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SYMMETRIC CLOSED OPERATORS COMMUTING WITH A UNITARY TYPE I REPRESENTATION OF FINITE MULTIPLICITY ARE SELF-ADJOINT

BY

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Introduction

In the first part of this paper we prove the following:

THEOREM. *Let τ be a type I unitary representation of a group G whose direct integral decomposition has (a.e.) finite multiplicity. Let $(T, \mathscr{D}_T)$ be a densely defined symmetric operator such that $\tau(x)\mathscr{D}_T = \mathscr{D}_T$ and $\tau(x)Tf = T\tau(x)f$ for all $x \in G$ and $f \in \mathscr{D}_T$. Then T is essentially self-adjoint.*

In [1] van der Ban proves a similar result about selfadjointness for symmetric spaces of semisimple groups "filling some gaps in the argument of [4]" and proves the finiteness of multiplicities in the corresponding representation and on more general symmetric spaces. The two parts of the paper are independent; thus our result can be used with the second part of [1] to get the main theorem in the first part of [1].

For another application of the theorem see [2], where invariant differential operators occur in connection with nilpotent groups and induced representations.

In the second part of the paper we analyse commutative algebras of unbounded invariant operators such as occur in the papers [1] and [2].

In [7] we give an example of a left invariant symmetric differential operator on the Heisenberg group which fails to be essentially selfadjoint. (If X, Y, Z is the usual basis of the Lie algebra the operator $X^4 + Y^2$ is such an example). This shows that even in the case of an invariant symmetric differential operator on a homogeneous space the assumption of finite multiplicity in the above theorem is essential.

If τ is multiplicity-free (so that the commutant of τ is abelian) the above theorem is easy to prove: Since the closure of T is invariant we may assume

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T to be closed. Let $V = (T - iI)(T + iI)^{-1}$ be the Cayley transform of T extended by 0 on the orthogonal complement of the domain of $(T + iI)^{-1}$. Then V and V^* commute with τ, and hence $VV^* = V^*V$. Since V^*V is the projection on $\mathrm{Ker}(T^* - iI)^{\perp}$ and VV^* is the projection on $\mathrm{Ker}(T^* + iI)^{\perp}$ we must have $\mathrm{Ker}(T^* - iI) = \mathrm{Ker}(T^* + iI)$. Therefore both spaces are (0) and T is selfadjoint. (This argument is found in [6] where it is also shown that in the case of the hyperbolic spaces such as those considered in [4] the situation is multiplicity free). The more general result above also uses Cayley transforms, but requires more argument.

I should like to thank Larry Corwin for mentioning the problem in [2], for the hospitality at Rutgers, and for helping to transform a letter into the present paper.

Section 1

We begin with a summary of results about direct integrals as they apply to direct integrals and partial isometries. All operators we deal with will be closed, but need not be densely defined; the domain of an operator T will be denoted by $\mathscr{D}_T$, and $\mathscr{R}_T = \mathscr{R}(T)$ will denote the image $T(\mathscr{D}_T)$.

Suppose that A: $\mathscr{H} \to \mathscr{H}$ is a partial isometry in the Hilbert space $\mathscr{H}$, so that A^*A and AA^* are orthogonal projections. Let $V = A|_{(\mathrm{Ker}\, A)} \perp$. Then V: $\mathscr{D}_V \to \mathscr{H}$ is an isometry, with $\mathscr{D}_V = \mathscr{R}(A^*A)$ and $\mathscr{R}_V = \mathscr{R}(AA^*)$. Conversely, if V: $\mathscr{D}_V \to \mathscr{H}$ is an isometry ($\mathscr{D}_V \subset \mathscr{H}$) we get a partial isometry $A = \tilde{V}$ on $\mathscr{H}$ by defining $\tilde{V}$ to be V on $\mathscr{D}_V$ and 0 on $\mathscr{D}_V^{\perp}$. Notice that 1 is not an eigenvalue of V iff it is not an eigenvalue of $\tilde{V}$.

