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EXAMPLES OF SOLVMANIFOLDS WITHOUT CERTAIN AFFINE STRUCTURE[1]

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Let G be a connected and simply connected solvable Lie group. J. Milnor [Milnor] in 1977 conjectured that G admits a complete affinely flat structure invariant under left translations. Recently Boyom [Boyom] has published a positive answer for nilpotent groups using complicated Lie algebra and left symmetric algebra constructions. The general question remains open. See the note added in proof at the end of the paper.

Suppose G admits a lattice π. Let N be the nil-radical of G and $\Gamma = N \cap \pi$. Then Γ is a lattice in N. G is diffeomorphic to $\mathbf{R}^m$ and it is known that $\pi \backslash \mathbf{R}^m$ is a bundle over a torus with a nil-manifold $\Gamma \backslash N$ a fiber. Now, if there is an embedding of π into $\mathrm{Aff}(m) = \mathbf{R}^m \rtimes GL(m, \mathbf{R})$, $m = \mathrm{rank}(\pi)$ so that π acts as a group of affine diffeomorphisms of $\mathbf{R}^m$, it is tempting to expect that the center of Γ is carried into pure translations on $\mathbf{R}^m$ and that π/Γ also induces pure translations on $\mathbf{R}^{m-\mathrm{rank}(\Gamma)}$. In fact, one can expect that π imbeds into $\mathrm{Aff}(m) = \mathbf{R}^m \rtimes GL(m, \mathbf{R})$ in the blocked form

$$\left(\begin{bmatrix} A & M \\ O & I \end{bmatrix}, \begin{bmatrix} a \\ b \end{bmatrix}\right) \in \mathbf{R}^m \rtimes GL(m, \mathbf{R})$$

where $A \in GL(k, \mathbf{R})$, $k = \mathrm{rank}(\Gamma)$ and O is the zero matrix; with the center of Γ mapping into pure translations.

This approach has proved successful for 2-step nilpotent Lie groups; see [Lee]. Many others, including [Nisse], have attempted to prove the conjecture using induction based on exploiting this idea. This technique, as we shall show, will, unfortunately, not work in general. We present here an example of a solvmanifold which is a torus bundle over a torus. We show that *it does not admit an affine structure of the expected type*. In particular, the expected type of embedding of this example would be "canonical" in the sense of Nisse. Therefore, our example shows that Proposition 2.1 and Théoreme B of [Nisse] is incorrect.

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Example. There is a finitely generated, torsion-free solvable group π *with the following property*:

(1) π *has discrete nil-radical* (*i.e., maximal normal nilpotent subgroup*) $\mathbf{Z}^{13}$ *and* $\pi/\mathbf{Z}^{13}$ *is* $\mathbf{Z}^4$ *so that* $1 \to \mathbf{Z}^{13} \to \pi \to \mathbf{Z}^4 \to 1$ *exact.*

(2) *There is no embedding of* π *into* Aff(17) *in the blocked form*

$$\left(\begin{bmatrix} A & M \\ O & I \end{bmatrix}, \begin{bmatrix} a \\ b \end{bmatrix}\right) \in \mathbf{R}^{17} \rtimes GL(17, \mathbf{R}),$$

where $A \in GL(13, \mathbf{R})$ *and* O *is the zero matrix*; *which sends the discrete nil-radical* $\mathbf{Z}^{13}$ *of* π *into pure translations*

$$\left(\begin{bmatrix} I & O \\ O & I \end{bmatrix}, \begin{bmatrix} e_i \\ 0 \end{bmatrix}\right) \in \mathbf{R}^{17} \rtimes GL(17, \mathbf{R})$$

for $i = 1, 2, \cdots, 13$; *and the quotient group* $\mathbf{Z}^4$ *acts on* $\mathbf{R}^4$ *as translations.*

(3) *There exists a connected, simply connected solvable Lie group* S *which contains* π *as a lattice. The Lie group* S *has a nilradical* $N = \mathbf{R}^{13}$ *and* $[S, S] = N$ *so that* $S = N \rtimes \mathbf{R}^4$. *There is no embedding* θ *of* S *into* Aff(17) *in such a way that* N *maps into the pure translational part of* Aff(17) *and* θ *induces an action of* $S/N = \mathbf{R}^4$ *on* $\mathbf{R}^4 = \mathbf{R}^{17}/\mathbf{R}^{13}$ *which is a pure translation.*

