ILLINOIS JOURNAL OF MATHEMATICS
Volume 36, Number 4, Winter 1992

ON QUOTIENTS OF BANACH SPACES HAVING SHRINKING UNCONDITIONAL BASES

BY

E. ODELL[1]

Introduction

We shall say that a Banach space Y has *property* (WU) if every normalized weakly null sequence in Y has an unconditional subsequence. The well known example of Maurey and Rosenthal [MR] shows that not every Banach space has property (WU) (see also [O]). W.B. Johnson [J] proved that if Y is a quotient of a Banach space X having a shrinking unconditional f.d.d. and the quotient map does not fix a copy of c_0, then Y has (WU). Our main result extends this (and solves Problem IV.1 of [J]).

THEOREM A. *Let X be a Banach space having a shrinking unconditional finite dimensional decomposition. Then every quotient of X has property* (WU).

Of course such an X will itself have property (WU). Furthermore, if (E_n) is an unconditional f.d.d. (finite dimensional decomposition) for X, then (E_n) is shrinking if and only if X does not contain l_1.

The proof of Theorem A yields:

THEOREM B. *Let Y be a Banach space which is a quotient of S, the Schreier space. Then Y is c_0-saturated.*

Y is said to be *c_0-saturated* if every infinite dimensional subspace of Y contains an isomorph of c_0.

Our notation is standard as may be found in the books of Lindenstrauss and Tzafriri [LT 1, 2]. The proof of Theorem A is given in §1 and the proof of Theorem B appears in §2. §3 contains some open problems. We thank H. Knaust, H. Rosenthal and T. Schlumprecht for useful conversations regarding the results contained herein.

Received February 4, 1991
1991 Mathematics Subject Classification. Primary 46B45; Secondary 46A35.
[1]Research partially supported by a grant from the National Science Foundation.

1. The proof of Theorem A

Let T be a bounded linear operator from X onto Y where X has a shrinking unconditional f.d.d., $(\tilde{\tilde{E}}_i)$. By renorming if necessary we may suppose that $(\tilde{\tilde{E}}_i)$ is 1-unconditional. Y^* is separable and so by a theorem of Zippin [Z] we may assume that Y is a subspace of a Banach space Z possessing a bimonotone shrinking basis, (z_i). Fix $C > 0$ such that

$$T(CB_aX) \supseteq B_aY \equiv \{y \in Y : \|y\| \leq 1\}.$$

Recall that $(\tilde{E}_i)$ is a *blocking* of $(\tilde{\tilde{E}}_i)$ if there exist integers $0 = q_0 < q_1 < q_2 < \cdots$ such that $\tilde{E}_i = [\tilde{\tilde{E}}_j]_{j=q_{i-1}+1}^{q_i}$ for all i (where $[\cdots]$ denotes the closed linear span). Similarly, $\tilde{F}_i = [z_j]_{j=q_{i-1}+1}^{q_i}$ defines a blocking of (z_i).

Fix a sequence $\varepsilon_{-1} > \varepsilon_0 > \varepsilon_1 > \varepsilon_2 > \cdots$ converging to 0 which satisfies

$$\sum_{i=-1}^{\infty} \varepsilon_i < 1/4 \quad \text{and} \quad \sum_{i=p}^{\infty} (4i+2)\varepsilon_i < \varepsilon_{p-1} \quad \text{for } p \geq 0. \tag{1.1}$$

Then choose $\tilde{\varepsilon}_0 > \tilde{\varepsilon}_1 > \cdots$ converging to 0 which satisfies

$$4p\tilde{\varepsilon}_p < \varepsilon_{p+2} \quad \text{for } p \geq 1 \quad \text{and} \quad \sum_{j=p+1}^{\infty} \tilde{\varepsilon}_j < \tilde{\varepsilon}_p \quad \text{for } p \geq 0. \tag{1.2}$$

Our first step is the blocking technique of Johnson and Zippin.

LEMMA 1.1 [JZ 1,2]. *There exist blockings $(\tilde{E}_i)$ and $(\tilde{F}_i)$ of $(\tilde{\tilde{E}}_i)$ and (z_i), respectively, such that if $(\tilde{Q}_i)$ is the natural projection of Z onto $\tilde{F}_i$ then*

(1.3) *for all $i \in \mathbf{N}$ and $x \in \tilde{E}_i$ with $\|x\| \leq C$, we have $\|\tilde{Q}_jTx\| < \tilde{\varepsilon}_{\max(i,j)}$ if $j \neq i, i-1$.*

Roughly, this says that $T\tilde{E}_i$ is essentially contained in $\tilde{F}_{i-1} + \tilde{F}_i$ (where $\tilde{F}_0 = \{0\}$). Let (y_i'') be a normalized weakly null sequence in Y. Choose a subsequence (y_i') of (y_i) and a blocking (F_i) of $(\tilde{F}_i)$, given by $F_i = [\tilde{F}_j]_{j=q_{i-1}+1}^{q_i}$, such that if $Q_i = \sum_{j=q_{i-1}+1}^{q_i} \tilde{Q}_j$ is the natural projection of Z onto F_i, then

$$\|Q_jy_i'\| < \tilde{\varepsilon}_{\max(i,j)} \text{ if } i \neq j. \tag{1.4}$$

Roughly, y_i' is essentially in F_i. Furthermore we may assume that

$$\left\|\sum a_iy_i'\right\| = 1 \text{ implies } \max|a_i| \leq 2. \tag{1.5}$$

Let (E_i) be the blocking of $(\tilde{E}_i)$ given by the *same* sequence (q_i) which defined (F_i), $E_i = [\tilde{E}_j]_{j=q_{i-1}+1}^{q_i}$.

