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LOCAL REAL ANALYTIC BOUNDARY REGULARITY OF THE $\bar{\partial}$-EQUATION ON CONVEX DOMAINS

BY

ZHENHUA CHEN

1. Introduction

The global C^∞ boundary regularity of the $\bar{\partial}$-equation was proved by Kohn [KOH1], [KOH2] for general pseudoconvex domains. In the local case, the necessary and sufficient condition for the local C^∞ boundary regularity of the $\bar{\partial}$-equation is still unknown. The real analytic analogue of the C^∞ boundary regularity of the $\bar{\partial}$-equation was studied by many researchers [D-T], [KOM], [C1], [C2], [TA], [TR], [D-T2], [D-T3]. These real analytic boundary regularity results were proved for the canonical solution or Kohn's solution of the $\bar{\partial}$-equation. It was believed that local real analytic boundary regularity of the $\bar{\partial}$-equation holds for pseudoconvex domains of finite type. Recent counterexample of Christ and Geller [C-G] shows the local real analytic boundary regularity of the $\bar{\partial}$-equation does not hold for general pseudoconvex domains of finite type. The necessary and sufficient condition for local real analytic boundary regularity of the $\bar{\partial}$-equation is still unknown.

Using a well-known integral solution operator of the $\bar{\partial}$-equation obtained by Henkin and others, we obtain the following local real analytic boundary regularity result of the $\bar{\partial}$-equation for convex domains at the totally convex boundary points (terminology will be defined in Section 2).

THEOREM. *Let Ω be a bounded convex domain in $\mathbf{C}^n$ with C^2 boundary and ξ be a boundary point of Ω. If $\partial\Omega$ is totally convex at ξ in the complex tangential directions, then for any $\bar{\partial}$-closed $f \in C^1_{(p,q+1)}(\bar{\Omega})$ $(p, q > 0)$ which is real analytic at ξ, there exists a solution u of the $\bar{\partial}$-equation $\bar{\partial}u = f$ in Ω such that u is also real analytic at ξ.*

Our ad-hoc solution exists even in some cases where the real analytic regularity of the canonical solution has yet to be proved. For example, our

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theorem applies to the domain

$$\Omega = \left\{z = (z_1, \ldots, z_n) \in \mathbf{C}^n \colon |z_1|^{2m_1} + \cdots + |z_n|^{2m_n} < 1\right\}, \ m_1, \ldots, m_n \geq 1,$$

which is a pseudoconvex domain of finite type. Notice that in our theorem we did not assume any real analyticity of the boundary. So our theorem also applies to the domain

$$\Omega = \left\{z = (z_1, z_2) \in \mathbf{C}^2 \colon |z_1|^2 + e \cdot \exp \frac{-1}{|z_2|^2} < 1\right\},$$

which is a pseudoconvex domain of infinite type.

However, the region on which our ad-hoc solution is shown to be real analytic may be smaller than the region on which the datum f is known to be real analytic.

Here we would like to point out that the domain used by Christ and Geller in their counterexample does not satisfy the total convexity at the boundary point $z = 0$.

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2. Definitions

Let h be a function on a domain $\Omega \subseteq \mathbf{C}^n$ and ξ be a boundary point of Ω. We say that h is *real analytic* at ξ if h has a convergent power series expansion in a neighborhood U of ξ in $\mathbf{C}^n$. We say that a (p, q) form is real analytic at ξ if its coefficients are all real analytic at ξ.

The following total convexity was used by Range [R1] for studying the Caratheodory metric and holomorphic mappings. Here we use it as the basic assumption for our local real analytic boundary regularity result.

Definition. Let Ω be a convex domain in $\mathbf{C}^n$ and ξ be a C^1 boundary point of Ω. We say that $\partial\Omega$ is *totally convex at ξ in the complex tangential directions* if there exists a neighborhood U of ξ in $\mathbf{C}^n$ such that $\overline{\Omega} \cap (H_\zeta(\partial\Omega) + \{\zeta\}) = \{\zeta\}$, for any $\zeta \in \partial\Omega \cap U$, where $H_\zeta(\partial\Omega)$ is the complex tangent space of $\partial\Omega$ at ζ.