After we construct and show that the example does not embed in Aff(17) in the expected way we shall embed π in Aff(17) in a different way. To begin the construction let Γ be the group generated by $E_1, E_2, \ldots, E_9$ with the relations (using $[a, b] = a \cdot b \cdot a^{-1} \cdot b^{-1}$)

$$\begin{aligned} &[E_6, E_4] = E_1 \\ &[E_7, E_5] = E_2, \qquad [E_7, E_6] = E_3^{-1} \\ &[E_8, E_3] = E_1^{-1}, \qquad [E_8, E_7] = E_4^{-1} \\ &[E_9, E_4] = E_2^{-1}, \qquad [E_9, E_8] = E_5^{-1} \\ &[E_i, E_j] = 1 \text{ for all others.} \end{aligned}$$

The commutator subgroup $[\Gamma, \Gamma]$ is generated by $E_1, E_2, \ldots, E_5$ and the quotient $\Gamma/[\Gamma, \Gamma]$ is isomorphic to $\mathbf{Z}^4$. From the above presentation, it is clear that $[\Gamma, \Gamma]$ is isomorphic to $\mathbf{Z}^5$. Thus, we have a short exact sequence

$$1 \to \mathbf{Z}^5 \to \Gamma \to \mathbf{Z}^4 \to 1$$

Let the group $\mathbf{Z}^5$ act on $\mathbf{R}^5$ as standard translations; $\mathbf{Z}^4$ act on $\mathbf{R}^4$ as standard translations, via ρ: $\mathbf{Z}^4 \to$ Aff(4); and act on $\mathbf{Z}^5$ by conjugations, via φ: $\mathbf{Z}^4 \to GL(5, \mathbf{Z})$. If we denote the images of $E_i \in \Gamma$ by $\overline{E_i} \in \mathbf{Z}^4$ for $i = 6, 7, 8$

and 9, then $\varphi\colon \mathbf{Z}^4 \to GL(5,\mathbf{Z})$ can be read off from the presentation of Γ above. The conjugation by E_i is denoted by $\mu(E_i)$ so that $\mu(E_i)(x) = E_i x E_i^{-1}$. From $[E_6, E_4] = E_1$, we have $\mu(E_6)(E_4) = E_1E_4$, and $\mu(E_6)(E_i) = E_i$ for $i = 1, 2, 3$ and 5. Thus,

$$\varphi(\overline{E_6}) = I + e_{1,4}$$

where I is the identity matrix; $e_{i,j}$ is the matrix whose (i, j)th entry is 1 and 0 elsewhere, both 5×5 matrices. Similarly, we have

$$\varphi(\overline{E_7}) = I + e_{2,5}$$
$$\varphi(\overline{E_8}) = I - e_{1,3}$$
$$\varphi(\overline{E_9}) = I - e_{2,4}$$

Suppose there is a homomorphism $\theta\colon \Gamma \to \mathrm{Aff}(9)$ such that

$$\theta(E_i) = \left(\begin{bmatrix} I & O \\ O & I \end{bmatrix}, \begin{bmatrix} e_i \\ 0 \end{bmatrix}\right)$$

for $i = 1,2,3,4,5$, and

$$\theta(E_l) = \left(\begin{bmatrix} \varphi(\overline{E_l}) & X_l \\ O & I \end{bmatrix}, \begin{bmatrix} x_l \\ \bar{e}_{l-5} \end{bmatrix}\right)$$

for $l = 6,7,8,9$, where $X_l = ((X_l)_{(i,j)})$, $x_l = ((x_l)_i)$ are 5×4, 5×1 matrices, respectively; $\{e_i\colon i = 1,2,3,4,5\}$ and $\{\bar{e}_l\colon l = 1,2,3,4\}$ are the standard bases for $\mathbf{R}^5, \mathbf{R}^4$, respectively.

From the commutator relations $[E_8, E_7] = E_4^{-1}$, $[E_7, E_6] = E_3^{-1}$ and $[E_9, E_8] = E_5^{-1}$, we have

$$E_7E_8 = E_4E_8E_7, \quad E_6E_7 = E_3E_7E_6, \quad E_8E_9 = E_5E_9E_8.$$

By comparing $(4,10), (1,8), (2,7)$ entry of these identities, we get

$$(X_7)_{(4,3)} - (X_8)_{(4,2)} = 1$$
$$(X_7)_{(4,3)} = 0$$
$$(X_8)_{(4,2)} = 0$$

which is an inconsistent system. In the above, we used the inclusion Aff(9) $\subset GL(10, \mathbf{R})$ via

$$(A, a) \mapsto \left(\begin{bmatrix} A & a \\ O & 1 \end{bmatrix}\right).$$

We have shown that there is no such a homomorphism θ.