We begin with a sequence of elementary technical yet necessary lemmas. For $I \subseteq \mathbf{N}$ we define $Q_I = \sum_{j \in I} Q_j$ and set $Q_\varnothing = 0$.

LEMMA 1.2. *Let $0 < n < m$ be integers and let $y = \sum_{i \notin (n,m)} a_i y_i'$ with $\|y\| = 1$. Then for $j \in (n,m)$, $\|Q_j y\| < \varepsilon_j$ and $\|Q_{(n,m)} y\| < \varepsilon_n$.*

Proof. Let $n < j < m$. Then by (1.5), (1.4), (1.2) and (1.3),

$$\begin{aligned}\|Q_j y\| &\le 2\Big(\sum_{i \le n} \|Q_j y_i'\| + \sum_{i \ge m} \|Q_j y_i'\|\Big)\\ &< 2\big(n\tilde{\varepsilon}_j + \tilde{\varepsilon}_{m-1}\big)\\ &\le (2j+2)\tilde{\varepsilon}_j \le 4j\tilde{\varepsilon}_j < \varepsilon_j\end{aligned}$$

Thus $\|Q_{(n,m)} y\| < \sum_{j \in (n,m)} \varepsilon_j < \varepsilon_n$ by (1.1). ■

LEMMA 1.3. *Let $0 = p_0 < r_0 = 1 < p_1 < r_1 < p_2 < r_2 < \cdots$ be integers and let $y = \sum_{i=1}^{\infty} a_i y_{p_i}'$ with $\|y\| = 1$. Then for $i \in \mathbf{N}$,*

$$\|Q_{[r_{i-1}, r_i)} y - a_i y_{p_i}'\| < \varepsilon_{p_{i-1}-1}.$$

Proof.

$$\begin{aligned}&\|Q_{[r_{i-1}, r_i)} y - a_i y_{p_i}'\|\\ &\quad \le \Big\|Q_{[r_{i-1}, r_i)} \sum_{j \ne i} a_j y_{p_j}'\Big\| + \|Q_{[r_{i-1}, r_i)} a_i y_{p_i}' - a_i y_{p_i}'\|\end{aligned}$$

which by Lemma 1.2 is

$$\begin{aligned}&< \varepsilon_{r_{i-1}-1} + \|Q_{[1, r_{i-1})} a_i y_{p_i}'\| + \|Q_{[r_i, \infty)} a_i y_{p_i}'\|\\ &< \varepsilon_{r_{i-1}-1} + 2\sum_{k < r_{i-1}} \|Q_k y_{p_i}'\| + 2\varepsilon_{r_i - 1} \text{ (by (1.5) and Lemma 1.2)}\\ &< \epsilon_{r_{i-1}-1} + 2(r_{i-1} - 1)\tilde{\varepsilon}_{p_i} + 2\varepsilon_{r_i-1} \text{ (by (1.4))}\\ &\le \varepsilon_{p_{i-1}} + 2p_i\tilde{\varepsilon}_{p_i} + 2\varepsilon_{p_i} < \varepsilon_{p_{i-1}} + 4\varepsilon_{p_i} \text{ (by (1.2))}\\ &< \varepsilon_{p_{i-1}-1} \text{ (by 1.1)}.\end{aligned}$$

■

LEMMA 1.4. *Let* $i \in \mathbf{N}$, $x \in E_i$ *and* $\|x\| \le C$. *Then*

$$\|Q_j Tx\| < \varepsilon_{\max(i,j)} \qquad \text{if } j \ne i, i-1,$$
$$\|Q_{[1,i-2]} Tx\| < \varepsilon_{i-1} \text{ and } \|Q_{(i,\infty)} Tx\| < \varepsilon_i.$$

Proof. Let $x = \sum_{l \in (q_{i-1}, q_i]} \omega_l$ with $\omega_l \in \tilde{E}_l$.

$$\|Q_j Tx\| \le \sum_{k \in (q_{j-1}, q_j]} \sum_{l \in (q_{i-1}, q_i]} \|\tilde{Q}_k T \omega_l\|.$$

If $j < i - 1$ this is

$$< \sum_{k \in (q_{j-1}, q_j]} \sum_{l \in (q_{i-1}, q_i]} \tilde{\varepsilon}_l \text{ (by (1.4))}$$
$$< q_j \tilde{\varepsilon}_{q_{i-1}} < q_{i-1} \tilde{\varepsilon}_{q_{i-1}} < \varepsilon_{q_{i-1}+2} < \varepsilon_i \text{ using (1.2);}$$

if $j > i$ this is

$$< \sum_{k \in (q_{j-1}, q_j]} \sum_{l \in (q_{i-1}, q_i]} \tilde{\varepsilon}_k$$
$$< \sum_{k \in (q_{j-1}, q_j]} q_i \tilde{\varepsilon}_k \le q_i \tilde{\varepsilon}_{q_{j-1}}$$
$$\le q_{j-1} \tilde{\varepsilon}_{q_{j-1}} < \varepsilon_{q_{j-1}+2} \le \varepsilon_{j+1} < \varepsilon_j.$$

