has $E_t(n, \Phi_\nu(n), K_\nu)$ for all $n \in \mathbf{N}$. Here $K_\nu = K(C\kappa)^\nu$, where κ is the property constant (H) and $C \geq 1$ is a universal constant.*

Proof. We proceed by induction on ν. For $\nu = 0$ the statement is obviously satisfied.

Let $\nu > 1$ and assume that for some constant $K_{\nu-1}$, (x_j) has $E_t(n, \Phi_{\nu-1}(n), K_{\nu-1})$ for all $n \in \mathbf{N}$. Fix $n \in \mathbf{N}$ and let $I \subseteq \{j \in \mathbf{N} | j > n\}$ with $|I| \leq \Phi_\nu(n)$. Let $x = \sum_{i \in I} t_i x_i$, with $\sum_{i \in I} |t_i|^2 = 1$. Define a partition of I into mutually disjoint sets $(\sigma_j)_{j=1}^s$ as follows: σ_1 consists of the first $\Phi_{\nu-1}(n)$ elements of I; if $j > 1$ and if $\sigma_1, \sigma_2, \ldots, \sigma_j$ are chosen then let σ_{j+1} consist of the first $\Phi_{\nu-1}^{(j+1)}(n)$ elements of $I \setminus \bigcup_{i=1}^j \sigma_i$, whenever the cardinality of the latter set is larger than or equal to $\Phi_{\nu-1}^{(j+1)}(n)$, otherwise set $\sigma_{j+1} = I \setminus \bigcup_{i=1}^j \sigma_i$, if this set is non-empty. Continue this process as long as possible to exhaust all of I.

It is clear from the definition of Φ_ν that the above construction ends after at most $s = \Phi_{\nu-1}(n)$ steps. Also, for all $1 \le i \le s$, we have $|\sigma_i| \le \Phi_{\nu-1}^{(i)}(n)$ and $\sigma_i \subseteq]\Phi_{\nu-1}^{(i-1)}(n), \infty[$. Therefore, by the inductive hypothesis, $\{x_j | j \in \sigma_i\}$ is $K_{\nu-1}$-equivalent to the unit vector basis of $l_2^{|\sigma_i|}$.

Set

$$(1) \qquad u_i = \sum_{j \in \sigma_i} t_j x_j \quad 1 \le i \le s.$$

Since $(x_j)_{j \in I}$ is $(\Phi_{\nu-1}(n), K_{\nu-1})$-Euclidean the blocks $(u_i)_{i=1}^s$ satisfy the assumptions of Proposition 3.6. Therefore there is a universal constant $C \ge 1$ such that

$$(2) \qquad (CK_{\nu-1}\kappa)^{-1} \le \left\| \sum_{i=1}^{s} u_i \right\| = \left\| \sum_{i \in I} t_i x_i \right\| \le CK_{\nu-1}\kappa.$$

This shows that (x_j) has $E_t(n, \Phi_\nu(n), K_\nu)$ with $K_\nu = CK_{\nu-1}\kappa$. By induction this completes the proof. ∎

We are now ready for the main theorem on the tail behaviour of unconditional bases in spaces with property (H). The fast growing functions φ_ν which control this behaviour are defined, for $\nu \in \mathbf{N}$, by

$$(**) \quad \varphi_\nu(0) = 1 \quad \text{and} \quad \varphi_0(n) = n^2, \qquad \varphi_\nu(n) = \Phi_\nu(n) \quad \text{for } n \in \mathbf{N}.$$

(Here the functions Φ_ν are defined by $(*)$, with $\Phi_0 = \varphi_0$.)