This implies also that there is no embedding θ' such that $\theta'(E_1), \ldots, \theta'(E_5)$ are translations (may be non-standard) and $\theta'(E_6), \ldots, \theta'(E_9)$ induce translations on $\mathbf{R}^4$ (again, this may be non-standard). For, let

$$A = [\theta'(E_1) \cdots \theta'(E_5)]$$

($\theta'(E_i)$ is a column vector) and

$$B = [\theta'(E_6) \cdots \theta'(E_9)].$$

Then

$$\theta'' = \left(\begin{bmatrix} A & O \\ O & B \end{bmatrix}, 0\right) \circ \theta' \circ \left(\begin{bmatrix} A & O \\ O & B \end{bmatrix}, 0\right)$$

is a new embedding such that $\theta''(E_1), \ldots, \theta''(E_5)$ are standard translations and $\theta''(E_6), \ldots, \theta''(E_9)$ induce pure translations on $\mathbf{R}^4$.

Note, however, that Γ is 3-step nilpotent and there does exist a nice embedding into Aff(9) if we iterate the Seifert Fiber Space Construction with the center of the nilpotent groups, see below.

We digress to Seifert fiberings for a moment. Let G be a Lie group and W be a smooth simply connected manifold. Consider the product $P = G \times W$. The group $G = l(G)$ acts as left translation on the first factor of $P = G \times W$. Consider the group $\mathrm{Diff}_G(P)$ of all weakly G-equivariant smooth diffeomorphisms of P onto itself. That is, $f \in \mathrm{Diff}_G(P)$ if and only if there exists an $a \in \mathrm{Aut}(G)$ such that $f(g \cdot x, w) = a(g) \cdot f(x, w)$ for all $(x, w) \in P$ and $g \in G$. It is known that $\mathrm{Diff}_G(P)$ is exactly the normalizer of $G = l(G)$ in $\mathrm{Diff}(P)$. Let $C(W, G^*)$ be the group of all smooth maps from W to G with the group operation $(\lambda * \eta)(w) = \eta(w) \cdot \lambda(w)$. Moreover, one can think of it as a subgroup of $\mathrm{Diff}_G(P)$ by $\lambda(x, w) = (x \cdot \lambda(w), w)$. The group $\mathrm{Aut}(G) \times \mathrm{Diff}(W)$ acts on $C(W, G^*)$ by: $(a, h) \cdot \lambda = a \circ \lambda \circ h^{-1}$. Then it turns out that

$$\mathrm{Diff}_G(P) = C(W, G^*) \rtimes (\mathrm{Aut}(G) \times \mathrm{Diff}(W)).$$

An element (λ, a, h) acts on $(x, w) \in G \times W$ by

$$(\lambda, a, h) \cdot (x, w) = (a(x)\lambda(h(w)), h(w))$$

There is an exact sequence

$$1 \to C(W, G^*) \rtimes \mathrm{Inn}(G) \to \mathrm{Diff}_G(P) \to \mathrm{Out}(G) \times \mathrm{Diff}(W) \to 1$$

See [LR] for more details. For $W = \mathbf{R}^n$ and $G = \mathbf{R}^k$, there is a nice subspace $L(\mathbf{R}^n, \mathbf{R}^k)$ of $C(W, G^*)$. Elements of $L(\mathbf{R}^n, \mathbf{R}^k)$ are maps of the form $x \mapsto A \cdot x + a$, where A is an $m \times n$ matrix and $a \in \mathbf{R}^k$.

For the extension

$$1 \to \mathbf{Z}^5 \to \Gamma \cdot \to \mathbf{Z}^4 \to 1$$

together with the representation

$$\varphi \times \rho \colon \mathbf{Z}^4 \to GL(5, \mathbf{Z}) \times \mathrm{Aff}(4),$$

we attempt the Seifert Fiber Space Construction to embed Γ into $\mathrm{Diff}_G(P)$. In the case at hand, $G = \mathbf{R}^5$, $W = \mathbf{R}^4$, $P = \mathbf{R}^5 \times \mathbf{R}^4$. We use the subgroup $L(\mathbf{R}^4, \mathbf{R}^5) \rtimes (GL(5, \mathbf{R}) \times \mathrm{Aff}(4))$ of $\mathrm{Aff}(9) \subset \mathrm{Diff}_G(P)$. Note that $L(\mathbf{R}^4, \mathbf{R}^5)$ is a subgroup of $C(\mathbf{R}^4, \mathbf{R}^5)$ which is invariant under the action of $GL(5, \mathbf{R}) \times \mathrm{Aff}(4)$.

