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INVOLUTIONS AND STATIONARY POINT FREE $\mathbf{Z}_4$-ACTIONS

BY

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1. Introduction

This paper studies fixed point sets of involutions and $\mathbf{Z}_2$-fixed point sets of stationary point free $\mathbf{Z}_4$-actions.

In Section 2, the interest is to determine which bordism classes in the unoriented bordism ring $\mathcal{N}_*$ can be realized as the fixed point set of an involution on an n-dimensional manifold. Denoting by I_n the subgroup of these classes in $\mathcal{N}_*$, we are going to prove that $I_n = \bigoplus_{j \leq n} \mathcal{N}_j$ if n is even; and for n odd I_n is the set of classes in $\bigoplus_{j \leq n} \mathcal{N}_j$ with zero Euler characteristic mod 2.

In Section 3, the $\mathbf{Z}_2$-fixed sets of stationary point free $\mathbf{Z}_4$-actions will be studied. Let $\mathcal{N}_m^{\mathbf{Z}_4}$ *(st. pt. free)* be the m-dimensional bordism group of manifolds with stationary point free $\mathbf{Z}_4$-action. Considering a $\mathbf{Z}_4$-action restricted to $\mathbf{Z}_2$ we get an involution, and the fixed set of this involution with the action induced by the $\mathbf{Z}_4$-action is an element in the bordism group of free involutions.

We are going to study the following question: Which classes in the bordism group of free in involutions $\mathcal{N}_*^{\mathbf{Z}_2}$ (free) can be realized as the $\mathbf{Z}_2$-fixed point set of a $\mathbf{Z}_4$-action in $\mathcal{N}_m^{\mathbf{Z}_4}$ (st. pt. free)?

Denoting by $I_m^{\mathbf{Z}_2}$ the set of these classes and considering $A_m = (\bigoplus_{j \leq m} \mathcal{N}_j) \cap \mathcal{X}_*$, where $\mathcal{X}_*$ is the set of classes in $\mathcal{N}_*$ with zero Euler characteristic mod 2, the main result of this section is the following theorem.

THEOREM. (a) *For m odd,*

$$I_m^{\mathbf{Z}_2} = \bigoplus_{\substack{j=1 \\ j \text{ odd}}}^{m} \mathcal{N}_j^{\mathbf{Z}_2}(\mathit{free}) + A_m[S^0, -1] + \left(\bigoplus_{j=0}^{m-1} \mathcal{N}_j \right)[S^1, -1]$$

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(b) *For m even*

$$I_m^{\mathbf{Z}_2} = \bigoplus_{\substack{j=0 \\ j \text{ even}}}^{m} \mathscr{N}_j^{\mathbf{Z}_2}(\mathit{free}) + A_m[S^0, -1] + A_{m-1}[S^1, -1]$$

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2. Involutions

Let $\mathscr{N}_*$ be the unoriented bordism ring of smooth manifolds and $\mathscr{N}_*^{\mathbf{Z}_2}$ the unrestricted bordism group of smooth manifolds with involution.

Being given a closed manifold M^n with an involution T, the fixed point set of $[M^n, T]$ is a disjoint union of closed manifolds F^j, $0 \leq j \leq n$.

Consider the homomorphism

$$F_n : \mathscr{N}_n^{\mathbf{Z}_2} \to \bigoplus_{j \leq n} \mathscr{N}^j$$

which assigns to $[M^n, T]$ the class $\oplus_{j \leq n}[F^j]$, where the disjoint union $\cup_{j \leq n} F^j$ is the fixed point set of T. Denote by I_n the image of F_n. In what follows, we are going to determine the image of the homomorphism F_n. To do this, we need the following lemmas.

LEMMA 2.1. *Let $[M^n]$ be in $\mathscr{N}_n$. If $\mathscr{X}[M^n] \equiv 0 \bmod 2$, then for every integer $k \geq 0$, there exists a $(n+k)$-manifold with involution $[W^{n+k}, T]$ such that the fixed point set is bordant to M^n.*

Proof. First, the lemma holds for $k = 0$ since the involution $[M^n, \mathrm{id}]$ fixes M^n.