3.11 THEOREM. *Let X be a Banach space with property (H) and let (x_j) be a normalized 1-unconditional basis for X. There is a permutation π of $\mathbf{N}$, so that for every $\nu \in \mathbf{N}$, there is a constant K_ν, so that $(x_{\pi(j)})$ has $E_t(n, \varphi_\nu(n), K_\nu)$ for all $n \in \mathbf{N}$. Moreover, $K_\nu \le C^\nu \kappa^*(X)^{\nu+19}$, where $C \ge 1$ is a universal constant.*

Proof. From Corollary 3.7 and Proposition 3.8 it follows that there is a universal constant C_1 so that (x_j) has $E(n, n^4, K)$ for all $n \in \mathbf{N}$ with $K = C_1\kappa^*(X)^{19}$. For each $n \in \mathbf{N}$ choose $J_n \subseteq \mathbf{N}$ so that $|J_n| \le n$ and that $(x_j)_{j \notin J_n}$ is (n^4, K)-Euclidean and set $I_n = \bigcup_{j=1}^n J_j$. Since $|I_n| \le n^2$ it follows that if π is a permutation with $\pi(I_n) = \{1, 2, \ldots, |I_n|\}$ for all $n \in \mathbf{N}$ then $(x_{\pi(j)})$ has $E_t(n^2, n^4, K)$ for all $n \in \mathbf{N}$. Proposition 3.10 now gives the existence of a universal constant C_2 so that $(x_{\pi(j)})$ has $E_t(n, \varphi_\nu(n), K_\nu)$ for all $n \in \mathbf{N}$, $\nu \in \mathbf{N}$ with $K_\nu = C_2^\nu K\kappa(X)^\nu \le C_1 C_2^\nu \kappa^*(X)^{\nu+19}$. ∎

4. Applications for subspaces of lattices with property (H)

Here we shall apply the tail theorems from the previous section to investigate some quantitative structure of subspaces of Banach lattices with property (H). We get the results on the dimension of their Euclidean sections, in particular on the weak Hilbert space property, on the existence of bases and on the uniform approximation property.

It was discovered by W.B. Johnson that the results of Section 3 imply the following proposition which we present here with his permission.

4.1 Proposition. *Let X be an order complete Banach lattice property (H), let $E \subseteq X$ be an n-dimensional subspace and let $\nu \in \mathbf{N}$. There exists a subspace $F \subseteq E$ of dimension k, where $\varphi_\nu(k) \geq n^{2n}$, and $K_\nu \geq 1$ such that $d(E) \leq K_\nu d(F)$. One has $K_\nu = C^\nu \kappa^*(X)^{38+2\nu}$, where $C \geq 1$ is a universal constant.*

Proof. From the order completeness of X it follows that there exist mutually disjoint normalized elements $x_1, x_2, \ldots, x_{n^{2n}} \in X$ and an operator $T\colon E \to [x_i | 1 \leq i \leq n^{2n}] = G$ such that

$$\|Tx - x\| \leq \frac{1}{2}\|x\| \quad \text{for all } x \in E. \tag{1}$$

Since (x_i) is a 1-unconditional basis for G, we get from Theorem 3.11 that (x_i) has $E(k, \varphi_\nu(k), \tilde{K})$, where $\tilde{K} = C_1^\nu \kappa^*(X)^{19+\nu}$ and C_1 is a universal constant. We may assume that the x_i's are enumerated so that $\{x_i | k+1 \leq i \leq n^{2n}\}$ is $\tilde{K}$-equivalent to the unit vector basis of $l_2^{n^{2n}-k}$. Put

$$G_1 = \left[x_i | k+1 \leq i \leq n^{2n}\right]; \quad F_1 = T^{-1}(T(E) \cap G_1). \tag{2}$$

G_1 is clearly 1-complemented in G and $T(E) \cap G_1$ is $\tilde{K}$-complemented in G_1; hence by (1), F_1 is $3\tilde{K}$-complemented in E. Let P be a projection of E onto F_1 with $\|P\| \leq 3\tilde{K}$ and put $F = P^{-1}(0)$. Clearly $\dim F \leq k$ and $E = F \oplus F_1$.