Finally,

$$\|Q_{[1,i-2]} Tx\| \le \sum_{k=1}^{i-2} \|Q_k Tx\| < \sum_{k=1}^{i-2} \varepsilon_i = (i-2)\varepsilon_i < \varepsilon_{i-1}$$

and

$$\|Q_{(i,\infty)} Tx\| \le \sum_{k=i+1}^{\infty} \|Q_k Tx\| < \sum_{k=i+1}^{\infty} \varepsilon_k < \varepsilon_i. \qquad \blacksquare$$

LEMMA 1.5. *Let* $\|x\| \le C$, $x = \sum_{k \ne j, j+1} \omega_k$ *where* $\omega_k \in E_k$ *for all* k. *Then*

$$\|Q_j Tx\| < \varepsilon_{j-1}.$$

Proof. By Lemma 1.4,

$$\|Q_jTx\| \le \sum_{k\neq j, j+1} \|Q_jT\omega_k\| < \sum_{k<j} \varepsilon_j + \sum_{k>j+1} \varepsilon_k$$
$$< (j-1)\varepsilon_j + \varepsilon_j = j\varepsilon_j < \varepsilon_{j-1}. \qquad \blacksquare$$

LEMMA 1.6. *Let $1 \le n < m$ and $x = \sum\omega_j$, $\|x\| \le C$, with $\omega_j \in E_j$ for all j. Suppose that $\|Q_jTx\| < 2\varepsilon_{j-1}$ for $n < j < m$. Let $a_{j-1} = Q_{j-1}T\omega_j$ and $b_j = Q_jT\omega_j$. Then*
(a) *$\|a_j + b_j\| < 3\varepsilon_{j-1}$ for $n < j < m$ and*
(b) *$\|\sum_{j\in(r,s]}T\omega_j - (a_r + b_s)\| < 5\varepsilon_{r-1}$ if $n < r < s < m$.*

Proof. (a) Let $n < j < m$. By Lemma 1.5,

$$\|Q_jTx - (a_j + b_j)\| = \left\|Q_j\left(\sum_{i\neq j, j+1} T\omega_i\right)\right\| < \varepsilon_{j-1}.$$

Since $\|Q_jTx\| < 2\varepsilon_{j-1}$, (a) follows.
(b) Let $n < r < s < m$ and let $j \in (r, s]$. Then $T\omega_j = a_{j-1} + b_j + \gamma_j$ where $\|\gamma_j\| < 2\varepsilon_{j-1}$ by Lemma 1.4. Thus

$$\begin{aligned}\left\|\sum_{r+1}^{s} T\omega_j - (a_r + b_s)\right\| &\le \|a_r + b_{r+1} + a_{r+1} + b_{r+2} \\ &\quad + \cdots + a_{s-1} + b_s - (a_r + b_s)\| + \sum_{j=r+1}^{s} 2\varepsilon_{j-1} \\ &< \sum_{r+1}^{s-1} \|a_j + b_j\| + 2\varepsilon_{r-1} \\ &< \sum_{r+1}^{s-1} 3\varepsilon_{j-1} + 2\varepsilon_{r-1}\,(\text{by (a)}) \\ &< 5\varepsilon_{r-1}. \qquad \blacksquare\end{aligned}$$

We next come to the key lemma. Let (P_j) be the natural sequence of finite rank projections of X onto (E_j). For $I \subseteq \mathbf{N}$, we let $P_I = \sum_{i\in I} P_i$.
Notation. If $x = \sum x_j \in X$ with $x_j \in E_j$ for all j and $\bar{x} \in X$, we define

$$\bar{x} \precsim x \qquad \text{if } \bar{x} = \sum a_j x_j \text{ with } 0 \le a_j \le 1 \text{ for all } j.$$

LEMMA 1.7. *Let $n \in \mathbf{N}$ and let $\varepsilon > 0$. There exists $m \in \mathbf{N}$, $m > n + 1$, such that whenever $x \in CBaX$ with $\|Q_jTx\| < 2\varepsilon_{j-1}$ for all $j \in (n, m)$ then*

there exists $\bar{x} \precsim x$ *with*
(1) $\|Tx - T\bar{x}\| < \varepsilon$ *and*
(2) $P_r\bar{x} = 0$ *for some* $r \in (n, m)$.

Remark. Lemma 1.7 is the main difference between our result and Johnson's earlier special case [J]. In the case where T does not fix a copy of c_0, Johnson showed that one could take $\bar{x} = x - P_r(x)$ for some $r \in (n, m)$.
The proof of Lemma 1.7 requires the following key result.