Now, suppose that $k \geq 1$. By [3, 4.5], we have the bordism class of M^n admits a representative fibred over the circle since $\mathscr{X}[M^n] \equiv 0 \bmod 2$, i.e., there exists a closed manifold F^{n-1} with involution t such that $[M^n] = [(F^{n-1} \times S^1)/(t \times -1)]$. Then, considering the manifold with involution

$$[W^{n+k}, T] = \left[(F^{n-1} \times S^{k+1})/(t \times -1), 1 \times T'\right] + \left[F^{n-1} \times RP^{k+1}, t \times T''\right]$$

where

$$T' : (x_0, x_1, x_2, \ldots, x_{k+1}) \mapsto (x_0, x_1, -x_2, \ldots, -x_{k+1})$$

and

$$T'': [x_0, x_1, x_2, \dots, x_{k+1}] \mapsto [x_0, x_1, -x_2, \dots, -x_{k+1}],$$

it is easy to see that the fixed point set of the involution T is bordant to

$$(F^{n-1} \times S^1)/(t \times -1).$$

Hence, the class $[M^n]$ is represented by a manifold which is the fixed point set of $[W^{n+k}, T]$. Therefore, the lemma holds for all k.

LEMMA 2.2. *Fix an integer $k \geq 0$. Let $[M^m]$ and $[N^n]$ be in $\mathcal{N}_m$ and $\mathcal{N}_n$ respectively, for $m, n \leq 2k+1$. If $\mathscr{X}[M^m] + \mathscr{X}[N^n] \equiv 0 \bmod 2$, then there exists a $(2k+1)$-manifold with involution $[W^{2k+1}, T]$ such that the class of the fixed point set of T is bordant to $[M^m] + [N^n]$.*

Proof. If $\mathscr{X}[M^m] \equiv \mathscr{X}[N^n] \equiv 0 \bmod 2$, it is clear that there exists $[W^{2k+1}, T]$ with fixed point set bordant to $[M^m] + [N^n]$, by (2.1).

Thus, we only need to consider the case $\mathscr{X}[M^m] \equiv \mathscr{X}[N^n] \equiv 1 \bmod 2$. In this case, we have $m = 2j$ and $n = 2l$, since $\mathscr{X}_m = \mathcal{N}_m$ if m is odd.

We may suppose $j \leq l$. Consider the involution

$$\left[W_1^{2k+1}, T_1\right] = \left[M^m \times RP^{2l-2j+1} \times RP^{2k-2l}, 1 \times t_1 \times t_2\right],$$

where

$$t_1: [x_0, \dots, x_{2l-2j+1}] \mapsto [-x_0, x_1, \dots, x_{2l-2j+1}]$$

and

$$t_2: [x_0, \dots, x_{2k-2l}] \mapsto [-x_0, x_1, \dots, x_{2k-2l}].$$

The fixed point set F of $1 \times t_1 \times t_2$ is

$$\begin{aligned} F &= M^m \times (RP^0 \cup RP^{2l-2j}) \times (RP^0 \cup RP^{2k-2l-1}) \\ &= M^m \cup (M^m \times RP^{2k-2l-1}) \cup (M^m \times RP^{2l-2j}) \\ &\quad \cup (M^m \times RP^{2l-2j} \times RP^{2k-2l-1}). \end{aligned}$$

Therefore, $[F] = [M^m] + [M^m \times RP^{2l-2j}]$ since $2k - 2l - 1$ is odd.