Since $d(F_1) \leq 3\tilde{K}$, an easy calculation shows that

$$d(E) \leq 6\tilde{K}(3\tilde{K}+1)d(F). \tag{3}$$

■

As an immediate consequence of Proposition 4.1 we obtain:

4.2 Theorem. *Let X be a Banach lattice with property (H).*

For every $\nu \in \mathbf{N}$ there is $K_\nu \geq 1$ such that for every finite dimensional subspace $E \subseteq X$ there exist a subspace $Y \subseteq E$ with $\varphi_\nu(\dim E - \dim Y) = (\dim E)^{2 \dim E}$ and a projection P of E onto Y such that $d(F) \leq K_\nu$ and $\|P\| \leq K_\nu$. One had $K_\nu = C^\nu \kappa^*(X)^{38+2\nu}$, where $C \geq 1$ is a universal constant.

In particular X is a weak Hilbert space.

Proof. Without loss of generality we can assume that X is order complete. Indeed, X^{**} is order complete and by the principle of local reflexivity [8], X^{**} is finitely representable in X, which implies that it has property (H) with the same constant. Now let $\nu \in \mathbf{N}$ and let $E \subseteq X$ be finite dimensional, say $\dim E = n$. Let K_ν be as in Lemma 4.1 and let $Y = F_1$ be the subspace constructed in the proof of that lemma. Then it follows that Y has the required properties.

Theorem 2.7 of [12] now gives that X is a weak Hilbert space. ∎

Remark. For an arbitrary Banach space X to be weak Hilbert requires, according to the definition, that given an n-dimensional subspace E of X, the dimension of a "nice" subspace $Y \subset E$ is proportional to n. The theorem above shows that for a weak Hilbert Banach lattice a much stronger fact is true: the dimension of a Hilbertian nicely complemented subspace $Y \subset E$ is "extremely large" with the same estimate as in the case of the 2-convexified Tsirelson space and its dual.

Since by the classical John's estimate (for example, see [15] Proposition 9.12), any k-dimensional space F satisfies $d(F) \leq \sqrt{k}$, Proposition 4.1 also immediately implies the following.

4.3 THEOREM. *Let X be a Banach lattice with property (H).*
For every $\nu \in \mathbf{N}$ there is $K_\nu \geq 1$ such that if $E \subseteq X$ is an n-dimensional subspace then $d(E) \leq K_\nu \sqrt{k}$, where the integer k satisfies $\varphi_\nu(k) \geq n^{2n}$. One has $\kappa_\nu = C^\nu \kappa^*(X)^{38+2\nu}$, where $C \geq 1$ is a universal constant.

Recall that a Banach space X is said to have *local unconditional structure* $(l.u.s.t)$, if there exists a $\lambda \geq 1$ so that every finite dimensional subspace $E \subseteq X$ is contained in a finite dimensional subspace $F \subseteq X$, which has a λ-unconditional basis. It is easy to see that the proof of Theorem 4.2 gives:

4.4 COROLLARY. *A Banach space X with l.u.s.t. has property (H) if and only if it is a weak Hilbert space.*

In the sequel we shall need two notions introduced by W.B. Johnson [4] to give a criterion for all subspaces of a given Banach space to have the uniform projection approximation property (defined a little latter in this paper).

4.5 DEFINITION. *Let X be a Banach space, let $n \in \mathbf{N}$, and let $K \geq 1$, $t \geq 1$.*
- (i) *X is said to satisfy $C(n,t,K)$, if there exist a subspace Y of X of codimension n, so that every subspace $E \subseteq Y$ with* $\dim E \leq t$ *is the range of a projection P on X with $\|P\| \leq K$.*
- (ii) *X is said to satisfy $H(n,t,K)$ if there is a subspace $Y \subseteq X$ of codimension n, so that for every subspace $E \subseteq Y$ with* $\dim E \leq t$ *we have $d(E) \leq K$.*

For later convenience, in the definition above t is not required to be a positive integer. Johnson [4], Proposition 1.8, proved that the two notions introduced in Definition 4.4 are in duality. In fact, if X has $H(n, t, K)$, then X^* has $H(5^n, \log_5 t, 12K)$ and $C(5^n, \log t, 12K)$. Hence, since these two notions are hereditary, we get that if E is a subspace of a quotient of X, then E has $H(5^{5^n}, \log_5 \log_5 t, 144K)$ and $C(5^{5^n}, \log_5 \log_5 t, 144K)$.