SUBLEMMA 1.8. *Let* $n \in \mathbf{N}$ *and* $\varepsilon > 0$. *There exists an integer* $m = m(n, \varepsilon) > n + 1$ *satisfying the following. Let* $x \in CBaX$, $x = \Sigma\omega_j$ *with* $\omega_j \in E_j$ *for all* j. *Assume in addition that* $\|Q_jTx\| < 2\varepsilon_{j-1}$ *for* $j \in (n, m)$ *and set* $a_{j-1} = Q_{j-1}T\omega_j$. *Then there exist* $k \in \mathbf{N}$ *and integers* $n < i_1 < \cdots < i_k < m$ *such that*

$$k^{-1}\|a_{i_1} + a_{i_2} + \cdots + a_{i_k}\| < \varepsilon. \tag{1.6}$$

Proof of Lemma 1.7. Let $n \in \mathbf{N}$ and $\varepsilon > 0$. Choose $n_0 \geq n$ such that

$$\varepsilon_{n_0} < \varepsilon/15. \tag{1.7}$$

Let $m_1 = m(n_0 + 1, \varepsilon/3)$ be given by the sublemma and let $m = m(m_1, \varepsilon/3)$.
Let $x = \Sigma\omega_j \in CBaX$ with $\omega_j \in E_j$ for all j and suppose that $\|Q_jTx\| < 2\varepsilon_{j-1}$, $a_{j-1} = Q_{j-1}T\omega_j$ and $b_j = Q_jT\omega_j$ for $j \in (n, m)$. By our choice of m there exist integers k and K and integers $n \leq n_0 < n_0 + 1 < i_1 < i_2 < \cdots < i_k < m_1 < j_1 < \cdots < j_K < m$ such that

$$k^{-1}\|a_{i_1} + \cdots + a_{i_k}\| < \varepsilon/3 \tag{1.8}$$

and

$$K^{-1}\|a_{j_1} + \cdots a_{j_K}\| < \varepsilon/3. \tag{1.9}$$

Define

$$\bar{x} = \sum_1^{i_1}\omega_j + \frac{k-1}{k}\sum_{i_1+1}^{i_2}\omega_j + \cdots + \frac{1}{k}\sum_{i_{k-1}+1}^{i_k}\omega_j + \frac{0}{k}\sum_{i_k+1}^{j_1}\omega_j$$
$$+ \frac{1}{K}\sum_{j_1+1}^{j_2}\omega_j + \cdots + \frac{K}{K}\sum_{j_k+1}^{\infty}\omega_j.$$

Clearly (2) holds and we are left to check (1).

$$\|Tx - T\bar{x}\| = \Bigg\| \frac{1}{k} \sum_{i_1+1}^{i_2} T\omega_j + \frac{2}{k} \sum_{i_2+1}^{i_3} T\omega_j + \cdots + \frac{k}{k} \sum_{i_k+1}^{j_1} T\omega_j + \frac{K-1}{K} \sum_{j_1+1}^{j_2} T\omega_j + \cdots + \frac{1}{K} \sum_{j_{K-1}+1}^{j_K} T\omega_j \Bigg\|.$$

Thus by Lemma 1.6,

$$\|Tx - T\bar{x}\| \le \Bigg\| \frac{1}{k} a_{i_1} + \frac{1}{k} b_{i_2} + \frac{2}{k} a_{i_2} + \frac{2}{k} b_{i_3} + \cdots + \frac{k}{k} a_{i_k} + \frac{K}{K} b_{j_1} + \frac{K-1}{K} a_{j_1} + \frac{K-1}{K} b_{j_2} + \cdots + \frac{1}{K} a_{j_{K-1}} + \frac{1}{K} b_{j_K} \Bigg\| + k^{-1} \sum_{j=1}^{k} 5j\varepsilon_{i_j-1} + K^{-1} \sum_{l=1}^{K} 5l\varepsilon_{j_l-1}.$$

Now

$$k^{-1} \sum_{j=1}^{k} 5j\varepsilon_{i_j-1} \le 5 \sum_{j=1}^{k} \varepsilon_{i_j-1} < \varepsilon_{i_1-2} \le \varepsilon_{n_0}$$

and

$$K^{-1} \sum_{l=1}^{K} 5l\varepsilon_{j_l-1} < \varepsilon_{n_0}$$

as well.

Thus

$$\|Tx - T\bar{x}\| < k^{-1}\|a_{i_1} + \cdots + a_{i_k}\| + K^{-1}\|b_{j_1} + \cdots + b_{j_K}\| + \sum_{j=2}^{k} \|b_{i_j} + a_{i_j}\| + \sum_{l=1}^{K-1} \|b_{j_l} + a_{j_l}\| + 2\varepsilon_{n_0}.$$

Now

$$K^{-1}\|b_{j_1} + \cdots + b_{j_K}\| \le K^{-1}\|a_{j_1} + \cdots + a_{j_K}\| + K^{-1} \sum_{l=1}^{K} \|b_{j_l} + a_{j_l}\|.$$

Hence from (1.8), (1.9) and Lemma 1.6 we obtain

$$\|Tx - T\bar{x}\| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \sum_{j=2}^{k} 3\varepsilon_{i_j-1} + 2\sum_{l=1}^{K} 3\varepsilon_{j_l-1} + 2\varepsilon_{n_0}$$
$$< \frac{2\varepsilon}{3} + \varepsilon_{n_0} + 2\varepsilon_{n_0} + 2\varepsilon_{n_0} < \varepsilon$$