According to the Seifert Theory, there exists a homomorphism $\theta \colon \Gamma \to \mathrm{Aff}(9)$ making the diagram

$$\begin{array}{ccccccccc}
1 \to & \mathbf{Z}^5 & \to & \Gamma & \to & \mathbf{Z}^4 & \to 1 \\
& \downarrow & & \downarrow & & \downarrow & \\
1 \to & L(\mathbf{R}^4, \mathbf{R}^5) & \to & L(\mathbf{R}^4, \mathbf{R}^5) \rtimes (GL(5, \mathbf{R}) \times \mathrm{Aff}(4)) & \to & GL(5, \mathbf{R}) \times \mathrm{Aff}(4) & \to 1
\end{array}$$

commutative if and only if $[\Gamma] \in H^2(\mathbf{Z}^4; \mathbf{Z}^5)$ is in the image of the connecting homomorphism

$$\delta \colon H^1(\mathbf{Z}^4; L(\mathbf{R}^4, \mathbf{R}^5)/\mathbf{Z}^5) \to H^2(\mathbf{Z}^4; \mathbf{Z}^5).$$

According to the action of $L(\mathbf{R}^4, \mathbf{R}^5) \rtimes (GL(5, \mathbf{R}) \times \mathrm{Aff}(4))$ on $\mathbf{R}^5 \times \mathbf{R}^4$,

$$(\lambda, g, h)(x, w) = (g(x) + \lambda(h(w)), h(w)).$$

Thus, $((D, d), A, (B, b))$ corresponds to

$$\left(\begin{bmatrix} A & DB \\ O & B \end{bmatrix}, \begin{bmatrix} Db + d \\ b \end{bmatrix} \right) \in \mathrm{Aff}(9).$$

However, the calculation earlier showed that there is no embedding of Γ into $\mathrm{Aff}(9)$ of the shape described above. This means that $[\Gamma]$ is not in the image of

$$\delta\colon H^1(\mathbf{Z}^4; L(\mathbf{R}^4,\mathbf{R}^5)/\mathbf{Z}^5) \to H^2(\mathbf{Z}^4;\mathbf{Z}^5);$$

or equivalently, $j_1[\Gamma] \neq 0$ under $j_1\colon H^2(\mathbf{Z}^4;\mathbf{Z}^5) \to H^2(\mathbf{Z}^4; L(\mathbf{R}^4,\mathbf{R}^5))$ which is induced from the coefficient inclusion.

To construct our solvable example, we do the following construction. Let Γ act on $\mathbf{Z}^8$ via $\Gamma \to \mathbf{Z}^4 \to GL(8,\mathbf{Z})$, where the second homomorphism ψ: $\mathbf{Z}^4 \to GL(8,\mathbf{Z})$ is given as follows:

$$\psi(\overline{E_6}) = J \oplus I \oplus I \oplus I,\ \psi(\overline{E_7}) = I \oplus J \oplus I \oplus I,\ \psi(\overline{E_8}) = I \oplus I \oplus J \oplus I,$$

and

$$\psi(\overline{E_9}) = I \oplus I \oplus I \oplus J,$$

where

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad\text{and}\quad J = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}.$$

Now form the semi-direct product $\pi = \mathbf{Z}^8 \rtimes \Gamma$. Then the discrete nil-radical of π is $\mathbf{Z}^{13} = \mathbf{Z}^8 \oplus \mathbf{Z}^5$ with quotient $\mathbf{Z}^4$ so that $1 \to \mathbf{Z}^{13} \to \pi \to \mathbf{Z}^4 \to 1$ is exact.

Let $i\colon L(\mathbf{R}^4,\mathbf{R}^5) \to L(\mathbf{R}^4,\mathbf{R}^{13})$ be the homomorphism sending (D,d) to

$$\left(\begin{bmatrix} O \\ D \end{bmatrix}, \begin{bmatrix} 0 \\ d \end{bmatrix}\right),$$

a $\mathbf{Z}^4$-equivariant injection. Let

$$i_1\colon H^2(\mathbf{Z}^4;\mathbf{Z}^5) \to H^2(\mathbf{Z}^4;\mathbf{Z}^{13}),$$
$$i_2\colon H^2(\mathbf{Z}^4; L(\mathbf{R}^4,\mathbf{R}^5)) \to H^2(\mathbf{Z}^4; L(\mathbf{R}^4,\mathbf{R}^{13}))$$

be the homomorphisms induced from i. Then $[\pi] = i_1([\Gamma])$.