We recall that if $\lambda \geq 1$ and E is a subspace of a Banach space X, then a subspace $F \subseteq X^*$ is called *λ-norming* over E, if

$$\|x\| \leq \lambda \sup\{|x^*(x)| \,|\, x^* \in F,\ \|x^*\| \leq 1\} \quad \text{for all } x \in E.$$

The following lemma is well known and standard.

4.6 LEMMA. *Let X be a Banach space and $E \subseteq X$ a subspace with* $\dim E = n$. *There exists a 5^n-dimensional subspace $F \subseteq X^*$, which is 2-norming over E.*

Proof. Let $(x_i)_{i=1}^{5^n}$ be a $(1/2)$-net in the unit space of E. For $1 \leq i \leq 5^n$ pick $y_i^* \in X^*$ with $\|y_i^*\| = y_i^*(x_i) = \|x_i\| = 1$ and set $F = [y_i^*]$. ∎

Let us introduce one more family of functions which plays a basic role in quantitative results we come to now. We define $\tau_\nu\colon \mathbf{N} \to \mathbf{R}_+$, for $\nu \in \mathbf{N}$, as follows. Set

$$S(0) = 1 \quad \text{and} \quad S(k) = \exp_2^{(3)}(4k) \quad \text{for } k \in \mathbf{N}.$$

For $n \in \mathbf{N}$ let k_n be the largest (non-negative) integer k satisfying

$$\text{(A)} \qquad \exp_2^{(S(k))}(k) \leq n.$$

Finally, for $\nu \in \mathbf{N}$ and $n \in \mathbf{N}$ define $\tau_\nu(n)$ by

$$\text{(B)} \qquad \tau_\nu(n) = \left(\log \varphi_\nu(k_n)/2\right)^{1/2};$$

where φ_ν is defined in $(**)$.

We postpone a technical discussion of the functions τ_ν until the end of this section. For the moment let us observe that $[\tau_\nu(n)] \leq 1$, whenever $\varphi_\nu(k_n) \leq 2^8$. Moreover, the τ_ν's are fast growing, as $n \to \infty$; in particular, already for $\nu = 3$ we have

$$\text{(C)} \qquad \tau_3(n) \geq \exp_2^{(2^{2^n})}(n).$$

4.7 THEOREM. *Let X be a Banach lattice with property (H) (or, equivalently, which is a weak Hilbert space). For every $\nu \in \mathbf{N}$ there exists K_ν so that X*

has $H(n, \tau_\nu(n), K)$ for all $n \in \mathbf{N}$. One has $K_\nu = C^\nu \kappa_X^*(21)^{12}\kappa^*(X)^{38+2\nu}$, where $C \geq 1$ is a universal constant.

Proof. Fix $\nu \in \mathbf{N}$ and $n \in \mathbf{N}$. Assume that for some $\tilde{K} \geq 1$, X does not have $H(n, \tau_\nu(n), \tilde{K})$. Put $k = k_n$ as defined above, let K'_ν be as in Proposition 4.1. Let r be the largest integer with $5^{kr} \leq n$.

Set $L = S(k)(\exp_2^{(S(k)-3)}(k))^{1/3}$ and observe that by (A),

$$kL \leq \exp_2^{(S(k)-3)}(k) \leq \log_5 n.$$

Therefore

$$r \geq L \geq S(k)\exp_2^{(s(k))}\big(2^k(2^k+3)\big), \tag{1}$$

where $s(k) = (3 \cdot 2^k)^{k2^k}$.

We shall construct by induction a finite sequence $(F_j)_{j=1}^r$ of k-dimensional subspaces of X and a sequence $(Y_l)_{l=0}^{r-1}$ of subspaces of X^* with $Y_{l-1} \subseteq Y_l$ for $1 \leq l \leq r-1$ such that:

(i) $F_j \subseteq Y_{j-1}^\perp$ and $d(F_j) > \tilde{K}(K'_\nu)^{-1}$ for all $1 \leq j \leq r$;

(ii) $\dim Y_l \leq 5^{kl}$ and Y_l is 2-norming over $\Sigma_{i=1}^l F_i$ for all $1 \leq l \leq r-1$.