Now, note that

$$\mathscr{X}[M^m \times RP^{2l-2j} \cup N^n] \equiv 0 \bmod 2,$$

since $\mathscr{X}[M^m] \equiv \mathscr{X}[RP^{2l-2j}] \equiv \mathscr{X}[N^n] \equiv 1 \bmod 2$. Then, there exists an involution $[W_2^{2k+1}, T_2]$ with fixed point set in the class $[M^m \times RP^{2l-2j}] + [N^n]$, by (2.1).

Finally, the class of the fixed point set of

$$[W, T] = [W_1^{2k+1}, T_1] + [W_2^{2k+1}, T_2]$$

is

$$[M^m] + [N^n].$$

THEOREM 2.3. (a) *The homomorphism F_n is onto for $n = 2k$ even*; *i.e.*,

$$I_n = \bigoplus_{j \leq n} \mathscr{N}_j.$$

(b) *The image of F_n is the subgroup of classes in $\bigoplus_{j \leq n} \mathscr{N}_j$ with zero Euler characteristic*, *if $n = 2k + 1$ is odd.*

Proof. (a) First considering the involution $[M^n, \mathrm{id}]$ we see that the class $[\mathrm{M}^n]$ belongs to I_n. This means that $\mathscr{N}_n \subset I_n$. Now, by Capobianco [2, p. 339] we have $\mathscr{N}_j \subset I_n$ for $k \leq j \leq 2k$ and $j \neq 2k - 1$. For $j = 2k - 1$, Lemma (2.1) implies that $\mathscr{N}_{2k-1} \subset I_n$ since $\mathscr{N}_{2k-1} = \mathscr{X}_{2k-1}$.

Finally, it remains to show that $\mathscr{N}_j \subset I_n$ for $0 \leq j \leq k$. To prove this, take $[M^j]$ in $\mathscr{N}_j$. Consider the involution $[RP^{2k-2j}, T]$ where

$$T \colon [x_0, \dots, x_{2k-2j}] \mapsto [-x_0, x_1, \dots, x_{2k-2j}].$$

So, the class of the fixed point set of the involution $[RP^{2k-2j} \times M^j \times M^j, T \times \mathit{twist}]$ is

$$[RP^0 \times M^j] + [RP^{2k-2j-1} \times M^j]$$

which is bordant to $[M^j]$ since $2k - 2j - 1$ is odd. Then, $\mathscr{N}_j \subset I_n$ for $0 \leq j \leq k$

(b) By [3, 27.2], the image is contained in $\mathscr{X}_*$, i.e., the subgroup with zero Euler characteristic. We use now the lemma (2.2) and (2.1) to conclude that the classes in $\bigoplus_{j \leq 2k+1} \mathscr{N}_j$ with zero Euler characteristic are in the image. Hence, the theorem follows at once.

3. Stationary point free $\mathbf{Z}_4$-actions

Let $\mathscr{N}_*^{\mathbf{Z}_4}$ (st. pt. free) be the unoriented bordism group of stationary point free $\mathbf{Z}_4$-actions and $\mathscr{N}_*^{\mathbf{Z}_2}$ (free) the unoriented bordism group of free involutions.

Consider the homomorphism

$$F_m^{\mathbf{Z}_2}\colon \mathcal{N}_m^{\mathbf{Z}_4}(st.pt.free) \to \bigoplus_{j\le m} \mathcal{N}_j^{\mathbf{Z}_2}(free)$$

which assigns to $[M^m, T]$ the class of the $\mathbf{Z}_2$-fixed point set of $[M^m, T]$. Recall the restriction homomorphism

$$\rho\colon \mathcal{N}_m^{\mathbf{Z}_4}(st.pt.free) \to \mathcal{N}_m^{\mathbf{Z}_2}$$

assigning to $[M, T]$ the involution $[M, T^2]$. The fixed point set of $[M, T^2]$ is the disjoint union of closed submanifolds $\bigcup_{j\le m} F^j$. Then, considering $t_j \equiv T/F_j$, $j = 0, \dots, m$, we have

$$F_m^{\mathbf{Z}_2}([M, T]) = \bigoplus_{j\le m} \left[F^j, t_j\right].$$

In this section we are going to study the image of the homomorphism $F_m^{\mathbf{Z}_2}$.