First we prove that i_2 is injective. Observe that the action of $\mathbf{Z}^4$ on $\mathbf{Z}^{13}$ is given by

$$\psi \oplus \varphi\colon \mathbf{Z}^4 \to GL(8,\mathbf{Z}) \times GL(5,\mathbf{Z}).$$

Therefore, $L(\mathbf{R}^4,\mathbf{R}^{13}) = L(\mathbf{R}^4,\mathbf{R}^8) \oplus L(\mathbf{R}^4,\mathbf{R}^5)$ as $\mathbf{Z}^4$-modules. This implies that

$$H^2(\mathbf{Z}^4;\mathbf{Z}^{13}) = H^2(\mathbf{Z}^4;\mathbf{Z}^8) \oplus H^2(\mathbf{Z}^4;\mathbf{Z}^5);$$

and

$$H^2(\mathbf{Z}^4; L(\mathbf{R}^4, \mathbf{R}^{13})) = H^2(\mathbf{Z}^4; L(\mathbf{R}^4, \mathbf{R}^8)) \oplus H^2(\mathbf{Z}^4; L(\mathbf{R}^4, \mathbf{R}^5)).$$

Consider the commutative diagram

$$\begin{array}{ccc} H^2(\mathbf{Z}^4;\mathbf{Z}^5) & \xrightarrow{j_1} & H^2(\mathbf{Z}^4; L(\mathbf{R}^4,\mathbf{R}^5)) \\ \downarrow{\scriptstyle i_1} & & \downarrow{\scriptstyle i_2} \\ H^2(\mathbf{Z}^4;\mathbf{Z}^{13}) & \xrightarrow{j_2} & H^2(\mathbf{Z}^4; L(\mathbf{R}^4,\mathbf{R}^{13})) \end{array}$$

If $j_2[\pi] = 0$, then $0 = j_2[\pi] = j_2 i_1[\Gamma] = i_2 j_1[\Gamma]$. Since i_2 is injective, we must have $j_1[\Gamma] = 0$, which is not true from earlier observation. This means that $j_2[\pi] \neq 0$. Consequently, π cannot be embedded into Aff(17) in the expected way because π is not in the image of

$$\delta\colon H^1(\mathbf{Z}^4; L(\mathbf{R}^4, \mathbf{R}^{13})/\mathbf{Z}^{13}) \to H^2(\mathbf{Z}^4;\mathbf{Z}^{13}).$$

On the other hand, we can embed the group Γ into Aff(9) by the Seifert Fiber Space Construction using the upper central series. We have central extensions

$$1 \to \mathcal{Z}(\Gamma) \to \Gamma \to \Gamma_1 \to 1, \qquad \mathcal{Z}(\Gamma) \cong \mathbf{Z}^2, \Gamma_1 = \Gamma/\mathcal{Z}(\Gamma),$$
$$1 \to \mathcal{Z}(\Gamma_1) \to \Gamma_1 \to \Gamma_2 \to 1, \qquad \mathcal{Z}(\Gamma_1) \cong \mathbf{Z}^3, \Gamma_2 = \Gamma_1/\mathcal{Z}(\Gamma_1) \cong \mathbf{Z}^4$$

where $\mathcal{Z}(\Gamma) \cong \mathbf{Z}^2$ is generated by E_1 and E_2; $\mathcal{Z}(\Gamma_1) \cong \mathbf{Z}^3$ is generated by E_3, E_4 and E_5; $\Gamma_2 \cong \mathbf{Z}^4$ is generated by E_6, E_7, E_8 and E_9.