Set $Y_0 = \{0\}$. By our assumption we can find a $[\tau_\nu(n)]$-dimensional subspace E_1 of X with $d(E_1) > \tilde{K}$.

$$\tau_\nu(n)^{2\tau_\nu(n)} \leq \exp_2\big(2\tau_\nu(n)^2\big) = \varphi_\nu(k_n).$$

Thus Lemma 4.1 yields the existence of a k-dimensional subspace $F_1 \subseteq E_1$ with $K'_\nu d(F_1) \geq d(E_1) > \tilde{K}$. From Lemma 4.6 we obtain a 5^k-dimensional subspace $Y_1 \subseteq X^*$ which is 2-norming over F_1.

Assume now that $1 \leq j < r$ and that $F_1, F_2, \dots, F_j$, $Y_0, Y_1, \dots, Y_j$ have been constructed to satisfy (i) and (ii). Since $Y_j^\perp$ is of codimension $5^{kj} < n$, our assumption gives a $\tau_\nu(n)$-dimensional subspace $E_{j+1} \subseteq Y_j^\perp$ with $d(E_{j+1}) > \tilde{K}$. Again an application of Proposition 4.1 yields a k-dimensional subspace $F_{j+1} \subseteq E_{j+1}$ with $d(F_{j+1}) > \tilde{K}(K'_\nu)^{-1}$. Using Lemma 4.6 we find a $5^{k(j+1)}$-dimensional subspace $Y'_{j+1} \subseteq X^*$, 2-norming over $\Sigma_{i=1}^j F_i$ and we set $Y_{j+1} = Y_j + Y'_{j+1}$.

Clearly the F_j's form a direct sum. Put $F = \Sigma_{i=1}^r F_i$ and let $P_j \colon F \to \Sigma_{i=1}^j F_j$ and $Q_j : F \to F_j$ be the natural projections. Then (i) and (ii) imply that $\|P_j\| \leq 2$, hence $\|Q_j\| \leq 4$.

Since $S(k) \geq k^{(1+16k)^k}$, by Lemma 1.3 there is a set $A \subseteq \{1, 2, \dots, r\}$ with $|A| \geq r/S(k)$ so that if $i_0 \in A$ then $d(F_{i_0}, F_j) \leq 4$ for all $j \in A$. By (1) and Corollary 2.4 we get that there is a subspace of $\Sigma_{j \in A} F_j$ which is 4-isomorphic to a space Y, say, where Y is an 21-unconditional sum of 2^k copies of a k-dimensional space $\tilde{E}$ with $d(\tilde{E}, F_{i_0}) \leq 8$.

In particular Y has property (H) with $\kappa_Y(\lambda) \le 4\kappa_X(\lambda)$, for every $\lambda \ge 1$. Thus the proof of Theorem 1.1 implies that $d(\tilde{E}) \le M = C\kappa_X^*(21)^{12}$ and hence $\tilde{K} < 8K'_\nu M = C'K'_\nu\kappa_X^*(21)^{12}$, where $C \ge 1$ and $C' \ge 1$ are universal constants. ∎

We now turn our attention to the uniform approximation property for Banach lattices which are weak Hilbert spaces. Recall that a Banach space X is said to have the *uniform approximation property* $(u.a.p.)$, provided there is a function $f\colon \mathbf{N} \to \mathbf{N}$ and a constant $K \ge 1$ such that whenever $E \subseteq X$ is a finite dimensional subspace, then there exists an operator T on X with $\|T\| \le K, Tx = x$ for all $x \in E$ and the rank $rk(T) \le f(\dim E)$. In this situation we shall also say that X has the (K, f)-u.a.p.