3. Proof of the Theorem

Suppose Ω is a bounded convex domain in $\mathbf{C}^n$ with C^2 boundary. We can construct an integral solution operator for the $\bar{\partial}$-equation on Ω as follows.

Let

$$\delta(\zeta) = \begin{cases} -\operatorname{dist}(\zeta, \partial\Omega), & \zeta \in \Omega; \\ \operatorname{dist}(\zeta, \partial\Omega), & \zeta \notin \Omega. \end{cases}$$

Set

$$\Phi(\zeta, z) = \sum_{j=1}^{n} \frac{\partial \delta}{\partial \zeta_j}(\zeta)(\zeta_j - z_j);$$

$$B = B(\zeta, z) = \frac{\partial_\zeta\left(|\zeta - z|^2\right)}{|\zeta - z|^2};$$

$$W(\zeta, z) = \sum_{j=1}^{n} \frac{\frac{\partial \delta}{\partial \zeta_j}(\zeta)}{\Phi(\zeta, z)} \, d\zeta_j;$$

$$\hat{W} = \hat{W}(\lambda, \zeta, z) = \lambda W(\zeta, z) + (1 - \lambda) B(\zeta, z);$$

$$\Omega_q(B(\zeta, z)) = C_{n,q} B \wedge \left(\bar{\partial}_\zeta B\right)^{n-q-1} \wedge \left(\bar{\partial}_z B\right)^q;$$

$$\Omega_q(\hat{W}) = C_{n,q} \hat{W} \wedge \left(\bar{\partial}_{\zeta,\lambda} \hat{W}\right)^{n-q-1} \wedge \left(\bar{\partial}_z \hat{W}\right)^q;$$

where

$$\bar{\partial}_{\zeta,\lambda} = \bar{\partial}_\zeta + d_\lambda, \quad \text{and} \quad C_{n,q} = \frac{(-1)^{q(q+3)/2}}{(2\pi i)^n} \binom{n-1}{q}.$$

Note that all of the preceding functions and forms in the last paragraph are real analytic in z away from the zeros of $\Phi(\zeta, z)$.

Let

$$T_q(h)(z) = \int_{(\zeta,\lambda) \in \partial\Omega \times I} h(\zeta) \wedge \Omega_q\left(\hat{W}(\lambda, \zeta, z)\right) - \int_{\zeta \in \Omega} h(\zeta) \wedge \Omega_q(B(\zeta, z)).$$

It is well known that for any given $\bar{\partial}$-closed form $f \in C^1_{(0,q+1)}(\overline{\Omega})$ $(q > 0)$,

$$u(z) = T_q(f)(z)$$

is a solution of the $\bar{\partial}$-equation $\bar{\partial} u = f$ in Ω (see, for example, [R2]).

Notice that $H_\xi(\partial\Omega) + \{\zeta\} = \{z \in \mathbf{C}^n\colon\ \Phi(\zeta, z) = 0\}$. By the continuity of the function $\Phi(\zeta, z)$, we have:

LEMMA 1. *Let Ω be a bounded convex domain in $\mathbf{C}^n$ with C^1 boundary. If $\partial\Omega$ is totally convex at a boundary point ξ of Ω in the complex tangential directions, then there exists a neighborhood U of ξ in $\mathbf{C}^n$ such that for any neighborhood $V \subset\subset U$ of ξ in $\mathbf{C}^n$ there exists a constant $m > 0$ such that*

$$|\Phi(\zeta, z)| \geq m \text{ for any } \zeta \in \partial\Omega \cap (\mathbf{C}^n \setminus U) \text{ and any } z \in \overline{\Omega} \cap \overline{V}.$$