(by (1.7)). ■

Proof of Sublemma 1.8. If the sublemma fails then by a standard compactness argument we obtain $\omega_j \in E_j$ for $j \in \mathbf{N}$ such that for all m,

$$\left\| \sum_{j=1}^{m} \omega_j \right\| \leq C \text{ and } \left\| Q_j T\left(\sum_{i=1}^{m} \omega_i \right) \right\| \leq 3\varepsilon_{j-1}$$

if $n < j < m$. The extra ε_{j-1} comes from an application of Lemma 1.5. Furthermore setting $Q_{j-1}T\omega_j = a_{j-1}$ and $Q_jT\omega_j = b_j$ for $j \in \mathbf{N}$, then for all k and all $n < i_1 < \cdots < i_k$ we have

$$k^{-1}\|a_{i_1} + \cdots + a_{i_k}\| \geq \varepsilon. \tag{1.10}$$

Now $a_j \in F_j$ and (F_j) is a shrinking f.d.d. Thus $(a_j)_{j>n}$ is a seminormalized weakly null sequence. By (1.10) any spreading model of a subsequence of (a_j) must be equivalent to the unit vector basis of l_1 (see [BL] for basic information on spreading models). In particular we can choose an even integer k and integers $n < i_1 < \cdots < i_k$ such that

$$\|a_{i_1} - a_{i_2} + \cdots + a_{i_{k-1}} - a_{i_k}\| > C\|T\| + 1. \tag{1.11}$$

However,

$$C\|T\| \geq \left\| T\left(\sum_{i_1+1}^{i_2} \omega_j + \sum_{i_3+1}^{i_4} \omega_j + \cdots + \sum_{i_{k-1}+1}^{i_k} \omega_j \right) \right\|$$
$$\geq \|a_{i_1} + b_{i_2} + a_{i_3} + b_{i_4} + \cdots + a_{i_{k-1}} + b_{i_k}\|$$
$$- 5\sum_{j=1}^{k} \varepsilon_{i_j-1} \quad \text{(by Lemma 1.6)}.$$

Now $5\sum_{j=1}^{k} \varepsilon_{i_j-1} < \varepsilon_{i_1-2}$ and by Lemma 1.6 and (1.11)

$$\|a_{i_1} + b_{i_2} + \cdots + a_{i_{k-1}} + b_{i_k}\| \geq \|a_{i_1} - a_{i_2} + a_{i_3} - a_{i_4} + \cdots + a_{i_{k-1}} - a_{i_k}\|$$
$$- \sum_{j=1}^{k/2} \|a_{i_{2j}} + b_{i_{2j}}\| > C\|T\| + 1 - \sum_{j=1}^{k/2} 3\varepsilon_{i_{2j}-1}.$$

Thus

$$\begin{aligned} C\|T\| &> C\|T\| + 1 - \varepsilon_{i_1-2} - \varepsilon_{i_2-2} \\ &\geq C\|T\| + 1 - 2\varepsilon_{i_1-2} \\ &> C\|T\|, \end{aligned}$$

which is impossible. ■

Completion of the proof of Theorem A. Let the integer m given by Lemma 1.7 be denoted by $m = m(n; \epsilon)$. Choose $1 < p_1 < p_2 < \cdots$ such that for all i, $p_{i+1} - 1 \geq m(p_i; \varepsilon_{p_i})$. Let $(y_i) = (y'_{p_i})$. We shall prove that (y_i) is unconditional.

Let $y = \sum a_i y_i$, $\|y\| = 1$, $x \in CBaX$, $Tx = y$ and let $x = \sum_{i=0}^{\infty} g_i$ where $g_0 = P_{[1, p_1)}x$ and $g_i = P_{[p_i, p_i+1)}x$ for $i \geq 1$. We shall apply Lemma 1.7 to each g_i for $i \geq 1$. Fix $i \geq 1$ and let $(n, m) = (p_i, p_{i+1} - 1)$. Let $j \in (n, m)$. Then $\|Q_j y\| < \varepsilon_j$ by Lemma 1.2. Thus

$$\|Q_j Tx\| = \left\| Q_j T g_i + Q_j T \sum_{k \neq i} g_k \right\| < \varepsilon_j.$$

However $\|Q_j T \sum_{k \neq i} g_k\| < \varepsilon_{j-1}$ by Lemma 1.5 so $\|Q_j T g_i\| < \varepsilon_{j-1} + \varepsilon_j < 2\varepsilon_{j-1}$. Thus by Lemma 1.7 there exist $\bar{g}_i \precsim g_i$ and $r_i \in (p_i, p_{i+1} - 1)$ such that $P_{r_i}\bar{g}_i = 0$ and $\|Tg_i - T\bar{g}_i\| < \varepsilon_{p_i}$ for all $i \in \mathbf{N}$.

Let $\bar{x} = \sum_{i=0}^{\infty} \bar{g}_i = \sum_{i=1}^{\infty} \bar{x}_i$ where $\bar{g}_0 = g_0$ and $\bar{x}_i = P_{[r_{i-1}, r_i)}\bar{x}$ for $i \in \mathbf{N}$ ($r_0 = 1$). Of course, $\bar{x}_i = P_{(r_{i-1}, r_i)}\bar{x}$ if $i > 1$.