Now, let $\mathcal{N}_*^{\mathbf{Z}_4}$ (*st. pt. free*, *free*) be the relative bordism group of stationary point free $\mathbf{Z}_4$-actions on manifolds with boundary for which the action is free on the boundary. There exist the isomorphism

$$\mathcal{N}_*^{\mathbf{Z}_4}(st.pt.free, free) \cong \bigoplus_{k=0}^{*} \mathcal{N}_{*-k}^{\mathbf{Z}_2}(free)(BO_k(C^\infty))$$

by [1, pp. 85], and the sequence

$$0 \to \mathcal{N}_*^{\mathbf{Z}_4}(st.pt.free) \xrightarrow{i_*} \bigoplus_{k=0}^{*} \mathcal{N}_{*-k}^{\mathbf{Z}_2}(free)(BO_k(C^\infty)) \xrightarrow{\partial} \mathcal{N}_*^{\mathbf{Z}_4}(free) \to 0$$

of $\mathcal{N}_*$-modules and homomorphisms is split exact, where ∂ is the boundary homomorphism.

Further, for all k odd, we have the isomorphism

$$\varphi\colon \mathcal{N}_*^{\mathbf{Z}_4}(free) \otimes_{\mathcal{N}_*} \mathcal{N}_*(BSO_k) \to \mathcal{N}_*^{\mathbf{Z}_2}(free)(BO_k(C^\infty)) \tag{3.1}$$

which assigns to $[N, t] \times [P, \xi]$ the class of

$$\left[(N \times D\xi)/(t^2 \times -1), t \times 1\right] \quad \text{(see [5, 4.1])}.$$

Also, we have the homomorphism

$$\overline{F}_{Z_2}\colon \mathcal{N}_*^{Z_4}(st.pt.free, free) \to \mathcal{N}_*^{Z_2}(free).$$

mapping the class $[M, T]$ into the class of Z_2-fixed point set of $[M, T]$, and the restriction homomorphism

$$\rho\colon \mathcal{N}_*^{Z_4}(free) \to \mathcal{N}_*^{Z_2}(free)$$

mapping the class $[M, T]$ into the class $[M, T^2]$.

Next, considering the homomorphism

$$\rho \circ \partial\colon \bigoplus_{k=0}^{m} \mathcal{N}_{m-k}^{\mathbf{Z}_2}(free)(BO_k(C^\infty)) \xrightarrow{\partial} \mathcal{N}_m^{\mathbf{Z}_4}(free) \xrightarrow{\rho} \mathcal{N}_m^{\mathbf{Z}_2}(free)$$

for m even, we are going to analyze the kernel of $\rho \circ \partial$ restricted to the summands with k odd.

THEOREM 3.2. *For m even, if α is in the kernel of the homomorphism $\rho \circ \partial$ restricted to the summands with k odd, then the $\mathbf{Z}_2$-fixed point set of α belongs to*

$$\mathscr{X}_*[S^0, -1] + \mathscr{X}_{*-1}[S^1, -1].$$

Proof. First, by [5; 5.1], $\bar{F}_{\mathbf{Z}_2}$ restricted to the summands with k odd maps into

$$\mathcal{N}_*[S^0, -1] + \mathcal{N}_{*-1}[S^1, -1].$$

Now, we are going to prove that if an element x belongs to the kernel of $\rho \circ \partial$ restricted to the summands with k odd, then the $\mathbf{Z}_2$-fixed point set of x is in

$$\mathscr{X}_*[S^0, -1] + \mathscr{X}_{*-1}[S^1, -1].$$

For k odd, we have the isomorphism

$$\mathcal{N}_*^{\mathbf{Z}_2}(free)(BO_k(C^\infty)) \simeq \mathcal{N}_*^{\mathbf{Z}_4}(free) \otimes_{\mathcal{N}_*} \mathcal{N}_*(BSO_k)$$