If the operator above can be chosen to be a projection we shall say that X has the *projection uniform approximation property* $(p.u.a.p.)$.

The smallest function f which can be used in the above definition is called the uniformity function and denoted by $k_X(K, n)$ (by $p_X(K, n)$ if we consider the p.u.a.p.).

It was proved by Pisier [12] that every weak Hilbert space has the u.a.p., and very recently by Johnson and Pisier [6] that proportional growth of the uniformity function characterizes weak Hilbert spaces.

We shall combine our Theorem 4.7 with Johnson's results in [4] to show that Banach lattices which are weak Hilbert spaces have the p.u.a.p. with extremely slow growth of p_X; i.e., a behaviour as the Tsirelson weak Hilbert spaces, see e.g. [2]. If $\nu \in \mathbf{N}$ and h_ν denotes the inverse function to $n \to [\log_5 \log_5 \tau_\nu(n) - \exp_5^{(3)}(n)]$, we define $\vartheta_\nu(n) = \exp_5^{(3)}(h_\nu(n))$. The definition of τ_ν shows that ϑ_ν has a very slow growth.

4.8 THEOREM. *Let Y be a subspace of a quotient of a Banach lattice X which is a weak Hilbert space.*

For every $\nu \in \mathbf{N}$ there exists a K_ν, so that $p_Y(K_\nu, n) \le n + \vartheta_\nu(n)$ for all $n \in \mathbf{N}$. One has $K_\nu = C^\nu\kappa_X^*(21)^{12}\kappa^*(X)^{38+2\nu}$, where $C \ge 1$ is a universal constant. In particular Y has p.u.a.p.

Proof. Fix $\nu \in \mathbf{N}$ and $n \in \mathbf{N}$. Let $\tilde{K}$ be equal to the constant K_ν from Theorem 4.6. From this theorem and the remarks after Definition 4.5 it follows that Y has

$$C\left(5^{5^n}, \log_5 \log_5 \tau_\nu(n), \tilde{K}\right).$$

By Proposition 1.3 of [4], if $F \subseteq Y$ is a subspace and $h \in \mathbf{N}$ is any integer such that

$$n = \dim F \le \log_5 \log_5 \tau_\nu(h) - 5^{5^{5h}},$$

then there is a projection P on Y with $Px = x$ for all $x \in F$, $\|P\| \leq 4\tilde{K} + 3$ and

$$rk(P) \leq \dim F + \exp_5^{(3)}(h).$$

Thus $rk(P) \leq \dim F + \vartheta_\nu(\dim F)$. ∎

We also get the result on bases of subspaces of quotients of Banach lattices which are weak Hilbert spaces.

4.9 THEOREM. *Let Y be a (finite or infinite dimensional) subspace of a quotient of a Banach lattice X which is a weak Hilbert space. Then Y has a basis with the constant less than or equal to K, where K depends on the property (H) constants of X. In fact, $K \leq C\kappa_X^*(21)^{112}$, where $C \geq 1$ is a universal constant.*

Proof. Let K'' denote the maximum of constants in Theorems 4.7 and 4.8, for $\nu = 3$. Let $K' = 144K''$, that is, $K' \leq C\kappa_X^*(21)^{56}$, where $C \geq 1$ is a universal constant. It is readily checked that $\vartheta_3(n) \leq (1/10)n$ and $\log_5 \log_5 \tau_3(\log_5 \log_5 n) \geq 3^{n+1}$. Hence, if Y is a subspace of a quotient of X, then Y has $H(n, 3^{n+1}, K')$ and $(K', 1.1n)$-p.u.a.p. The argument of Johnson's on page 23 of [4] now shows that Y has a basis with constant less than or equal to $K'(2 + K')$. ∎

4.10 COROLLARY. *If X is a weak Hilbert space with l.u.s.t., then the conclusions of Theorems* 4.8 *and* 4.9 *hold for every subspace of a quotient of X.*

Proof. Since X has l.u.s.t. and is reflexive it is isomorphic to a complemented subspace of a Banach lattice Z, finitely representable in X. Hence Z is a weak Hilbert space and the corollary follows immediately from Theorems 4.8 and 4.9. ∎