Each embedding of $\Gamma_1 \to$ Aff(7) makes $L(\mathbf{R}^7, \mathbf{R}^2)/\mathbf{Z}^2$ a Γ_1-module. According to Scheuneman's result employed by [Lee; p. 1445], for every element $[\Gamma] \in H^2(\Gamma_1; \mathbf{Z}^2)$, there exists $\Gamma_1 \to$ Aff(7) for which $[\Gamma]$ lies in the image of $\delta\colon H^1(\Gamma_1; L(\mathbf{R}^7, \mathbf{R}^2)/\mathbf{Z}^2) \to H^2(\Gamma_1; \mathbf{Z}^2)$. Consequently we can embed Γ into Aff(9) resulting in a Seifert Construction where the typical fiber is a principal affinely flat 2-torus bundle over a 7 dimensional nilmanifold. Using the inclusion Aff(9) $\subset GL(10, \mathbf{R})$ via

$$(A, a) \mapsto \left(\begin{bmatrix} A & a \\ O & 1 \end{bmatrix}\right),$$

we give an explicit embedding which arises from the above construction:

$$E_1 = I + e_{1,10}, \qquad E_2 = I + e_{2,10},$$
$$E_3 = I + e_{1,8} + e_{3,10}, \qquad E_4 = I + e_{2,9} + e_{4,10}, \qquad E_5 = I + e_{5,10},$$
$$E_6 = I + e_{1,4} + e_{3,7} + e_{6,10}, \quad E_7 = I + e_{2,5} + e_{4,8} + e_{7,10},$$
$$E_8 = I + e_{5,9} + e_{8,10}, \quad E_9 = I + e_{9,10}$$

where I denotes the identity matrix in $GL(10, \mathbf{R})$.

To embed π into Aff(17), first embed $\mathbf{Z}^8$ as the standard translations on the first 8 factors of $\mathbf{R}^{17}$. Then combine the embedding of Γ described above. A general element of π will be of the form

$$\left(\begin{bmatrix} B & O \\ O & A \end{bmatrix}, \begin{bmatrix} b \\ a \end{bmatrix}\right)$$

where $B \in GL(8, \mathbf{R})$ comes from $\psi(\overline{E_i})$'s and $A \in GL(9, \mathbf{R})$ comes from the embedding of Γ from above.

In order to find a solvable Lie group S which contains π as a lattice, we extend the representation $\mathbf{Z} \to GL(2, \mathbf{R})$ sending 1 to the matrix J as follows: Let $a = \frac{1}{2}(3 + \sqrt{5})$; $b = \log a$;

$$P = \begin{bmatrix} a-2 & a-1 \\ 1 & -1 \end{bmatrix}.$$

Define φ: $\mathbf{R} \to GL(2, \mathbf{R})$ by

$$\varphi(t) = P\begin{bmatrix} e^{tb} & 0 \\ 0 & e^{-tb} \end{bmatrix}P^{-1}$$

Then

$$\varphi(1) = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} = J$$

and $\mathbf{R}^2 \rtimes_\varphi \mathbf{R}$ is a solvable Lie group. Thus we get $\mathbf{R}^4 \to (GL(2, \mathbf{R}))^4$. Let N be the Malcev completion of Γ; that is, the connected, simply connected, nilpotent Lie group which contains Γ as a lattice. The above homomorphism induces a representation $N \to \mathbf{R}^4 \to (GL(2, \mathbf{R}))^4$. Let S be the semi-direct product $S = \mathbf{R}^8 \rtimes_\varphi N$. Then S is a solvable Lie group containing π as a lattice. The nilradical of S is $\mathbf{R}^{13} = \mathbf{R}^8 \times \mathbf{R}^5$, where $\mathbf{R}^5$ is the subgroup of N generated by $E_1, \ldots, E_5$. Clearly $[S, S] = \mathbf{R}^{13}$.

Such a connected, simply connected solvable Lie group S cannot be embedded into Aff(17) in a "canonical" way (i.e., "canonical" for a solvable Lie group on analogy to the "canonical" for poly $\mathbf{Z}$-groups as defined by [Nisse]). For, such an embedding would restrict to an embedding of the lattice π into Aff(17) such that $\theta(E_1), \ldots, \theta(E_5)$ are the standard translations and $\theta(E_6), \ldots, \theta(E_9)$ induce the standard translations on $\mathbf{R}^4$.

Added in proof. Paul Igodt and Karel Dekimpe constructed a nilpotent example which does not admit a "canonical type" affine structure. See *Computational aspects of affine representations for torsion free nilpotent groups via the Seifert construction*, to appear in Journal of Pure and Applied Algebra.

Boyom claims that the conjecture is true in general; see N. Boyom, *Sur les structures affines homotopes à zéro des groupes de Lie*, J. Diff. Geometry **31** (1990), 859–911. In the meantime, there is a recent preprint, Y. Benoist, *Une nilvariete non affine*, claiming that there exists an 11-dimensional nilpotent Lie group which does not admit affine structure, thereby disproving Boyom's result. It is not yet known at this time if the example is correct.

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