(see [5; 4.1]); and recall that $\mathcal{N}_*^{\mathbf{Z}_4}(free)$ is freely generated as an $\mathcal{N}_*$ module by extensions of the antipodal action on even dimensional spheres and by restrictions of circle actions on odd dimensional spheres. Therefore, for k odd, we can take as generators of $\mathcal{N}_{m-k}^{\mathbf{Z}_2}(free)(BO_k(C^\infty))$ the classes

$$y_{(2l,J)} = \left(\left[S^{2l} \times_{\mathbf{Z}_2} \mathbf{Z}_4, 1 \times i\right], \left[RP^J, \xi^J\right]\right)$$

and

$$y_{(2l+1,J')} = \left([S^{2l+1}, i], \left[RP^{J'}, \xi^{J'}\right]\right)$$

where $[RP^J, \xi^J]$ and $[RP^{J'}, \xi^{J'}]$ are generators of $\mathscr{N}_{n-2l}(BSO_k)$ and $\mathscr{N}_{n-2l-1}(BSO_k)$ respectively (obs. $m = n + k$).

Thus, as in [5; 6.2] we have

$$\rho \circ \partial(\alpha) = \begin{cases} 0 & \text{if } \alpha = y_{(2l, J)} \\ [S^{2l+1}, -1][S(\xi^{J'}), -1] & \text{if } \alpha = y_{(2l+1, J')} \end{cases}$$

Moreover, the $\mathbf{Z}_2$-fixed point set of the generators are

$$\overline{F}_{\mathbf{Z}_2}(\alpha) = \begin{cases} [RP^{2l} \times RP^J][S^0, -1] & \text{if } \alpha = y_{(2l, J)} \\ [\mathbf{C}P^l \times RP^{J'}][S^1, -1] & \text{if } \alpha = y_{(2l+1, J')} \end{cases}$$

Now, taking the map

$$f\colon RP^{2l+1} \times RP^{J'} \to RP^\infty$$

that classifies the bundle $[RP^{2l+1} \times RP^{J'}, \gamma_1 \otimes \gamma_2]$ with γ_1 the line bundle over RP^{2l+1} and γ_2 the line bundle over $RP^{J'}$, we have that the Whitney number $\langle cw_{m-2}, \sigma_{m-1} \rangle$ of the map f, where $c = \alpha_{2l+1} \times 1$ and α_{2l+1} is the generator of $H^1(RP^{2l+1}; \mathbf{Z}_2)$, is given by

$$\begin{aligned} \langle cw_{m-2}, \sigma_{m-1} \rangle &= \langle (\alpha_{2l+1} \times 1) w_{m-2}, \sigma_{m-1} \rangle \\ &= \left\langle (\alpha_{2l+1} \times 1) \binom{2l+2}{2l} \alpha_{2l+1}^{2l} \times \mathscr{X}(RP(\xi^{J'}), \sigma_{m-1} \right\rangle \\ &= \left\langle (\alpha_{2l+1} \times 1) \binom{2l+2}{2l} \times \mathscr{X}(RP^{J'}) \mathscr{X}(RP^{k-1}), \sigma_{m-1} \right\rangle \end{aligned}$$

Further, we have

$$\mathscr{X}(\mathbf{C}P^l \times RP^{J'}) \equiv \binom{l+1}{l} \beta^l \times \mathscr{X}(RP^{J'}) \mod 2,$$

where β is the generator of $H^2(\mathbf{C}P^l; \mathbf{Z}_2)$.

Next, observe that $\mathscr{X}(\mathbf{C}P^l \times RP^{J'}) \equiv \langle cw_{m-2}, \sigma_{m-1} \rangle$ and $\mathscr{X}(RP^{2l} \times RP^J) \equiv 0 \mod 2$, since the dimension of $RP^{2l} \times RP^J$ is $2l + (n - 2l) = n$ odd.

