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ON ONE-DIMENSIONAL METRIC FOLIATIONS IN EINSTEIN SPACES

BY

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1. Introduction

Let $(M, g, \mathscr{F})$ be a compact Einstein manifold with a one-dimensional metric foliation $\mathscr{F}$. We shall give a sufficient condition that the leaf space $M/\mathscr{F}$ admits a Kähler-Einstein Satake manifold structure in a natural way. The main result (Theorem 5) somewhat improves a well-known result of L. Bérard Bergery [Be, Theorem 9.76] for principal S^1-bundles with a compact Einstein basis.

2. A structure theorem of $(M, g, \mathscr{F})$

Let $\mathscr{F}$ be a one-dimensional metric foliation (equivalently, $\mathscr{F}$ is a Riemannian flow) on a compact Riemannian manifold (M, g) of dimension $m := k + 1$. The metric g induces the orthogonal splitting $TM = \mathscr{F} \oplus \mathscr{H}$, where $\mathscr{H}$ is identified with the normal bundle $Q := TM/\mathscr{F}$, by means of g.

Let ∇ be the Levi-Civita connection with curvature tensor R on (M, g). Let D be the canonical transversal Levi-Civita connection on $\mathscr{H}$ with respect to the metric $g|_{\mathscr{H}}$ and R^D its curvature tensor [KT2], [TV]. The O'Neill structure tensors T and A of type $(1,2)$ for $\mathscr{F}$ and $\mathscr{H}$ are naturally defined by

$$T_{E_1}E_2 := \mathscr{H}\nabla_{\mathscr{V}E_1}\mathscr{V}E_2 + \mathscr{V}\nabla_{\mathscr{V}E_1}\mathscr{H}E_2 \tag{2.1}$$

and

$$A_{E_1}E_2 := \mathscr{H}\nabla_{\mathscr{H}E_1}\mathscr{V}E_2 + \mathscr{V}\nabla_{\mathscr{H}E_1}\mathscr{H}E_2, \tag{2.2}$$

for arbitrary vector fields E_1 and E_2 on M. Here we denote by $\mathscr{V}(\ )$ and $\mathscr{H}(\ )$ the $\mathscr{F}$-part and $\mathscr{H}$-part of $(\ )$ respectively. Let $N := \operatorname{Trace} T$ be the mean curvature vector field for $\mathscr{F}$. Hereafter, we denote by V one of the two vertical vector fields of unit length, and by X, Y, Z basic vector fields.

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LEMMA 1. *If $\mathcal{F}$ is isoparametric, i.e., the mean curvature one-form κ, or equivalently N, of $\mathcal{F}$ is basic (see* [GG] *), then $A_X V$ is basic for any basic vector field X.*

Proof. We note that $A_X V$ is basic if and only if $Vg(A_X V, Y) = 0$ for any basic vector fields X and Y. Then we have, by using O'Neill's formulas,

$$\begin{aligned} -2Vg(A_X V, Y) &= 2Vg(A_X Y, V) \\ &= 2g((\nabla_V A)_X Y, V) + 2g(A_{\nabla_V X} Y + A_X \nabla_V Y, V) \\ &= g(\nabla_Y N, X) - g(\nabla_X N, Y) \\ &= d\kappa(Y, X), \end{aligned}$$

so that $A_X V$ is basic if and only if $d\kappa(X, Y) = 0$. The well-known fact [KT2] that κ is closed whenever it is basic completes the proof.

LEMMA 2. *There exists on M a Riemannian metric g with respect to which $\mathcal{F}$ is metric and the cohomology class $[\kappa] = 0$ if and only if there exists on M a Riemannian metric $\bar{g}$ with respect to which $\mathcal{F}$ is a geodesic, metric foliation.*

Proof. By assumption, $\kappa = df$ for a basic function f on M. We claim that $e^{-f}V$ is Killing. Obviously $(L_{e^{-f}V} g)(V, V) = 0$, and the vanishing of $(L_{e^{-f}V} g)(X, Y)$ follows from the fact that $[\Gamma(\mathcal{F}), \mathcal{B}] \subset \Gamma(\mathcal{F})$. Here and hereafter, we denote by $\mathcal{B}$ the space of basic vector fields for $\mathcal{F}$ and by $\Gamma(\quad)$ the space of sections of $(\quad)$. Since

$$g(V, [V, X]) = g(V, T_V X) = -\kappa(X),$$

we have

$$\begin{aligned} (L_{e^{-f}V} g)(V, X) &= -g([e^{-f}V, V], X) - g(V, [e^{-f}V, X]) \\ &= -g(V, e^{-f}[V, X] - X(e^{-f})V) \\ &= 0. \end{aligned}$$

Finally, if we renormalize the metric by

$$\bar{g} := e^{2f} g|_{\mathcal{F}} + g|_{\mathcal{H}},$$

$\bar{g}$ is a bundle-like metric and $e^{-f}V$ is a unit Killing vector field for $(M, \mathcal{F})$. Thus $\mathcal{F}$ is a geodesic, metric foliation with respect to $\bar{g}$.