Now let $(T, \mathscr{D}_T)$ be a closed symmetric operator, not necessarily densely defined (that is, we assume $\langle Tf, g\rangle = \langle f, Tg\rangle$ for all f, $g \in \mathscr{D}_T$). The Cayley transform $V = (T - iI)(T + iI)^{-1}$ is an isometry with $\mathscr{D}_V = \mathscr{R}(T + iI)$ and $\mathscr{R}_V = \mathscr{R}(T - iI)$, and 1 is not an eigenvalue of V. The correspondence $T \mapsto V$ is bijective since

$$(1) \qquad T = i(I + V)(I - V)^{-1}.$$

Clearly V determines $\tilde{V}$. Thus we get a bijection between closed symmetric operators T and partial isometries $\tilde{V}$ that do not have 1 as an eigenvalue. (This is all standard; see e.g. [5, 13.19].)

Suppose τ is a unitary representation of the group G on $\mathscr{H}$; let T, V and $\tilde{V}$ be related as above. It is easy to check that the following are equivalent:

(a) $\tau(x)\mathscr{D}_T = \mathscr{D}_T$ and $\tau(x)T = T\tau(x)$, $\forall x \in G$.
(b) $\mathscr{D}_T$ and $\mathscr{R}_T$ are τ-invariant and $\tau(x)V = V\tau(x)$, $\forall x \in G$.
(c) $\tilde{V}\tau(x) = \tau(x)\tilde{V}$, $\forall x \in G$.

Now assume that $\mathscr{H} = \int^{\oplus} \mathscr{H}_\lambda \, d\mu(\lambda)$, $\tau = \int^{\oplus} \tau_\lambda \, d\mu(\lambda)$ is a standard direct integral decomposition of $\mathscr{H}$ giving a primary decomposition of τ. Let $\mathscr{A}$, $\mathscr{A}_\lambda$ be the commutants of τ, τ_λ in $\mathscr{H}, \mathscr{H}_\lambda$ respectively. Then (see [3, Ch II, §3]) $\mathscr{A} = \int^{\oplus} \mathscr{A}_\lambda \, d\mu(\lambda)$, that is, the bounded operators commuting with τ are precisely those of the form

$$A = \int^{\oplus} A_\lambda \, d\mu(\lambda) \tag{2}$$

with $A_\lambda \in \mathscr{A}_\lambda$ and $\|A\| = \operatorname{ess\,sup}_\lambda \|A_\lambda\| < \infty$.

The operator A is an orthogonal projection iff A_λ is an orthogonal projection for a.a. λ. Since $A^*A = \int^{\oplus} A_\lambda^* A_\lambda \, d\mu(\lambda)$ and $AA^* = \int^{\oplus} A_\lambda A_\lambda^* \, d\mu(\lambda)$ we see that A is a partial isometry iff A_λ is a partial isometry for a.a. λ. We also have

$$\operatorname{Ker} A = \int^{\oplus} \operatorname{Ker} A_\lambda \, d\mu(\lambda); \quad (\operatorname{Ker} A)^\perp = \int^{\oplus} (\operatorname{Ker} A_\lambda)^\perp \, d\mu(\lambda). \tag{3}$$

Suppose that (T_λ) is a family of closed operators with appropriate measurability properties (one property that suffices is that the $\tilde{V}_\lambda$ are a measurable family). Then we define an operator $T = \int^{\oplus} T_\lambda \, d\mu(\lambda)$ as follows:

$$\mathscr{D}_T = \left\{ f = (f_\lambda) : f_\lambda \in \mathscr{D}_{T_\lambda} \mu \text{ a.a. } \lambda, \ \lambda \mapsto f_\lambda, \ \lambda \mapsto T_\lambda f_\lambda \text{ are in } L^2 \right\},$$
$$Tf = (T_\lambda f_\lambda).$$

The operator T is closed since its graph G_T equals $\int^{\oplus} G_{T_\lambda} \, d\mu(\lambda)$. If $A = \int^{\oplus} A_\lambda \, d\mu(\lambda)$, the A_λ being bounded injective operators such that $\operatorname{ess\,sup}_\lambda \|A_\lambda\| < \infty$, then $A^{-1} = \int^{\oplus} A_\lambda^{-1} \, d\mu(\lambda)$ (even if the A_λ have domains that are proper closed subspaces of the $\mathscr{H}_\lambda$).