Finally, it is easy to see that these facts don't depend on k, since k is odd. Hence, if

$$x = \sum (a_{l,J} y_{(2l,J)} + b_{l,J'} y_{(2l+1,J')}),$$

with $a_{l,J}, b_{l,J'} \in \mathbf{Z}_2$ is in the kernel of $cw_{m-2} \circ \rho \circ \partial$ restricted to the summands with k odd, then we can see that the $\mathbf{Z}_2$-fixed point set of x is in $\mathscr{X}_*[S^0, -1] + \mathscr{X}_{*-1}[S^1, -1]$.

Next, consider the homomorphism

$$cw_{m-2} \circ \rho \circ \partial \colon \bigoplus_{k \le m} \mathscr{N}_{m-k}^{\mathbf{Z}_2}(free)(BO_k) \to \mathbf{Z}_2$$

where $cw_{m-2}\colon \mathscr{N}_{m-1}^{\mathbf{Z}_2}(free) \to \mathbf{Z}_2$ maps α into the Whitney number $\langle cw_{m-2}, [\alpha] \rangle$.

THEOREM 3.3. *For m even, the homomorphism $cw_{m-2} \circ \rho \circ \partial$ restricted to the summands with k even is the zero homomorphism.*

Proof. Take $m = n + k$, $k = 2j$ even. Let ξ^k be a k-bundle over M^n with M^n having a $\mathbf{Z}_4$-action such that the restriction to $\mathbf{Z}_2$ acts trivially. Further, this $\mathbf{Z}_4$-action is covered by a $\mathbf{Z}_4$-action on the total space of ξ^k and the induced $\mathbf{Z}_2$-action acts by multiplication by -1 in the fibers of ξ^k covering a free $\mathbf{Z}_2$-action on the base.

Observe that $\rho \circ \partial([\xi^k, M^n]) = [RP(\xi^k), \lambda]$, where $RP(\xi^k)$ is the associated $(m-1)$-dimensional projective space and λ is the canonical line bundle over $RP(\xi^k)$. Next, the total Stiefel-Whitney class of $RP(\xi^k)$ is given by

$$W(RP(\xi^k)) = W(M) + \left(\sum_{i=0}^{k} (1+c)^{k-i} v_i \right)$$

where $v = \sum_{i=0}^{k} v_i$ is the total Whitney class of ξ^k. Moreover, we have the relation $\sum_{i=0}^{k} c^{k-i} v_i = 0$.

Therefore, the Whitney number $\langle cw_{m-2}, [RP(\xi^k)] \rangle$ is

$$\begin{aligned} \langle cw_{m-2}, [RP(\xi^k)] \rangle = {} & \left\langle cw_n(M) \left\{ \binom{k}{k-2} c^{k-2} + \binom{k-1}{k-3} c^{k-3} v_1 \right.\right. \\ & \left.\left. \qquad + \cdots + v_{k-2} \right\}, [RP(\xi^k)] \right\rangle \\ & + \left\langle cw_{n-1}(M) \left\{ \binom{k}{k-1} c^{k-1} \right.\right. \\ & \left. \qquad + \binom{k-1}{k-2} c^{k-2} v_1 + \cdots + v_{k-1} \right\}, \\ & \left. [RP(\xi^k)] \right\rangle \end{aligned}$$

Now, since $k = 2j$ and M is n-dimensional, we have

$$\begin{aligned}\langle cw_{m-2}, [RP(\xi^k)]\rangle &\equiv \langle jw_n(M)c^{k-1} + v_1 w_{n-1}(M)c^{k-1}, [RP(\xi^k)]\rangle \\ &\equiv j\mathscr{X}[M] + \langle v_1 w_{n-1}(M), [M]\rangle \\ &\equiv \langle v_1 w_{n-1}, (M), [M]\rangle\end{aligned}$$

since $\mathscr{X}[M] \equiv 0 \bmod 2$ due to the fact that we have a free $\mathbf{Z}_2$-action on M.