Claim. $\|T\bar{x}_i - a_i y_i\| < 4\varepsilon_{p_{i-1}-1}$ for $i \in \mathbf{N}$.

Indeed $\|Q_{[r_{i-1}, r_i)} y - a_i y_i\| < \varepsilon_{p_{i-1}-1}$ by Lemma 1.3. Thus the claim follows from the following:

Subclaim. $\|Q_{[r_{i-1}, r_i)} Tx - T\bar{x}_i\| < 3\varepsilon_{p_{i-1}-1}$.

To see this we first note that

$$\begin{aligned} &\|Q_{[r_{i-1}, r_i)} Tx - Q_{[r_{i-1}, r_i)} T(g_{i-1} + g_i + g_{i+1})\| \\ &\leq \sum_{k \in [r_{i-1}, r_i)} \left\| Q_k \sum_{j \neq i-1, i, i+1} Tg_j \right\| \\ &< \sum_{k \in [r_{i-1}, r_i)} \varepsilon_{k-1} \text{ (by Lemma 1.5)} \\ &< \varepsilon_{r_{i-1}-1}. \end{aligned}$$

Also

$$\begin{aligned}\|Q_{[r_{i-1},r_i)}T(g_{i-1}+g_i+g_{i+1}) - Q_{[r_{i-1},r_i)}T(\bar{g}_{i-1}+\bar{g}_i+\bar{g}_{i+1})\| \\ \leq \|T(g_{i-1}+g_i+g_{i+1}-\bar{g}_{i-1}-\bar{g}_i-\bar{g}_{i+1})\| \\ < \varepsilon_{p_{i-1}}+\varepsilon_{p_i}+\varepsilon_{p_{i+1}} < \varepsilon_{p_{i-1}-1}.\end{aligned}$$

Finally, applying Lemma 1.5 again we have

$$\begin{aligned}\left\|Q_{[r_{i-1},r_i)}\left[T(\bar{g}_{i-1}+\bar{g}_i+\bar{g}_{i+1}) - T(\bar{x}_i)\right]\right\| \\ < \varepsilon_{r_{i-1}-1},\end{aligned}$$

and the subclaim follows.

Let $\delta_i = \pm 1$. Then

$$\begin{aligned}\left\|\sum \delta_i a_i y_i\right\| &\leq \left\|\sum \delta_i (a_i y_i - T\bar{x}_i)\right\| + \left\|\sum \delta_i T\bar{x}_i\right\| \\ &< \sum 4\varepsilon_{p_{i-1}-1} + \|T\| \left\|\sum \delta_i \bar{x}_i\right\| \quad \text{(by the claim)} \\ &\leq 1 + C\|T\|.\end{aligned}$$

■

The proof of Theorem A yields the following:

PROPOSITION 1.9. *Let X have a shrinking K-unconditional f.d.d.* (E_i) *and let T be a bounded linear operator from X onto Y. Let* $T(CBaX) \supseteq BaY$. *Then if* $\varepsilon_i \downarrow 0$ *and if* (y_i') *is a normalized weakly null basic sequence in Y there exists a subsequence* (y_i) *of* (y_i') *and integers* $p_1 < p_2 < \cdots$ *with the following property. Let* $\|\sum a_i y_i\| \leq 2$. *Then there exists* $x = \sum x_i \in 2CKBaX$, (x_i) *a block basis of* (E_i), *such that*

$$\|Tx_i - a_i y_i\| < \varepsilon_i \quad \textit{for all } i.$$

Moreover there exist (r_i) *with* $0 = r_0 < p_1 < r_1 < p_2 < r_2 < \cdots$ *such that* $x_i \in [E_j]_{j \in (r_{i-1}, r_i)}$ *for all i.*

COROLLARY 1.10. *Let X have a shrinking K-unconditional f.d.d. and let T be a bounded linear operator from X onto the Banach space Y. Then Y contains* c_0 *if and only if T fixes a copy of* c_0.

Proof. If Y contains c_0 then there exists (see [Ja]) (y_i), a normalized sequence in Y, with $2^{-1} \leq \|\sum a_i y_i\| \leq 2$ if $(a_i) \in S_{c_0}$, the unit sphere of c_0. Let $\varepsilon_i \downarrow 0$ with $\sum \varepsilon_i < 1$. We may assume that (y_i) satisfies the conclusion of Proposition 1.9. Thus for all $n \in \mathbf{N}$ there exist

$$0 = r_0^n < p_1 < r_1^n < p_2 < r_2^n < \cdots$$

and $x_i^n \in [E_j]_{j\in(r_{i-1}^n, r_i^n)}$ such that if $x^n = \sum_{i\le n} x_i^n$, then $\|x^n\| \le 2CK$ and $\|Tx_i^n - y_i\| < \varepsilon_i$ for $i \le n$.