Now let T be the given densely defined symmetric operator commuting with τ. Without restricting the generality we may assume that T is closed. Then if V is its Cayley transform the corresponding partial isometry $\tilde{V}$ is bounded and commutes with τ. Therefore we have $\tilde{V} = \int^{\oplus} \tilde{V}_\lambda \, d\mu(\lambda)$ where the $\tilde{V}_\lambda$ are partial isometries commuting with τ_λ. Let V_λ be the isometry corresponding to $\tilde{V}_\lambda$. Then $\mathscr{D}_V = \int^{\oplus} \mathscr{D}_{V_\lambda} \, d\mu(\lambda)$ from (3), and

$$V = \int^{\oplus} V_\lambda \, d\mu(\lambda), \quad (I - V)^{-1} = \int^{\oplus} (I_\lambda - V_\lambda)^{-1} \, d\mu(\lambda). \tag{4}$$

Now (1) immediately gives

$$T \subseteq \int^{\oplus} T_\lambda \, d\mu(\lambda) \tag{5}$$

where $T_\lambda = i(I + V_\lambda)(I - V_\lambda)^{-1}$.

In the case of interest to us T will turn out to be self-adjoint (as will the T_λ), and we will get equality in (5) because self-adjoint operators are maximal symmetric. But it does not seem to be obvious whether equality generally holds in (5).

In the theorem, τ_λ is a finite multiple of an irreducible representation π_λ. Thus $\mathscr{H}_\lambda = \mathscr{K}_\lambda \otimes \mathbf{C}^{n_\lambda}$ and $\tau_\lambda = \pi_\lambda \otimes I_{n_\lambda}$, while the commutant $\mathscr{A}_\lambda = I_\lambda \otimes M_{n_\lambda}(\mathbf{C})$. We write a typical element $A_\lambda \in \mathscr{A}_\lambda$ as $I_\lambda \otimes a_\lambda$; then $A_\lambda^* = I_\lambda \otimes a_\lambda^*$ and A_λ is a partial isometry if and only if a_λ is. It is also clear that $\operatorname{Ker}(A_\lambda) = I_\lambda \otimes \operatorname{Ker}(a_\lambda)$. Let T_λ, V_λ and $\tilde{V}_\lambda$ be as above. Then we have $\tilde{V}_\lambda = I_\lambda \otimes \tilde{v}_\lambda, \tilde{v}_\lambda$ being a partial isometry in $\mathbf{C}^{n_\lambda}$, and $V_\lambda = I_\lambda \otimes v_\lambda$ where $v_\lambda = \tilde{v}_\lambda|_{\operatorname{Ker}(\tilde{v}_\lambda)} \perp$. Since 1 is not an eigenvalue of V_λ (or $\tilde{V}_\lambda$), it is also not an eigenvalue of v_λ. Let t_λ be the symmetric operator in $\mathbf{C}^{n_\lambda}$ whose Cayley transform is v_λ. Then

$$\mathscr{D}_{T_\lambda} = \mathscr{R}(I_\lambda - V_\lambda) = \mathscr{K}_\lambda \otimes \mathscr{R}(I_\lambda - v_\lambda) = \mathscr{K}_\lambda \otimes \mathscr{D}_{t_\lambda}$$