Proof of the theorem. Let f be a $\bar{\partial}$-closed $(p, q+1)$ form with coefficients in $C^1(\overline{\Omega})$ and real analytic at ξ. Without loss of generality, we may assume $p = 0$. Let U_ξ be a small ball with center ξ such that f is real analytic in a neighborhood of $\overline{U}_\xi$. Let u_ξ be the canonical solution of the $\bar{\partial}$-equation $\bar{\partial}u_\xi = f$ in U_ξ. By the global real analytic regularity of the canonical solution on balls, we know that u_ξ is real analytic in $\overline{U}_\xi$. We now construct a global solution u of the equation $\bar{\partial}u = f$ in Ω such that u is real analytic at ξ. Let U be a neighborhood of ξ in $\mathbf{C}^n$ such that $U \subset\subset U_\xi$. Let $\psi \in C_0^\infty(U_\xi)$, such that $\psi = 1$ in U. Let $g = f - \bar{\partial}(\psi u_\xi)$. Then g is a $\bar{\partial}$-closed $(0, q+1)$ form with coefficients in $C^1(\overline{\Omega})$ and g is zero in $\Omega \cap U$. The kernel of T_q is obviously real analytic in the z variables when restricted to the support of g by Lemma 1. Thus $T_q(g)(z)$ is real analytic at ξ. Let $u = \psi u_\xi + T_q(g)$. Then u is a solution of the equation $\bar{\partial}u = f$ in Ω. Since both u_ξ and $T_q(g)$ are real analytic at ξ, and $\psi \equiv 1$ in U, it follows that u is real analytic at ξ.

This completes the proof.

REFERENCES

[C-G] M. CHRIST and D. GELLER, *Counterexamples to analytic hypoellipticity for domains of finite type*, preprint.

[C1] S.-C. CHEN, *Global analytic hypoellipticity of the $\bar{\partial}$-Neumann problem on circular domains*, Invent. Math, vol. 92 (1988), pp. 173–185.

[C2] S.-C. CHEN, *Global real analyticity of solutions to the $\bar{\partial}$-Neumann problem on Reinhardt domains*, Indiana Univ. Math. J., vol. 37 (1988), pp. 421–430.

[D-T1] M. DERRIDJ and D.S. TARTAKOFF, *On the global real analyticity for the $\bar{\partial}$-Neumann problem*, Comm. Partial Differential Equations, vol. 5 (1976), pp. 401–435.

[D-T2] ______, *Local Analyticity for $\square_b$ and the $\bar{\partial}$-Neumann problem at certain weakly pseudoconvex points*, preprint.

[D-T3] ______, *Local analyticity for the $\bar{\partial}$-Neumann problem and for $\square_b$ in some model domains without maximal estimates*, preprint.

[KOH1] J.J. KOHN, *Global regularity for $\bar{\partial}$ on weakly pseudoconvex manifolds*, Trans. Amer. Math. Soc., vol. 181 (1973), pp. 273–292.

[KOH2] ______, *Methods of partial differential equations in complex analysis*, Proc. Symp. Pure Math, number 2 (1977), pp. 215–237.

[KOM] G. KOMATSU, *Global analytic hypoellipticity of the $\bar{\partial}$-Neumann problem*, Tohoku Math. J., vol. 28 (1976), pp. 145–156.

[R1] R.M. RANGE, *The Caratheodory metric and holomorphic maps on a class of weakly pseudoconvex domains*, Pacific J. Math., vol. 78 (1978), pp. 173–188.
[R2] ______, *Holomorphic functions and integral representations in several complex variables*, Springer-Verlag, New York, 1986.
[TA] D.S. TARTAKOFF, *On the local real analyticity of solutions on $\square_b$ and the $\bar{\partial}$-Neumann problem*, Acta Math., vol. 145 (1980), pp. 117–204.
[TR] F. TRÈVES, *Analytic hypo-ellipticity of a class of pseudodifferential operators with double characteristics and application to the $\bar{\partial}$-Neumann problem*, Comm. in Partial Differential Equations, vol. 3 (1978), pp. 475–642.

UNIVERSITY OF MICHIGAN
ANN ARBOR, MICHIGAN