To conclude this section let us make some observations on the growth of functions $\tau_\nu(n)$, as $n \to \infty$. Observe that

$$\exp_2^{(S(k))}(k) \leq \exp_2^{(S(k)+k)}(2) \leq \exp_2^{(\exp_2^{(4)}(k))}(2),$$

for $k \geq 4$, hence (A) implies that

$$\text{(D)} \qquad \exp_2^{(\exp_2^{(4)}(k_n))}(2) \geq n.$$

Furthermore, $(**)$ implies that for $\nu \geq 2$, and all $k \in \mathbf{N}$,

$$\text{(E)} \qquad \varphi_\nu(k) \geq \exp_2^{(\varphi_{\nu-1}(k))}(k).$$

Also $\varphi_1(k) = k^{2^{k^2}} \geq \exp_2^{(2)}(k)$, for all $k \in \mathbf{N}$. Combining with (D) and using (E) twice, we get, for n sufficiently large,

$$\varphi_3(k_n) \geq \exp_2(\varphi_2(k_n))(k_n) \geq \exp_2^{(2^{2^n}+2)}(n).$$

By (B), this implies (C). Estimates for the functions $\varphi_\nu(k_n)$, for $\nu \geq 4$, can then be obtained, by induction, from (E).

5. Some additional remarks

If X is a Banach space with a normalized unconditional basis (x_j), and $n, m \in \mathbf{N}$, and $K \geq 1$, we shall say that (x_j) has property $H_t(n, m, K)$ if every m-dimensional subspace of $[x_j | j > n]$ is K-isomorphic to l_2^m. Let in the following X be a Banach space with a normalized unconditional basis (x_j).

The following result holds:

5.1 Proposition. *If X is a Banach space with property (H), then there is a constant K and a sequence $(m(n))$ of natural numbers with $m(n) \to \infty$, so that X has $H_t(n, m(n), K)$ for all $n \in \mathbf{N}$.*

Indeed, if for some K this does not hold, then by induction we can construct an infinite sequence (E_n) of mutually disjoint subspaces of the same dimension with $d(E_n) \geq K/2$. This contradicts Theorem 1.1.

Using Lemma 3.1 twice, combined with the fact that every m-dimensional subspace of X is almost contained in a subspace spanned by m^{2m} disjointly supported vectors, one can easily prove

5.2 Proposition. *If X has property (H) then there is $\tilde{K}$ such that whenever (x_j) has $H_t(n, m(n), K)$ for all $n \in \mathbf{N}$, then (x_j) has $H_t(n, 2^{m(n)}, K\tilde{K})$ for all $n \in \mathbf{N}$.*

This shows that if X has property (H) and there is a K, so that the sequence $(m(n))$ from Proposition 5.1 tends to infinity faster than some iteration of the logarithm, then to every $s \in \mathbf{N}$ there is a constant C so that (x_j) has $H_t(n, \exp_2^{(s)}(n), C)$ for all $n \in \mathbf{N}$. Then we could argue like in Section 4 to get the results there. However it seems impossible to obtain any growth condition of the sequence $(m(n))$ for a suitable K directly from property (H). An argument by contradiction like in Proposition 3.5 will break down if the low dimensional spaces constructed there are supported on extremely long blocks. Hence we can pose:

5.3 Proposition. *Assume that X has property (H). Does X have an unconditional basis (y_j), which is a permutation of (x_j), so that (y_j) has*

$H_t(n, m(n), K)$ for some K and $m(n) \to \infty$ faster than some iteration of the logarithm?

In [4], Johnson constructed Banach spaces $X(k_n)$ where (k_n) is a sequence increasing to ∞ ($k_n = n$ corresponds to the Tsirelson space). Calculations in these spaces show that $X((k_n))$ has property (H) if and only if $k_n \to \infty$ faster than some iteration of the logarithm, so it is not possible to use these spaces as eventual counterexamples to Problem 5.3.

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