and $T_\lambda = I_\lambda \otimes t_\lambda$. The domain $\mathscr{D}_{T_\lambda}$ being a closed subspace of $\mathscr{H}_\lambda$, $\int^\oplus \mathscr{D}_{T_\lambda}\, d\mu(\lambda)$ is a closed subspace $\mathscr{H}_0$ of $\mathscr{H}$. From (5), $\mathscr{D}_T \subset \mathscr{H}_0$. Since T is densely defined, $\mathscr{H}_0 = \mathscr{H}$ and $\mathscr{D}_{T_\lambda} = \mathscr{H}_\lambda$ for a.a. λ. Thus $\mathscr{D}_{t_\lambda} = \mathbf{C}^{n_\lambda}$ and v_λ is unitary a.e. Therefore V_λ is unitary a.e. and V is unitary. But this means that T is self adjoint.

We have also shown that the T_λ are bounded self-adjoint operators commuting with the τ_λ and that if T is closed,

$$T = \int^\oplus T_\lambda\, d\mu(\lambda). \tag{6}$$

Section 2

In this section we first prove:

THEOREM 2. *Under the assumption of finite multiplicity in the primary decomposition* $\tau = \int^\oplus \tau_\lambda\, d\mu(\lambda)$, *any densely defined closed operator* T *commuting with* τ *has an essentially unique representation* (6), *the* T_λ *being bounded operators commuting with the* τ_λ.

Proof. Consider the operators $B = (I + T^*T)^{-1}$ and $C = T(I + T^*T)^{-1}$. These are bounded operators and they determine T as follows: Let T_0 be the restriction of T to the domain of the self-adjoint operator T^*T. Then $T_0 = CB^{-1}$ and $T = \overline{T_0}$. It is sufficient to prove the last statement. Let

(f, Tf) be an element of the graph of T orthogonal to the graph of T_0. Then

$$\langle f, g\rangle + \langle Tf, Tg\rangle = 0 \quad \text{for all } g \in \mathscr{D}_{T_0},$$

that is, $\langle f, g + T^*Tg\rangle = 0$ for all $g \in \mathscr{D}_{T^*T}$. Thus $I + T^*T$ being self-adjoint and one-to-one, its range is dense and $f = 0$.

Now if T commutes with τ clearly B and C commute with τ. Consequently we have

$$B = \int^{\oplus} B_\lambda \, d\mu(\lambda); \quad C = \int^{\oplus} C_\lambda \, d\mu(\lambda) \tag{7}$$

where B_λ and C_λ commute with τ_λ and so are of the form $I_\lambda \otimes b_\lambda$ and $I_\lambda \otimes c_\lambda$ respectively. Now B being one to one, B_λ and therefore b_λ is one to one for a.a. λ. Let $t_\lambda = c_\lambda b_\lambda^{-1}$ and $T_\lambda = I_\lambda \otimes t_\lambda = C_\lambda B_\lambda^{-1}$. (Note that B_λ is bounded and invertible). It follows that (T_λ) is a measurable family of bounded operators and that $T_0 \subseteq \int^{\oplus} T_\lambda \, d\mu(\lambda)$; hence

$$T \subseteq \int^{\oplus} T_\lambda \, d\mu(\lambda). \tag{8}$$

To prove the reverse inclusion let $\Delta_n = \{\lambda\colon \|B_\lambda^{-1}\| \leq n\}$. Let S_0 be the operator $S = \int^{\oplus} T_\lambda \, d\mu(\lambda)$ restricted to the set of fields which vanish on the complement of some set Δ_n. Then $S_0 \subseteq T$. In fact $S_0 \subseteq T_0$, for if f belongs to $\mathscr{D}_{S_0}$ we have $f \in D_{B^{-1}}$, $B^{-1}f = \int^{\oplus} B_\lambda^{-1} f_\lambda \, d\mu(\lambda)$ and so $f \in D_{T_0}$ and

$$T_0 f = CB^{-1}f = \int^{\oplus} C_\lambda B_\lambda^{-1} f_\lambda \, d\mu(\lambda) = S_0 f.$$

It remains to show that S is the closure of S_0. If $f \in \mathscr{D}_S$ the field $f_n = 1_{\Delta_n} f = f_{\Delta_n}$ converges to f and Sf_n converges to Sf. Thus $S = \overline{S_0} \subseteq T$, i.e., we have proved the existence of a representation (7). The essential uniqueness of the family (T_λ) is fairly obvious. It will also follow from the proof of corollary 2 below.