By passing to a subsequence $(x_i^{n_k})$ we may assume $\lim_{k\to\infty} r_i^{n_k} = r_i$ and $\lim_{k\to\infty} x_i^{n_k} = x_i$ exist for all $i \in \mathbf{N}$. Thus $x_i \in [E_j]_{j\in(r_{i-1}, r_i)}$ with $r_0 = 0 < r_1 < r_2 < \cdots, \|Tx_i - y_i\| < \varepsilon_i$ for all i and $\sup_n \|\sum_1^n x_i\| < \infty$. It follows that (x_i) is equivalent to the unit vector basis of c_0. Moreover if we choose $\omega_i \in \varepsilon_i CBaX$ with $T\omega_i = y_i - Tx_i$ then $T(x_i + \omega_i) = y_i$ and some subsequence of $(x_i + \omega_i)$ is also a c_0 basis. Hence T fixes c_0. ■

2. The proof of Theorem B

We begin by recalling the definition of the Schreier space S [S]. Let c_{00} be the linear space of all finitely supported real valued sequences. For $x = (c_i) \in c_{00}$ set

$$\|x\| = \max\left\{ \sum_{i=1}^{p} |c_{k_i}| : p \in \mathbf{N} \quad \text{and} \quad p \le k_1 < \cdots < k_p \right\}.$$

S is the completion of $(c_{00}, \|\cdot\|)$. We let $\|x\|_0$ denote the c_0-norm of x. The unit vector basis (e_n) is a shrinking 1-unconditional basis of S. S can be embedded into $C(\omega^\omega)$ and thus S is c_0-saturated.

Theorem B will follow from a quantitative version, Theorem B′ (below). Given a sequence (x_n), $\lambda > 0$ and F a finite nonempty subset of $\mathbf{N}$, $y = \lambda \sum_{n\in F} x_n$ is said to be a 1-*average* of (x_n). We say that a Banach space X has property-$S(1)$ if every normalized weakly null sequence in X admits a block basis of 1-averages which is equivalent to the unit vector basis of c_0. S has property-$S(1)$.

THEOREM B′. *Let Y be a quotient of S. Then Y has property-$S(1)$.*

We shall use the following result:

LEMMA 2.1. *Let (x_n) be a normalized weakly null sequence in S with $\lim_n \|x_n\|_0 = 0$. Then some subsequence of (x_n) is equivalent to the unit vector basis of c_0.*

Let T be a bounded linear operator from S onto a Banach space Y and let (y_i') be a normalized weakly null basic sequence in Y. Let $T(CBaS) \supseteq BaY$.

LEMMA 2.2. *If no block basis of 1-averages of (y_i') is equivalent to the unit vector basis of c_0, then there exists $\delta > 0$ such that if $x \in 3CBaS$, Tx is a 1-average of (y_i') and $\|Tx\| > 1/3$ then $\|x\|_0 > \delta$.*

Proof. If no such δ exists then there exists $(x_i) \subseteq 3CBaS$ with $\lim_i \|x_i\|_0 = 0, \|Tx_i\| > \frac{1}{3}$ and Tx_i a 1-average of (y_i') for all i. By Lemma 2.1

there exists a subsequence (x_i') of (x_i) which is equivalent to the unit vector basis of c_0. By passing to a further subsequence we may assume that (Tx_i') is a seminormalized weakly null basic sequence in $[(y_i')]$. Thus (Tx_i') is also equivalent to the unit vector basis of c_0. ■

Proof of Theorem B'. Let (y_i') be a normalized weakly null sequence in Y. If (y_i') fails the $S(1)$ property, choose $\delta > 0$ by Lemma 2.2. Let $(\varepsilon_i)_{i=1}^{\infty}$ be a sequence of positive numbers satisfying (recall $T(CBaS) \supseteq BaY$)

$$\sum_{i=1}^{\infty} \varepsilon_i < \min(\delta/(2C), 1). \tag{2.1}$$

Let (y_i) be the subsequence of (y_i') given by Proposition 1.9 for the sequence (ε_i).

Choose an even integer $m \in \mathbf{N}$ with

$$m > 8C/\delta. \tag{2.2}$$

From the theory of spreading models there exists $(z_i)_{i=1}^{2m}$, a finite subsequence of (y_i), such that setting $\lambda = \|\sum_{i=1}^{2m} z_i\|^{-1}$,

$$2 > \lambda \left\| \sum_{i \in F} z_i \right\| > 1/3. \tag{2.3}$$

whenever $F \subseteq \{1, \ldots, 2m\}$ with $|F| \geq m$.

Thus there exists

$$x = \sum_{i=1}^{2m} x_i \in 2CBaS$$

with (x_i) a block basis of (e_i) and $\|Tx_i - \lambda z_i\| < \varepsilon_i$ for $i \leq 2m$. For $i \leq 2m$ choose $\omega_i \in S$ with $T\omega_i = \lambda z_i - Tx_i$ and $\|\omega_i\| \leq C\varepsilon_i$. Hence $T(x_i + \omega_i) = \lambda z_i$.

Since $\|T(\sum_1^{2m}(x_i + \omega_i))\| > 1/3$, and

$$\left\| \sum_1^{2m} (x_i + \omega_i) \right\| \leq \left\| \sum_1^{2m} x_i \right\| + \sum_1^{2m} \|\omega_i\| < 2C + \sum_1^{\infty} \varepsilon_i C < 3C,$$

by Lemma 2.2 we have $\|\sum_1^{2m}(x_i + \omega_i\|_0 > \delta$. Since $\|\sum_1^{2m} \omega_i\|_0 \leq \|\sum_1^{2m} \omega_i\| < \delta/2$ by (2.1) there exists $i_1 \leq 2m$ with $\|x_{i_1}\|_0 > \delta/2$.