Next, we are going to see that $\langle v_1 w_{n-1}(M), [M]\rangle \equiv 0 \bmod 2$. First, recall that $v_1 = w_1$ (det ξ^k), where det ξ^k is the determinant bundle of ξ^k. Moreover, we have det $\xi^k = \wedge^k \xi^k$ the k-exterior power of the bundle ξ^k. So, we can see that the $\mathbf{Z}_4$-action T on ξ^k induce a $\mathbf{Z}_2$-action on det ξ^k. In fact, let $x = x_1 \wedge x_2 \wedge \cdots \wedge x_k$ be in $\wedge^k \xi^k$ with $x_i \in \xi^k$. Then $T^2(x) = (-x_1) \wedge (-x_2) \wedge \cdots \wedge (-x_k) = x$ since k is even.

Therefore, we get the commutative diagram

$$\begin{array}{ccc} \det \xi^k & \longrightarrow & (\det \xi^k)/\mathbf{Z}_2 \\ \downarrow & & \downarrow \\ M & \xrightarrow{\pi} & M/\mathbf{Z}_2 \end{array}$$

with det ξ^k having a $\mathbf{Z}_2$-action covering a free $\mathbf{Z}_2$-action on M. Thus,

$$\begin{aligned}\langle v_1 w_{n-1}(M), [M]\rangle &= \langle w_1(\det \xi^k) w_{n-1}(M), [M]\rangle \\ &= \langle \pi^*\big(w_1((\det \xi^k)/\mathbf{Z}_2) w_{n-1}(M/\mathbf{Z}_2)\big), [M]\rangle \\ &= \langle w_1((\det \xi^k)/\mathbf{Z}_2) w_{n-1}(M/\mathbf{Z}_2), \pi_*[M]\rangle \\ &\equiv 0 \quad \bmod 2,\end{aligned}$$

since $\pi_*[M] = 2[M/\mathbf{Z}_2] \equiv 0 \bmod 2$.

THEOREM 3.4. *For m even, if α is in the kernel of the boundary homomorphism ∂, then the $\mathbf{Z}_2$-fixed point set of α is in*

$$\bigoplus_{\substack{j=0 \\ j \text{ even}}} \mathscr{N}_j^{\mathbf{Z}_2}(\mathit{free}) + \mathscr{X}_*[S^0, -1] + \mathscr{X}_{*-1}[S^1, -1].$$

Proof. We have $cw_{m-2} \circ \rho \circ \partial(\alpha) = 0$, since $\partial(\alpha) = 0$ Therefore, by (3.2) and (3.3) the result follows at once.

LEMMA 3.5. *Let $[N^n, t]$ be in $\mathscr{N}_*^{\mathbf{Z}_2}(\mathit{free})$. For $k \geq 0$, there exists a stationary point free $\mathbf{Z}_4$-action $[W^{n+2k}, T]$ such that the $\mathbf{Z}_2$-fixed point set is $[N^n, t]$.*

Proof. Suppose $k > 0$ and consider the $\mathbf{Z}_4$-action $[RP^{2k} \times N, T \times t]$, where

$$T\colon [x_0, x_1, \ldots, x_{2k}] \mapsto [x_0, -x_2, x_1, \ldots, -x_{2k}, x_{2k-1}].$$

The $\mathbf{Z}_2$-fixed point set is the class $[N, t] + [RP^{2k-1} \times N,\ i \times t]$ which is equal to $[N, t]$ since the free involution $[RP^{2k-1}, i]$ bounds as involution and then $[RP^{2k-1} \times N,\ i \times t]$ bounds as free involution.

Finally, for $k = 0$, taking $[N, t]$ as stationary point free $\mathbf{Z}_4$-action, the $\mathbf{Z}_2$-fixed point set is $[N, t]$.