Corollary 1. *Under the assumption of Theorem 2 we have*

$$T^* = \int^{\oplus} T_\lambda^* \, d\mu(\lambda). \tag{9}$$

Proof. Let $S = \int^{\oplus} T_\lambda^* \, d\mu(\lambda)$. The inclusion $S \subseteq T^*$ is obvious. Let $f \mapsto P_\Delta f = 1_\Delta f$ be the spectral projection corresponding to the measurable set Δ. If $\Delta = \Delta_n = \{\lambda\colon \|T_\lambda\| \leq n\}$, TP_Δ is bounded and $P_\Delta T^* \subseteq (TP_\Delta)^* = T^*P_\Delta$. If T_0^* is the restriction of T^* to the set of fields vanishing off some set $\Delta = \Delta_n$, $T_0^* f = T^* f = \int_\Delta^{\oplus} T_\lambda^* f_\lambda \, d\mu(\lambda) = Sf$. Thus $T_0^* \subseteq S$. But T^* equals $\overline{T_0^*}$. In fact if f belongs to D_{T^*} and $f_n = P_{\Delta_n} f$, f_n converges to f and $T^* f_n = T^* P_{\Delta_n} f = P_{\Delta_n} T^* f$ converges to $T^* f$. Thus $T^* \subseteq S$.

Remark. If T is symmetric the operators T_λ are obviously symmetric a.e. and so $T = T^*$. Thus we recover the first theorem from Theorem 2.

COROLLARY 2. *Let S and T be densely defined closed invariant operators such that $S \subseteq T$. Then $S = T$.*

Proof. We have $S = \int^{\oplus} S_\lambda \, d\mu(\lambda)$ and $T = \int^{\oplus} T_\lambda \, d\mu(\lambda)$. Using cutoff sets

$$\Delta_n = \{\lambda \colon \|T_\lambda\| \leq n \text{ and } \|S_\lambda\| \leq n\}$$

it is easy to see that $S \subseteq T$ implies $S_\lambda \subseteq T_\lambda$ a.a. λ. Thus S_λ and T_λ being bounded, $S_\lambda = T_\lambda$ a.e. and $S = T$.

COROLLARY 3. *Let T be a closed invariant densely defined operator. Then if $\mathscr{D}$ is a dense invariant subspace contained in $\mathscr{D}_T$ and T_0 is the restriction of T to $\mathscr{D}$ we have $T = \overline{T_0}$.*

Let $\mathscr{D}$ be a dense invariant subspace of $\mathscr{H}$. Let $\mathscr{A}$ be a set of invariant operators T_0 defined on $\mathscr{D}$, such that $T_0(\mathscr{D}) \subset \mathscr{D}$ and such that for each $T_0 \in \mathscr{A}$ the domain of its adjoint contains $\mathscr{D}$ and $T_0^*|_{\mathscr{D}}$ belongs to $\mathscr{A}$. We assume that $\mathscr{A}$ is closed under addition and products. Thus $\mathscr{A}$ is a $*$-algebra of operators commuting with τ.