The converse is trivial.

Hereafter, we denote g and V instead of $\bar{g}$ and $e^{-f}V$ given in the proof of Lemma 2 respectively. Then $\mathcal{F}$ is a one-dimensional geodesic, metric foliation generated by a unit Killing vector field V on (M, g). Now, we assume that $(M, g, \mathscr{F})$ is Einstein and transversally Einstein, i.e., $\mathrm{Ric} = c_1 g$ for a real constant c_1 and the transversal Ricci curvature tensor Ric^D satisfies $\mathrm{Ric}^D = c_2 g|_{\mathscr{H}}$ for a real constant c_2. Since $\mathscr{F}$ is a geodesic foliation, the O'Neill formulas for the curvatures imply that

$$\begin{aligned} c_1 g(X, X) &= \mathrm{Ric}(X, X) = g(R_{XV}X, V) + \sum_\alpha g(R_{XY_\alpha} X, Y_\alpha) \\ &= \sum_\alpha g\left(R^D_{XY_\alpha} X, Y_\alpha\right) - 2g(A_X V, A_X V) \\ &= c_2 g(X, X) - 2|A_X V|^2. \end{aligned}$$

Then, setting $c := (c_2 - c_1)/2$, where c is nonnegative and using the polarization trick, we have

$$g(A_X V, A_Y V) = c g(X, Y). \tag{2.3}$$

Note that $c = 0$ if and only if A is identically zero. In this case (M, g) is locally a Riemannian product. In what follows we exclude this case and then we may put $c = 1$.

Let J be the endomorphism of $\mathscr{H}$ defined by

$$J(X) := A_X V \quad (V \text{ is fixed}). \tag{2.4}$$

Then (2.3) implies that

$$g(J^2 X, Y) = g\left(A_{A_X V} V, Y\right) = -g(A_X V, A_Y V) = -g(X, Y),$$

i.e., $J^2 = -\mathrm{Id}$. Thus in this way we have an almost complex structure on $\mathscr{H}$, constant along the leaves by Lemma 1. Therefore, we can suppose that k is even, e.g., $k := 2n$.

LEMMA 3. *$\Omega(X, Y) := g(X, JY)$ is a basic closed 2-form for X and $Y \in \mathscr{B}$.*

Proof. Clearly Ω is a basic 2-form. Let θ be the dual one-form of V. Then we have

$$d\theta(X, Y) = -\theta([X, Y]) = -2g(A_X Y, V) = -2\Omega(X, Y),$$

which implies that Ω is closed.

LEMMA 4. *The transversal scalar curvature* Scal^D *of $\mathscr{F}$ is nonnegative.*

Proof. By the O'Neill formulas for the Ricci curvatures, we have

$$\int_M \mathrm{Ric}(V,V)\, d\mathrm{vol}_M = \int_M |A|^2\, d\mathrm{vol}_M, \tag{2.5}$$

$$\sum_\alpha \mathrm{Ric}(X_\alpha, X_\alpha) = \mathrm{Scal}^D - 2|A|^2. \tag{2.6}$$

By (2.5), we have $c_1 \geq 0$, and hence

$$\mathrm{Scal}^D = 2nc_1|X_\alpha|^2 + 2|A|^2 \geq 0. \tag{2.7}$$

Define a tensor $\mathscr{N}$ of type $(1,2)$ on $\mathscr{H}$ by

$$\mathscr{N}(X,X) := \mathscr{H}\{[X,Y] + J[JX,Y] + J[X,JY] - [JX,JY]\}, \quad X,Y \in \mathscr{B}, \tag{2.8}$$

which is a basic vector field. We say that J is integrable if $\mathscr{N}$ identically vanishes. Such a foliation $\mathscr{F}$ has a complex structure only in the normal direction [KT1].

Now by Lemma 4, $\mathscr{F}$ is a geodesic, transversally Einstein, metric foliation with nonnegative transversal scalar curvature. With an argument similar to the proof of Sekigawa [Se], we have

$$DJ = 0. \tag{2.9}$$

Thus by Lemma 3 and (2.9), $\mathscr{N}$ vanishes identically, i.e., J is a complex structure.

Now if all the leaves of $\mathscr{F}$ are closed, the leaf space $M/\mathscr{F}$ is a compact Satake manifold [Mo] with the almost Kähler structure $\tilde{J} := \pi_* J$, where $\pi : M \to M/\mathscr{F}$ is the canonical projection. Moreover, we have proved that J is a complex structure in the sense of Kamber-Tondeur [KT1]. Thus $M/\mathscr{F}$ is a compact Kähler-Einstein Satake manifold.