Next, denote by $I_m^{\mathbf{Z}_2}$ the image of the homomorphism $F_m^{\mathbf{Z}_2}$. Considering $A_m = (\bigoplus_{j \le m} \mathcal{N}_j) \cap \mathcal{X}_*$, we have the following lemma.

LEMMA 3.6. $A_m[S^0, -1] + A_{m-1}[S^1, -1] \subset I_m^{\mathbf{Z}_2}$

Proof. If $[N] \in A_m$, by Theorem (2.3) there exists an involution $[W_1^m, t_1]$ with the fixed point set bordant to N. Thus, the stationary point free $\mathbf{Z}_4$-action $[W_1^m \times_{\mathbf{Z}_2} \mathbf{Z}_4, t_1 \times i]$ has $\mathbf{Z}_2$-fixed point set bordant to $[N][S^0, -1]$. Therefore, $A_m[S^0, -1] \subset I_m^{\mathbf{Z}_2}$.

Now, if $[M] \in A_{m-1}$, again by (2.3) there exists an involution $[W_2^{m-1}, t_2]$ such that the fixed point set is $[M]$. Then, the $\mathbf{Z}_4$-action $[(W_2^{m-1} \times S^1)/(t_2 \times -1), 1 \times i]$ has the class $[M][S^1, -1]$ as $\mathbf{Z}_2$-fixed point set. Hence, $A_{m-1}[S^1, -1] \subset I_m^{\mathbf{Z}_2}$ and the lemma holds.

Now, we can state the main result of this section.

THEOREM 3.7. (a) *For m odd*,

$$I_m^{\mathbf{Z}_2} = \bigoplus_{\substack{j=1\\ j\ \mathrm{odd}}}^{m} \mathcal{N}_j^{\mathbf{Z}_2}(\mathit{free}) + A_m[S^0, -1] + \left(\bigoplus_{j=0}^{m-1} \mathcal{N}_j\right)[S^1, -1]$$

(b) For m even,

$$I_m^{\mathbf{Z}_2} = \bigoplus_{\substack{j=0\\ j\ \mathrm{even}}}^{m} \mathcal{N}_j^{\mathbf{Z}_2}(\mathit{free}) + A_m[S^0, -1] + A_{m-1}[S^1, -1]$$

Proof. First, since m is odd, then using [5; 5.1] and [3; 27.2], it is easy to see that

$$I_m^{Z_2} \subset \bigoplus_{\substack{j=1\\ j\ \mathrm{odd}}}^{m} \mathcal{N}_j^{Z_2}(\mathit{free}) + \mathcal{X}_*[S^0, -1] + \mathcal{N}_*[S^1, -1]$$

Now, if j is odd then $m-j$ is even and Lemma 3.5 implies that $\mathcal{N}_j^{\mathbf{Z}_2}(free) \subset I_m^{\mathbf{Z}_2}$.

Further, note that if we have N^{j-1}, $j\, odd$, then $[N^{j-1}][S^1, -1]$ belongs to $I_m^{\mathbf{Z}_2}$ by Lemma 3.5 since the codimension is even; and if j is even $[N^{j-1}][S^1, -1]$ belongs to $I_m^{\mathbf{Z}_2}$ by Lemma 3.6 since $\mathscr{X}(N^{j-1}) \equiv 0 \bmod 2$.

Hence, applying Lemma (3.6) again, part (a) of the theorem follows at once.

(b) By Theorem 3.4 we have

$$I_m^{\mathbf{Z}_2} \subset \bigoplus_{\substack{j=0 \\ j \text{ even}}} N_j^{\mathbf{Z}_2}(free) + \mathscr{X}_*[S^0, -1] + \mathscr{X}_{*-1}[S^1, -1].$$

Now, considering j even, Lemma 3.5 implies $\mathcal{N}_j^{\mathbf{Z}_2}(free) \subset I_m^{\mathbf{Z}_2}$ since the codimension is even. Therefore, applying Lemma 3.6 we have the result.

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