Now

$$\left\| T\left(\sum_{\substack{i=1\\ i\neq i_1}}^{2m} (x_i + \omega_i) \right) \right\| = \left\| \sum_{\substack{i=1\\ i\neq i_1}}^{2m} \lambda z_i \right\| > \frac{1}{3}$$

and so we may repeat the argument above finding $i_2 \neq i_1$ with $\|x_{i_2}\|_0 > \delta/2$. In fact by (2.3) we can repeat this m-times obtaining distinct integers $(i_k)_{k=1}^m \subseteq \{1,2,\dots,2m\}$ with $\|x_{i_k}\|_0 > \delta/2$ for $k \leq m$. But then

$$2C \geq \|x\| = \left\| \sum_{i=1}^{2m} x_i \right\| \geq \left\| \sum_{k=1}^{m} x_{i_k} \right\| \geq \sum_{k=m/2+1}^{m} \|x_{i_k}\|_0 \geq \delta m/4$$

which contradicts (2.2). ∎

3. Open problems

Our work suggests a number of problems, of which we list a few. For a more extensive list of related problems and an overview of the current state of infinite dimensional Banach space theory, see [R].

Problem 1. Let X be a Banach space having property (WU) which does not contain l_1 and let Y be a quotient of X. Does Y have property (WU)?

In light of Theorem A it is worth noting that $C(\omega^\omega)$ has property (WU) [MR] but does not embed into any space having a shrinking unconditional f.d.d. In fact $C(\omega^\omega)$ is not even a subspace of a quotient of such a space. Indeed $C(\omega^\omega)$ fails property (U) (for example, see [HOR]) while any quotient of a space with a shrinking unconditional f.d.d. will have property (U). In fact if X has property (U) and does not contain l_1, then any quotient of X will have property (U) [R]. The next problem is due to H. Rosenthal.

Problem 2. Let X have a shrinking unconditional f.d.d. and let Y be a quotient of X. Does Y embed into a Banach space having a shrinking unconditional f.d.d.?

We say that a Banach space Y has uniform-(WU) if there exists $K < \infty$ such that every normalized weakly null sequence in Y has a K-unconditional subsequence. Our proof of Theorem A showed that the quotient space Y has uniform-(WU).

Problem 3. If Y has property (WU) does Y have uniform-(WU)?

Theorem B solved a special case of the following well known problem.

Problem 4. Let Y be a quotient of $C(\omega^\omega)$ (or more generally $C(K)$ where K is a compact countable metric space). Is Y c_0-saturated?

Regarding this problem, T. Schlumprecht [Sc] has observed that if Y is a quotient of $C(\omega^\omega)$, then the closed linear span of any normalized weakly null sequence in Y which has l_1 as a spreading model must contain c_0.

It is not true that the quotient of a c_0-saturated space must also be c_0-saturated. The separable Orlicz function space $H_M(0,1)$, with $M(x) = (e^{x^4} - 1)/(e - 1)$, considered in [CKT] is c_0-saturated and yet has l_2 as a quotient. We wish to thank S. Montgomery-Smith for bringing this fact to our attention. However this space does not have an unconditional basis and so we ask:

Problem 5. Let X be a c_0-saturated space with an unconditional basis and let Y be a quotient of X. Is Y c_0-saturated?

A more restricted and perhaps more accessible question is the following (S_n is defined below).

Problem 6. Let Y be a quotient of S_n, the nth-Schreier space, where $n \geq 2$. Is Y c_0-saturated? Does Y have property-$S(n)$?

S_n is defined as follows. Let $\|x\|_1$ be the Schreier norm. If $(S_n, \|\cdot\|_n)$ has been defined, set for $x \in c_{00}$, the finitely supported real sequences,

$$\|x\|_{n+1} = \max\left\{ \sum_{k=1}^{p} \|E_k x\|_n : p \leq E_1 < E_2 < \cdots E_p \right\}.$$

(Here $p \leq E_1$ means $p \leq \min E_1$ and $E_1 < E_2$ means $\max E_1 < \min E_2$. Also $Ex(i) = x(i)$ if $i \in E$ and 0 otherwise.) S_{n+1} is the completion of $(c_{00}, \|\cdot\|_{n+1})$. The unit vector basis (e_n) is a 1-unconditional shrinking basis for every S_n and S_n embeds into $C(\omega^{\omega^n})$.

Property-$S(n)$ is defined as follows. *n-averages* of a sequence (y_m) are defined inductively: an $n+1$-average of (y_m) is a 1-average of a block basis of normalized n-averages. Y has *property*-$S(n)$ if every normalized weakly null basic sequence in Y admits a block basis of n-averages equivalent to the unit vector basis of c_0. S_n has property-$S(n)$.

Added in proof. Denny Leung has solved Problem 5 in the negative.

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UNIVERSITY OF TEXAS
AUSTIN, TEXAS