Summing up, we have:

THEOREM 5. *Let $(M, \mathscr{F})$ be a manifold of dimension $k+1$ with a one-dimensional foliation $\mathscr{F}$. Then there exists on M a Riemannian metric g with respect to which $\mathscr{F}$ is metric and the cohomology class $[\kappa] = 0$ if and only if there exists on M a Riemannian metric $\bar{g}$ with respect to which $\mathscr{F}$ is a geodesic, metric foliation. Moreover, if $(M, \bar{g}, \mathscr{F})$ is locally irreducible, Einstein and transversally Einstein, and if all the leaves of $\mathscr{F}$ are closed, then the leaf space $M/\mathscr{F}$ admits a natural compact Kähler-Einstein Satake metric whose Kähler form ω is given by $\Omega = \pi^*\omega$ up to a scalar factor, and the projection $\pi : M \to M/\mathscr{F}$ is a Satake morphism.*

Remarks. (A) In case that a leaf of $\mathscr{F}$ is not closed in the Theorem 5, the leaves of $\mathscr{F}$ are all diffeomorphic to $\mathbf{R}^1$. In this case, the behavior of leaves is more complicated. That is, since each leaf $\mathscr{L}$ is generated by a nonsingular unit Killing vector field on a compact Riemannian manifold (M, g), the closure $\bar{\mathscr{L}}$ of $\mathscr{L}$ is a compact, Abelian subgroup in the compact isometry group of (M, g), so a torus T^r of dimension r $(2 \leq r \leq m)$ (we refer to [Mo, Appendix A], [Ka]). But our arguments are not applicable when $\dim \mathscr{F} > 1$.

(B) A. Ranjan [R] proved that if Ric < 0, a compact Riemannian manifold M cannot have a one-dimensional metric foliation.

(C) Theorem 5 is related to the following result.

PROPOSITION 6 (See L. Bérard Bergery [Be, Theorem 9.76]). *Let (B, h) be a compact Einstein manifold and $\pi: M \to (B, h)$ be a principal S^1-bundle, classified by the integral cohomology of B. Then M admits a (unique) S^1-invariant Einstein Riemannian metric g such that π is a Riemannian submersion with totally geodesic fibres if and only if we have either* (a) $A = 0$, *and a finite covering of M is the Riemannian product $B \times S^1$, or* (b) $A \neq 0$, *and there exists on B a Kähler structure $(\tilde{J}, h, \omega)$ such that $\pi^*\omega = \Omega$.*

(D) Ph. Tondeur-L. Vanhecke [TV] proved that if a one-dimensional metric foliation on a locally irreducible symmetric space (M, g) with geodesic leaves is transversally symmetric, the ambient space (M, g) is of constant curvature, and conversely. And D. Gromoll-K. Grove [GG] also proved that if $\mathscr{F}$ is a one-dimensional metric foliation on a nonnegative constant curvature space, $\mathscr{F}$ is either flat or homogeneous, equivalently isoparametric.

(E) In case that (B, h, J, ω) is a Kähler-Einstein space of negative Ricci curvature, the metric g on M may be replaced by an Einstein Lorentz metric with signature $(1, \dim B)$. For such examples, see [Be], [Ma], [NT].

REFERENCES

[Be] A.L. BESSE, *Einstein manifolds*, Erg. Mat., vol. 3, Folge 10, Springer-Verlag, New York, 1987.

[GG] D. GROMOLL and K. GROVE, "One-dimensional metric foliations in constant curvature spaces" in *Differential geometry and complex analysis*, ed. I. Chavel and H.M. Farkas, Springer-Verlag, New York, 1985, 165–168.

[Ka] S. KASHIWABARA, *A fibering of Riemannian manifolds admitting* 1*-parameter groups of motions*, Tôhoku Math. J., vol. 17 (1965), pp. 266–270.

[KT1] F. KAMBER and PH. TONDEUR, *G-foliations and their characteristic classes*, Bull. Amer. Math. Soc., vol. 84 (1978), pp. 1086–1124.

[KT2] ______, "Foliations and metrics" in *Differential geometry*, eds. R. Brooks, A. Gray, B.L. Reinhart, Progress in Math., vol. 32, Birkhaüser, Boston, 1983, pp. 163–193.

[Ma] M.A. MAGID, *Submersion from anti-de Sitter space with totally geodesic fibres*, J. Differential Geom., vol. 16 (1981), pp. 323–331.

[Mo] P. MOLINO, *Riemannian foliations*, Progress in Math., vol. 73, Birkhäuser, Boston, 1988.

[NT] S. NISHIKAWA and PH. TONDEUR, *Transversal infinitesimal automorphisms for harmonic Kähler foliations*, Tohoku Math. J., vol. 40 (1988), pp. 599–611.

[O] B. O'NEILL, *The fundamental equations of a submersion*, Mich. Math. J., vol. 13 (1966), pp. 459–469.

[R] A. RANJAN, *Structural equations and integral formula for foliated manifolds*, Geom. Dedicata, vol. 20 (1986), pp. 85–91.

[Se] K. SEKIGAWA, *On some compact Einstein almost Kähler manifolds*, J. Math. Soc. Japan, vol. 39 (1987), pp. 677–684.

[TV] PH. TONDEUR and L. VANHECKE, *Transversally symmetric Riemannian foliations*, Tohoku Math. J., vol. 42 (1990), pp. 307–317.

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