THEOREM 3. *Let $\mathscr{A}$ be a $*$-algebra of invariant operators as described above, which is moreover commutative. Then:*

(1) *The closure $T = \overline{T_0}$ of each operator $T_0 \in \mathscr{A}$ is normal.*

(2) *If T_0 and S_0 belong to $\mathscr{A}$ their closures T and S commute strongly, i.e., they have commuting spectral resolutions.*

(3) *In the primary decomposition of τ, $\mathscr{H} = \int^{\oplus} \mathscr{H}_\lambda \, d\mu(\lambda)$, with $\mathscr{H}_\lambda = \mathscr{K}_\lambda \otimes \mathbf{C}^{n_\lambda}$ it is possible to choose the coordinates in such a way that for every $T_0 \in \mathscr{A}$ the closure T has an expression (6), where $T_\lambda = I_\lambda \otimes t_\lambda$, with t_λ a diagonal matrix.*

Proof. (1) The domain of T_0^* is dense because it contains $\mathscr{D}$. Therefore T_0 is closable. Let T be its closure. Then T is invariant and so $T = \int^{\oplus} T_\lambda \, d\mu(\lambda)$. If $S_0 \in \mathscr{A}$ we similarly have $S = \int^{\oplus} S_\lambda \, d\mu(\lambda)$. Now $STf = TSf$ for all $f \in \mathscr{D}$, implies $T_\lambda S_\lambda f_\lambda = S_\lambda T_\lambda f_\lambda$ for all $f \in \mathscr{D}$ and almost all λ. A standard argument using the separability of $\mathscr{H}$ shows that this implies $S_\lambda T_\lambda = T_\lambda S_\lambda$ for a.a. λ. If $S_0 = T_0^*|_{\mathscr{D}}$ we have $S \subseteq T^*$ and so $S = T^*$ by Corollary 2. Thus (Cor. 1) $S_\lambda = T_\lambda^*$ a.a. λ, which implies that T_λ is normal for a.a. λ. We obviously have

$$T^*T \subseteq \int^{\oplus} T_\lambda^* T_\lambda \, d\mu(\lambda),$$

but T^*T being closed and invariant we actually have equality:

$$T^*T = \int^{\oplus} T_\lambda^* T_\lambda \, d\mu(\lambda) \quad (\text{Cor. } 2).$$

Similarly $TT^* = \int^{\oplus} T_\lambda T_\lambda^* \, d\mu(\lambda)$. Thus $T^*T = TT^*$, i.e., T is normal.

(2) If $\Phi\colon \mathbf{C} \to \mathbf{C}$ is a Borel function it is clear that $\Phi(T) = \int^{\oplus} \Phi(T_\lambda) \, d\mu(\lambda)$. In particular, if Φ is bounded this implies that $\Phi(T)$ commutes with $\Phi(S)$, which implies that T and S strongly commute.

(3) This involves some separability property of the algebra $\mathscr{A}$. Because $\mathscr{A}$ does not have any topology we sketch an indirect method. Let $\mathscr{M}$ be the von Neumann algebra generated by the bounded operators TP_Δ (cf. proof Cor. 1). Then $\mathscr{H}$ being separable $\mathscr{M}$ is weakly separable. The operators $A \in \mathscr{M}$ have expressions (2) with $A_\lambda = I_\lambda \otimes a_\lambda$ and a_λ normal for a.a. λ. Also, if $B \in \mathscr{M}$, $a_\lambda b_\lambda = b_\lambda a_\lambda$ for a.a. λ. If $(A_n)_{n \in \mathbf{N}}$ is a sequence in $\mathscr{M}$ which is weakly dense, we can find a null-set N such that for $\lambda \in CN$ the matrices $a_{n,\lambda}$ are normal and commute. Now choose the coordinates so that these matrices are diagonal. Then for all $A \in \mathscr{M}$ a_λ is diagonal for a.a. λ, and after an appropriate modification a_λ is diagonal for all A and λ. But then t_λ is diagonal for all $T_0 \in \mathscr{A}$ and all λ.

Note added in proof. I should like to thank Dr. N.P. Landsman for kindly pointing out that the main result of this paper can also be deduced from a theorem of I.E. Segal [8] (Theorem 5), according to which a closed symmetric operator affiliated with a finite von Neumann algebra is essentially self-adjoint. However the proof of the fact that we are in the case of a finite von Neumann algebra seems to be essentially equivalent to our direct proof, which may therefore still be of interest.

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